1. **For a certain RR manipulator, the equations of motion are given by**

$$\begin{bmatrix} 4+c_2 & 1+c_2 \\ 1+c_2 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -s_2(\dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) \\ s_2\dot{\theta}_1^2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

(a) *Assume that joint 2 is locked at some value θ_2 using brakes and joint 1 is controlled with a PD controller, $\tau_1 = -40\dot{\theta}_1 - 400(\theta_1 - \theta_{1d})$. What is the minimum and maximum inertia perceived at joint 1 as we vary θ_2? What are the corresponding closed-loop frequencies?*

For joint 2 locked ($\ddot{\theta}_2 = \dot{\theta}_2 = 0$), the equation of motion for joint 1 is:

$$(4 + c_2)\ddot{\theta}_1 = \tau_1$$

The inertia seen at joint 1 is the coefficient of the $\ddot{\theta}_1$ term, $(4 + c_2)$. So, this inertia achieves its maximum and minimum values at $\theta_2 = 0$ and $\theta_2 = 180°$:

$$m_{max} = 5, \; m_{min} = 3$$

The closed-loop equation for joint 1 is

$$(4 + c_2)\ddot{\theta}_1 + 40\dot{\theta}_1 + 400(\theta_1 - \theta_{1d}) = 0$$

To get an expression for closed loop frequency, we compare our closed loop equation with the generic system of Equation 7.9 ($m\ddot{x} + b\dot{x} + kx = 0$).

The closed loop frequency is then given by:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{400}{(4 + c_2)}}$$

So, we have

$$m = m_{max} \quad \Rightarrow \quad \omega_{min} = \frac{20}{\sqrt{5}}$$
$$m = m_{min} \quad \Rightarrow \quad \omega_{max} = \frac{20}{\sqrt{3}}$$

(b) *Still assuming that joint 2 is locked, at what values of θ_2 do the minimum and maximum damping ratios occur? What are the minimum and maximum damping ratios?*

To get an expression for damping ratio, we once again compare our closed loop equation with the generic system of Equation 7.9 ($m\ddot{x}+b\dot{x}+kx = 0$). In this case, using Equation 7.12 the closed-loop damping ratio is given by:

$$\xi = \frac{b}{2\sqrt{km}} = \frac{40}{2\sqrt{400m}} = \frac{1}{\sqrt{m}} = \frac{1}{\sqrt{4 + c_2}}$$

So, the minimum and maximum values of ξ occur at $\theta_2 = 0$ and $\theta_2 = 180°$:

$$\begin{aligned} \xi_{min} &= \frac{1}{\sqrt{m_{max}}} = \frac{1}{\sqrt{5}} \\ \xi_{max} &= \frac{1}{\sqrt{m_{min}}} = \frac{1}{\sqrt{3}} \end{aligned}$$

(c) *Now assume that both joints are free to move, and that this system is controlled by a partitioned PD controller, $\tau = \alpha\tau' + \beta$. Design a partitioned, trajectory-following controller (one that tracks a desired position, velocity and acceleration) which will provide a closed-loop frequency of 10 rad/sec on joint 1 and 20 rad/sec on joint 2 and be critically damped over the entire workspace. That is, let*

$$\tau' = \ddot{\theta}_d - \begin{bmatrix} k'_{v_1} & 0 \\ 0 & k'_{v_2} \end{bmatrix} (\dot{\theta} - \dot{\theta}_d) - \begin{bmatrix} k'_{p_1} & 0 \\ 0 & k'_{p_2} \end{bmatrix} (\theta - \theta_d),$$

then find the matrices α and β and the vector τ, along with the necessary gains k'_{v_i} and k'_{p_i}.

The equations of motion are of the form

$$M(\theta)\ddot{\theta} + V(\dot{\theta}, \theta) = \tau$$

to which we apply a vector of torques τ of the form

$$\tau = \alpha\tau' + \beta$$

To make this look like a unit-mass system, we let

$$\alpha = M(\theta), \ \beta = V(\dot{\theta}, \theta)$$

which gives the unit-mass system

$$\ddot{\theta} = \tau'$$

To this system, we apply the control

$$\tau' = \ddot{\theta}_d - \begin{bmatrix} k'_{v_1} & 0 \\ 0 & k'_{v_2} \end{bmatrix} (\dot{\theta} - \dot{\theta}_d) - \begin{bmatrix} k'_{p_1} & 0 \\ 0 & k'_{p_2} \end{bmatrix} (\theta - \theta_d),$$

This yields two closed-loop equations

$$\begin{aligned} \ddot{e}_1 + k'_{v_1}\dot{e}_1 + k'_{p_1}e_1 &= 0 \\ \ddot{e}_2 + k'_{v_2}\dot{e}_2 + k'_{p_2}e_2 &= 0 \end{aligned}$$

where e_i is the error at joint i, $e_i = (\theta_i - \theta_{id})$. Now, we need to choose k'_{v_i} and k'_{p_i} to achieve critical damping, and to achieve our desired closed-loop frequencies. For a unit-mass system, we choose

$$\begin{aligned} k'_{p_i} &= \omega_i^2 \\ k'_{v_i} &= 2\xi_i\omega_i \end{aligned}$$

So, we get

$$k'_{p_1} = 100, \ k'_{v_1} = 20$$
$$k'_{p_2} = 400, \ k'_{v_2} = 40$$

(d) *If $\theta_2 = 180°$, what is the steady-state error vector for a given disturbance torque, $\tau_{dist} = [2\ 4]^T$?*
The controlled system, with a disturbance torque τ_{dist} is

$$M(\theta)\ddot{\theta} + V(\dot{\theta}, \theta) = \tau + \tau_{dist}$$

Substituting in our form for $\tau = \alpha\tau' + \beta$ yields

$$M(\theta)\ddot{\theta} - M(\theta)\tau' = \tau_{dist}$$

This has the form

$$M(\theta)\left[\ddot{\mathbf{e}} + K_v'\dot{\mathbf{e}} + K_p'\mathbf{e}\right] = \tau_{dist}$$

where $\mathbf{e}$ is the error vector $\mathbf{e} = \theta - \theta_d$, and K_v' and K_p' are the matrices given by

$$K_v' = \begin{bmatrix} k_{v_1}' & 0 \\ 0 & k_{v_2}' \end{bmatrix}, \ K_p' = \begin{bmatrix} k_{p_1}' & 0 \\ 0 & k_{p_2}' \end{bmatrix}$$

In the steady state ($\ddot{\mathbf{e}} = \dot{\mathbf{e}} = \mathbf{0}$), the equation is

$$M(\theta)K_p'\mathbf{e} = \tau_{dist}$$

which means that the steady state error is

$$\mathbf{e} = (M(\theta)K_p')^{-1}\tau_{dist}$$

For our values, this is:

$$\mathbf{e} = \left(\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 100 & 0 \\ 0 & 400 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 300 & 0 \\ 0 & 400 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{150} \\ \frac{1}{100} \end{bmatrix}$$

2. *Consider the 1-DOF system described the equation of motion, $4\ddot{x} + 20\dot{x} + 25x = f$.*

(a) *Find the natural frequency ω_n and the natural damping ratio ζ_n of the natural (passive) system ($f = 0$). What type of system is this (oscillatory, overdamped, etc.) ?*

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{25}{4}} = 2.5$$

$$\zeta_n = \frac{b}{2\sqrt{km}} = \frac{20}{2\sqrt{25 \cdot 4}} = 1$$

Since $\zeta_n = 1$, this system is **critically damped**.

(b) *Design a PD controller that achieves critical damping with a closed-loop stiffness $k_{CL} = 36$. In other words, let $f = -k_v\dot{x} - k_px$, and determine the gains k_v and k_p. Assume that the desired position is $x_d = 0$.*
The controlled system is:

$$4\ddot{x} + 20\dot{x} + 25x = -k_v\dot{x} - k_px$$

So, the closed loop equation is

$$4\ddot{x} + (20 + k_v)\dot{x} + (25 + k_p)x = m\ddot{x} + b\dot{x} + kx = 0$$

The closed loop stiffness is given by k, the coefficient of the positional term, so:

$$k = 25 + k_p = k_{CL} = 36$$

For critical damping, the coefficient b of $\dot{x}$ must satisfy

$$b = 20 + k_v = 2\sqrt{km} = 2\sqrt{36 \cdot 4} = 24$$

So, the gains that we need are

$$k_p = 11,\ k_v = 4$$

So, the PD controller is

$$f = -4\dot{x} - 11x$$

(c) *Assume that the friction model changes from linear ($20\dot{x}$) to Coulomb friction, $30sign(\dot{x})$. Design a control system which uses a non-linear model-based portion with trajectory following to critically damp the system at all times and maintain a closed-loop stiffness of $k_{CL} = 36$. In other words, let $f = \alpha f' + \beta$ and $f' = \ddot{x}_d - k_v'(\dot{x} - \dot{x}_d) - k_p'(x - x_d)$. Then, find f,α,β,f',k_p' and k_v'. Note that f is an m-mass control, and f' is a unit-mass control. Use the definition of error, $e = x - x_d$.*

The differential equation for the system is now

$$4\ddot{x} + 30sign(\dot{x}) + 25x = f$$

In order to linearize it, we apply a force f of the form

$$f = \alpha f' + \beta$$

where

$$\alpha = 4,\ \beta = 30sign(\dot{x}) + 25x$$

For purposes of control, this makes the system look like the unit-mass system:

$$\ddot{x} = f'$$

to which we apply the control

$$f' = \ddot{x}_d - k_v'(\dot{x} - \dot{x}_d) - k_p'(x - x_d)$$

Substituting into our unit-mass system yields the equation

$$\ddot{e} + k_v'\dot{e} + k_p'e = 0$$

where e is the position error, $e = x - x_d$.

Now, we want to choose our gains k_p' and k_v' so that we achieve critical damping and the desired closed-loop stiffness. Remember that the closed-loop stiffness of our system is actually m times the closed-loop stiffness of our unit-mass system, so we want

$$K_{CL} = mk_p' = 36$$

So, $k_p' = 9$. In order to have critical damping in our unit-mass system (and therefore in our original system), we need

$$k_v' = 2\sqrt{k_p'}$$

So, $k_v' = 6$. Thus, the control is

$$\begin{aligned} f &= \alpha f' + \beta \\ \alpha &= 4 \\ \beta &= 30 sign(\dot{x}) + 25x \\ f' &= \ddot{x}_d - 6(\dot{x} - \dot{x}_d) - 9(x - x_d) \end{aligned}$$

(d) *Given a disturbance force $f_{dist} = 4$, what is the steady-state ($\ddot{e} = \dot{e} = 0$) error of the system in part (c)?*

We can analyze the error by observing the error in the unit-mass system. With a disturbance force added, the system's equation of motion becomes

$$4\ddot{x} + 30 sign(\dot{x}) + 25x = f + f_{dist}$$

To linearize the system, we apply a force of the same form as before:

$$\begin{aligned} f + f_{dist} &= 4f' + 30 sign(\dot{x}) + 25x + f_{dist} \\ &= 4\left(f' + \frac{f_{dist}}{4}\right) + 30 sign(\dot{x}) + 25x \end{aligned}$$

This yields a unit-mass system as before, but now it has a disturbance force of $f_{dist}/4$, so the unit-mass system now looks like

$$\ddot{x} = f' + \frac{f_{dist}}{4}$$

With the control from before, we get a unit-mass closed-loop system of

$$\ddot{e} + 6\dot{e} + 9e = \frac{f_{dist}}{4}$$

For the steady state, when $\ddot{x} = \dot{x} = 0$, we get

$$9e = \frac{f_{dist}}{4}$$

So, the steady state error is given by

$$e = \frac{f_{dist}}{4 \cdot 9} = \frac{4}{36} \approx 0.111$$

3. *C*onsider the 1-DOF system with equation of motion:

$$f = ml^2\ddot{\theta} + v\dot{\theta} + mlg\cos(\theta)$$

We are using a control strategy which compensates for the non-linear part of the system and has a unit-mass linear controller for trajectory tracking:

$$\begin{aligned} f &= \alpha f' + \beta \\ \alpha &= \hat{m}l^2 \\ \beta &= v\dot{\theta} + \hat{m}lg\cos(\theta) \\ f' &= \ddot{\theta}_d - k_v'(\dot{\theta} - \dot{\theta}_d) - k_p'(\theta - \theta_d) \end{aligned}$$

(where $\hat{m}$ is the estimate of the mass m of our system.)

If there is an error in our mass estimate, given by $\psi = m - \hat{m}$, then what is the resultant *steady-state* position error of the controlled system? Assume position error is given by $e = \theta - \theta_d$. Your answer should be in terms of ψ, $\hat{m}$, l, k_p', $\ddot{\theta}_d$, θ, and g.

Unfortunately, the wording in this question was vague as to the meaning of "steady-state," therefore, there are *2 possible solutions*, both valid but different.

SOLUTION A:
This solution assumes steady-state means *only* $\ddot{e} = 0$, $\dot{e} = 0$.
Apply the given controller to the system's equation of motion, and start rearranging terms:

$$\begin{aligned}
ml^2\ddot{\theta} + v\dot{\theta} + mlg\cos(\theta) &= f \\
ml^2\ddot{\theta} + v\dot{\theta} + mlg\cos(\theta) &= \hat{m}l^2 f' + v\dot{\theta} + \hat{m}lg\cos(\theta) \\
\Rightarrow ml^2\ddot{\theta} + (m - \hat{m})lg\cos(\theta) &= \hat{m}l^2 f' \\
\Rightarrow m\ddot{\theta} + \frac{m - \hat{m}}{l} g\cos(\theta) &= \hat{m}f' \\
\Rightarrow m\ddot{\theta} + \frac{m - \hat{m}}{l} g\cos(\theta) &= \hat{m}(\ddot{\theta}_d - k_v'(\dot{\theta} - \dot{\theta}_d) - k_p'(\theta - \theta_d)) \\
\Rightarrow m\ddot{\theta} + \frac{m - \hat{m}}{l} g\cos(\theta) &= \hat{m}(\ddot{\theta}_d - k_v'\dot{e} - k_p' e) \\
\Rightarrow m\ddot{\theta} - \hat{m}\ddot{\theta}_d + \hat{m}k_v'\dot{e} + \hat{m}k_p' e + \frac{m - \hat{m}}{l} g\cos(\theta) &= 0 \\
\Rightarrow m\ddot{\theta} - m\ddot{\theta}_d + m\ddot{\theta}_d - \hat{m}\ddot{\theta}_d + \hat{m}k_v'\dot{e} + \hat{m}k_p' e + \frac{m - \hat{m}}{l} g\cos(\theta) &= 0 \\
\Rightarrow (m\ddot{\theta} - m\ddot{\theta}_d) + (m\ddot{\theta}_d - \hat{m}\ddot{\theta}_d) + \hat{m}k_v'\dot{e} + \hat{m}k_p' e + \frac{m - \hat{m}}{l} g\cos(\theta) &= 0 \\
\Rightarrow m\ddot{e} + (m - \hat{m})\ddot{\theta}_d + \hat{m}k_v'\dot{e} + \hat{m}k_p' e + \frac{m - \hat{m}}{l} g\cos(\theta) &= 0 \\
\Rightarrow m\ddot{e} + \psi\ddot{\theta}_d + \hat{m}k_v'\dot{e} + \hat{m}k_p' e + \frac{\psi}{l} g\cos(\theta) &= 0
\end{aligned}$$

Now take steady-state error ($\ddot{e} = 0$, $\dot{e} = 0$) to get:

$$\begin{aligned}
\psi\ddot{\theta}_d + \hat{m}k_p' e + \frac{\psi}{l} g\cos(\theta) &= 0 \\
\Rightarrow \hat{m}k_p' e &= -\psi\ddot{\theta}_d - \frac{\psi}{l} g\cos(\theta) \\
\Rightarrow e &= \boxed{-\frac{\psi}{\hat{m}k_p'}\left(\ddot{\theta}_d + \frac{g}{l}\cos(\theta)\right)}
\end{aligned}$$

SOLUTION B:
This solution assumes steady-state means $\ddot{\theta} = 0$, as well as $\ddot{e} = 0$ and $\dot{e} = 0$.
This means:

$$\begin{aligned}
ml^2\ddot{\theta} + v\dot{\theta} + mlg\cos(\theta) &= f \\
ml^2\ddot{\theta} + v\dot{\theta} + mlg\cos(\theta) &= \hat{m}l^2 f' + v\dot{\theta} + \hat{m}lg\cos(\theta) \\
\Rightarrow ml^2\ddot{\theta} + (m - \hat{m})lg\cos(\theta) &= \hat{m}l^2 f'
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow m\ddot{\theta} + \frac{m-\hat{m}}{l} g\cos(\theta) &= \hat{m}f' \\
\Rightarrow m\ddot{\theta} + \frac{m-\hat{m}}{l} g\cos(\theta) &= \hat{m}(\ddot{\theta}_d - k_v'(\dot{\theta} - \dot{\theta}_d) - k_p'(\theta - \theta_d)) \\
\Rightarrow m\ddot{\theta} + \frac{m-\hat{m}}{l} g\cos(\theta) &= \hat{m}(\ddot{\theta}_d - k_v'\dot{e} - k_p'e) \\
\Rightarrow m\ddot{\theta} - \hat{m}\ddot{\theta}_d + \hat{m}k_v'\dot{e} + \hat{m}k_p'e + \frac{m-\hat{m}}{l} g\cos(\theta) &= 0
\end{aligned}
$$

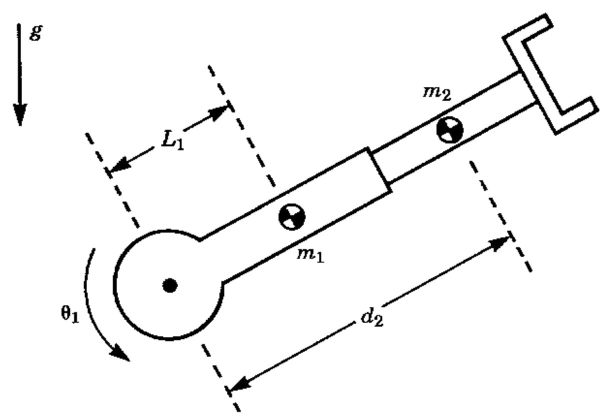
Taking steady-state:

$$
\begin{aligned}
-\hat{m}\ddot{\theta}_d + \hat{m}k_p'e + \frac{m-\hat{m}}{l} g\cos(\theta) &= 0 \\
-\hat{m}\ddot{\theta}_d + \hat{m}k_p'e + \frac{\psi}{l} g\cos(\theta) &= 0 \\
e &= \boxed{\frac{1}{k_p'}\ddot{\theta}_d - \frac{\psi}{\hat{m}lk_p'} g\cos(\theta)}
\end{aligned}
$$

Note: this can be simplified further, if you notice that together $\ddot{e} = 0$ and $\ddot{\theta} = 0$ imply $\ddot{\theta}_d = 0$. This means:

$$
e = \boxed{-\frac{\psi}{\hat{m}lk_p'} g\cos(\theta)}
$$

4. *C*onsider the 2-link RP manipulator shown below:



Its equations of motion were derived in the Lecture Notes and are shown here:

$$
\begin{aligned}
\tau_1 &= (m_1L_1^2 + I_{zz1} + I_{zz2} + m_2d_2^2)\,\ddot{\theta}_1 + 2m_2d_2\dot{\theta}_1\dot{d}_2 + (m_1L_1 + m_2d_2)g\cos(\theta_1) \\
\tau_2 &= m_2\ddot{d}_2 - m_2d_2\dot{\theta}_1^{\,2} + m_2 g\sin(\theta_1)
\end{aligned}
$$

The manipulator parameters have the following numerical values: $L_1 = 0.2m$, $m_1 = 1.0kg$, $m_2 = 0.8kg$, $I_{zz1} = 0.1kgm^2$, $I_{zz2} = 0.07kgm^2$, and the range of d_2 is between $0.5m$ and $1.0m$.

(a) *The* system is controlled by a joint-space dynamic decoupling control, $\tau = \alpha\tau' + \beta$, which compensates the non-linear part of the system, decouples the dynamics, and tracks a desired trajectory (ie. position, velocity and acceleration) separately for each joint. Leaving only the feedback gains (k'_{p1}, k'_{p2}, k'_{v1}, k'_{v2}) as symbols, give values for the matrix α, vector β, and vectors τ and τ'. Note: you should also leave the joint variables (θ_1, d_2) and joint velocities ($\dot{\theta}_1$, $\dot{d}_2$) as symbols.

$$
\begin{aligned}
\alpha &= \text{mass matrix} \\
&= \begin{bmatrix} m_1 L_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 0.21 + 0.8 d_2^2 & 0 \\ 0 & 0.8 \end{bmatrix}} \\
\beta &= \text{the rest of the terms in the equations of motion} \\
&= \begin{bmatrix} 2 m_2 d_2 \dot{\theta}_1 \dot{d}_2 + (m_1 L_1 + m_2 d_2) g \cos(\theta_1) \\ -m_2 d_2 \dot{\theta}_1^2 + m_2 g \sin(\theta_1) \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 1.6 d_2 \dot{\theta}_1 \dot{d}_2 + (0.2 + 0.8 d_2) g \cos(\theta_1) \\ -0.8 d_2 \dot{\theta}_1^2 + 0.8 g \sin(\theta_1) \end{bmatrix}} \\
\tau &= \boxed{\begin{bmatrix} 0.21 + 0.8 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix} \tau' + \begin{bmatrix} 1.6 d_2 \dot{\theta}_1 \dot{d}_2 + (0.2 + 0.8 d_2) g \cos(\theta_1) \\ 0.8 d_2 \dot{\theta}_1^2 + 0.8 g \sin(\theta_1) \end{bmatrix}} \\
\tau' &= \boxed{\begin{bmatrix} \ddot{\theta}_{1d} - k'_{v1}(\dot{\theta}_1 - \dot{\theta}_{1d}) - k'_{p1}(\theta_1 - \theta_{1d}) \\ \ddot{d}_{2d} - k'_{v2}(\dot{d}_2 - \dot{d}_{2d}) - k'_{p2}(d_2 - d_{2d}) \end{bmatrix}}
\end{aligned}
$$

(b) *F*ind the values for the gains k'_{p1}, k'_{p2}, k'_{v1}, k'_{v2} such that the closed-loop system for joint 1 is critically damped with natural frequency of 20 rad/sec, and the closed-loop system for joint 2 is critically damped with natural frequency of 25 rad/sec.

The controlled system is decoupled, and τ' is used to control a unit-mass system. Therefore, the controlled system looks like:

$$
\begin{aligned}
\ddot{\theta}_1 &= \ddot{\theta}_{1d} - k'_{v1}(\dot{\theta}_1 - \dot{\theta}_{1d}) - k'_{p1}(\theta_1 - \theta_{1d}) \\
\ddot{d}_2 &= \ddot{d}_{2d} - k'_{v2}(\dot{d}_2 - \dot{d}_{2d}) - k'_{p2}(d_2 - d_{2d})
\end{aligned}
$$

Bringing all the terms to the left side of the equality, and considering only error on each joint ($e_1, \dot{e}_1, \ddot{e}_1$ for joint 1 and $e_2, \dot{e}_2, \ddot{e}_2$ for joint 2) we get:

$$
\begin{aligned}
\ddot{e}_1 + k'_{v1}\dot{e}_1 + k'_{p1}e_1 &= 0 \\
\ddot{e}_2 + k'_{v2}\dot{e}_2 + k'_{p2}e_2 &= 0
\end{aligned}
$$

For joint 1, we want $w_n = 20 rad/sec$ and $\xi_n = 1$. This means, we must choose k'_{v1} and k'_{p1} to make the upper equation look like:

$$
\ddot{e}_1 + 2\xi_n w_n \dot{e}_1 + w_n^2 e_1 = 0
$$

Thus:

$$k'_{p1} = w_n^2 = \boxed{400}$$
$$k'_{v1} = 2\xi_n w_n = \boxed{40}$$

Similarly, for joint 2 we want $w_n = 25 rad/sec$ and $\xi_n = 1$, so:

$$k'_{p2} = w_n^2 = \boxed{625}$$
$$k'_{v2} = 2\xi_n w_n = \boxed{50}$$

(c) *C*onsider the original equations of motion (ie. without a controller), when $d_2 = 0.6m$. For joint 1, what is the effective inertia "seen" by the joint if we have gearing with ratio $\eta = 5$ and rotor inertia $I_m = 0.004 kgm^2$?

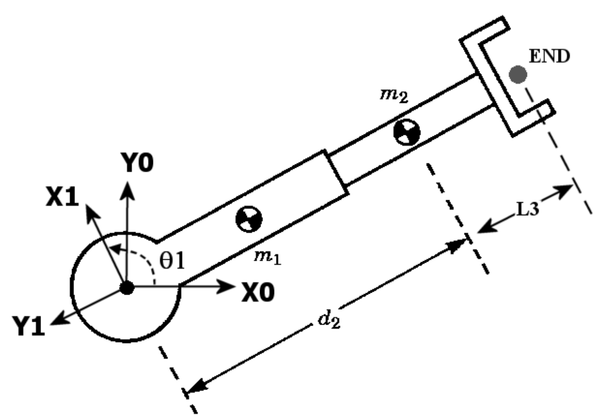
In equation 7.49 of the Lecture Notes reader, the coefficient of $\ddot{\theta}$ gives us the formula for "effective inertia":

$$I_{eff} = I_L + \eta^2 I_m$$

where I_L is the inertia of the link attached to the joint, and I_m is the motor inertia. For joint 1, the link inertia must include the rest of the manipulator (since that is also attached to the joint, via link 1 itself). Therefore, I_L is simply the coefficient of $\ddot{\theta}_1$ in the system's equations of motion. For $d = 0.6$ this gives:

$$\begin{aligned} I_{eff} &= I_L + \eta^2 I_m \\ I_{eff} &= (0.21 + 0.8d_2^2) + (5)^2(0.004) \\ &= \boxed{0.598} \end{aligned}$$

*C*onsider again the original system (ie. no controller or gearing). You are given DH coordinate frames as shown below:



The length from the center of mass of link 2 to the end-effector is L_3. In this case, the end-effector position in the plane is:

$$^0\mathbf{P}_{end} = \begin{bmatrix} s_1(d_2 + L_3) \\ -c_1(d_2 + L_3) \end{bmatrix}$$

(d) Use ${}^{0}\mathbf{P}_{end}$ to compute the Jacobian for linear velocity at the end-effector in frame $\{0\}$.

Take derivatives to compute J_v:

$$
{}^{0}J_v = \boxed{\begin{bmatrix} c_1(d_2+L_3) & s_1 \\ s_1(d_2+L_3) & -c_1 \end{bmatrix}}
$$

(e) Using your answer from part (d), the configuration $\theta_1 = 45°$, $d_2 = 0.6m$, and assuming $L_3 = 0.2m$, compute the system's mass matrix M_x in frame $\{0\}$ when the dynamics are written in terms of the operational space coordinates.

Equation 7.80 in Lecture Notes reader says:

$$
M_x = J^{-T} M J^{-1}
$$

Use the equations of motion for our system to compute M, and the answer from part (d) to compute J_v for the given configuration:

$$
M = \begin{bmatrix} m_1 L_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 0.498 & 0 \\ 0 & 0.8 \end{bmatrix}
$$

$$
\text{and } J_v = \begin{bmatrix} c_1(d_2+L_3) & s_1 \\ s_1(d_2+L_3) & -c_1 \end{bmatrix} = \begin{bmatrix} 0.566 & 0.707 \\ 0.566 & -0.707 \end{bmatrix}
$$

$$
\Rightarrow M_x = \boxed{\begin{bmatrix} 0.7891 & -0.0109 \\ -0.0109 & 0.7891 \end{bmatrix}}
$$