1. (a) *Derive a formula that transforms an inertia tensor given in some frame $\{C\}$ into a new frame $\{A\}$. The frame $\{A\}$ can differ from frame $\{C\}$ by both translation and rotation. You may assume that frame $\{C\}$ is located at the center of mass.*

Solving this problem involves using the Parallel Axis Theorem to translate the inertia tensor to a frame at a different location, and a similarity transformation to rotate it into the new frame. These operations can be done in either order, as long as we're careful that the vectors we use are expressed in the correct frame. However, it is definitely easier to do the rotation first.

Assume that we have ${}^A_CT$, the transformation from frame $\{C\}$ coordinates to frame $\{A\}$ coordinates, which contains the rotation matrix ${}^A_CR$ and the translation vector ${}^A\mathbf{p}_C$ which locates the origin of frame $\{C\}$ with respect to $\{A\}$. Let's first solve the problem by a rotation followed by a translation. Consider an intermediate frame $\{C'\}$ which has the same origin as $\{C\}$, but whose axes are parallel to frame $\{A\}$. Using a similarity transformation (see p. 134-135 of Lecture Notes), we know that

$$^{C'}I =^{C'}_C R^C I^{C'}_C R^T$$

However, since frame $\{C'\}$ has the same orientation as frame $\{A\}$, we know that ${}^{C'}_C R ={}^A_C R$, so

$$^{C'}I =^A_C R^C I^A_C R^T$$

We now have the inertia tensor expressed in the intermediate frame $\{C'\}$. Since $\{C'\}$ is parallel to $\{A\}$, we can use the Parallel Axis Theorem to transform ${}^{C'}I$ to ${}^AI$. To use this theorem, we just need the vector ${}^A\mathbf{p}_{C'}$ that locates the center of frame $\{C'\}$ with respect to $\{A\}$, expressed in frame $\{A\}$, which yields the formula

$$^AI = {}^{C'}I + m\left[({}^A\mathbf{p}_{C'}^T{}^A\mathbf{p}_{C'})I_3 - {}^A\mathbf{p}_{C'}{}^A\mathbf{p}_{C'}^T\right]$$

where m is the total mass of the object and I_3 is the 3×3 identity matrix. Since $\{C'\}$ and $\{C\}$ have the same origin, the vector ${}^A\mathbf{p}_{C'}$ is just ${}^A\mathbf{p}_C$. Substituting this value and our previous expression for ${}^{C'}I$ yields:

$$^AI =^A_C R^C I^A_C R^T + m\left[({}^A\mathbf{p}_C^T{}^A\mathbf{p}_C)I_3 - {}^A\mathbf{p}_C{}^A\mathbf{p}_C^T\right]$$

Equivalently, we could do this problem with a translation first, and then a rotation. To do that, we can define an intermediate frame $\{A'\}$, which has the same origin as $\{A\}$, but whose axes are parallel to $\{C\}$. We can get the intertia tensor in the intermediate frame by using the Parallel Axis Theorem. To use it, however, we need the vector ${}^{A'}\mathbf{p}_C$ which locates the origin of frame $\{C\}$ with respect to frame $\{A'\}$, *expressed in frame*

$\{A'\}$. Using this formula with the vector expressed in frame $\{A\}$ is incorrect. We can get ${}^{A'}\mathbf{p}_C$ by rotating ${}^{A}\mathbf{p}_C$ with ${}^{A'}_{A}R = {}^{C}_{A}R$, and then simplify:

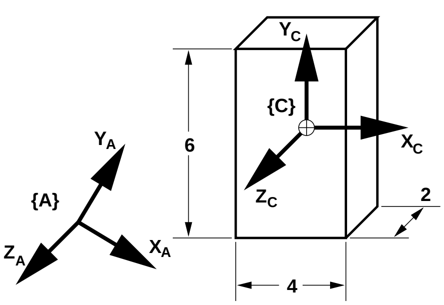
$$\begin{aligned}
{}^{A'}I &= {}^{C}I + m\left[({}^{A'}\mathbf{p}_C^T\,{}^{A'}\mathbf{p}_C)I_3 - {}^{A'}\mathbf{p}_C\,{}^{A'}\mathbf{p}_C^T\right] \\
&= {}^{C}I + m\left[({}^{C}_{A}R\,{}^{A}\mathbf{p}_C)^T({}^{C}_{A}R\,{}^{A}\mathbf{p}_C)I_3 - ({}^{C}_{A}R\,{}^{A}\mathbf{p}_C)({}^{C}_{A}R\,{}^{A}\mathbf{p}_C)^T\right] \\
&= {}^{C}I + m\left[{}^{A}\mathbf{p}_C^T({}^{C}_{A}R^T\,{}^{C}_{A}R^{A}\,\mathbf{p}_C I_3 - {}^{C}_{A}R({}^{A}\mathbf{p}_C\,{}^{A}\mathbf{p}_C^T){}^{C}_{A}R^T\right] \\
&= {}^{C}I + m\left[{}^{A}\mathbf{p}_C^T\,{}^{A}\mathbf{p}_C I_3 - {}^{C}_{A}R({}^{A}\mathbf{p}_C\,{}^{A}\mathbf{p}_C^T){}^{C}_{A}R^T\right]
\end{aligned}$$

Then, to get the inertia tensor in frame $\{A\}$, we can use a similarity transformation to rotate ${}^{A'}I$:

$$\begin{aligned}
{}^{A}I &= {}^{A}_{A'}R\,{}^{A'}I\,{}^{A}_{A'}R^T = {}^{A}_{C}R\,{}^{A'}I\,{}^{A}_{C}R^T \\
&= {}^{A}_{C}R\left({}^{C}I + m\left[{}^{A}\mathbf{p}_C^T\,{}^{A}\mathbf{p}_C I_3 - {}^{C}_{A}R({}^{A}\mathbf{p}_C\,{}^{A}\mathbf{p}_C^T){}^{C}_{A}R^T\right]\right){}^{A}_{C}R^T \\
&= {}^{A}_{C}R\,{}^{C}I\,{}^{A}_{C}R^T + m\,{}^{A}_{C}R\left[{}^{A}\mathbf{p}_C^T\,{}^{A}\mathbf{p}_C I_3 - {}^{C}_{A}R({}^{A}\mathbf{p}_C\,{}^{A}\mathbf{p}_C^T){}^{C}_{A}R^T\right]{}^{A}_{C}R^T \\
&= {}^{A}_{C}R\,{}^{C}I\,{}^{A}_{C}R^T + m\left[({}^{A}\mathbf{p}_C^T\,{}^{A}\mathbf{p}_C){}^{A}_{C}R I_3\,{}^{A}_{C}R^T - {}^{A}_{C}R\,{}^{C}_{A}R({}^{A}\mathbf{p}_C\,{}^{A}\mathbf{p}_C^T){}^{C}_{A}R^T\,{}^{A}_{C}R^T\right] \\
{}^{A}I &= {}^{A}_{C}R\,{}^{C}I\,{}^{A}_{C}R^T + m\left[({}^{A}\mathbf{p}_C^T\,{}^{A}\mathbf{p}_C)I_3 - {}^{A}\mathbf{p}_C\,{}^{A}\mathbf{p}_C^T\right]
\end{aligned}$$

This is the same expression that we got from the other approach.

(b) *C*onsider, for example, the uniform density box shown below. It has mass $m = 12kg$, and dimensions $6 \times 4 \times 2$.



Frame $\{C\}$ lies at the center of mass of the box, and the coordinate axes are lined up with the principal axes of the box. In other words, $\mathbf{Y}_C$ is aligned with the long axis of the box, and $\mathbf{X}_C$ and $\mathbf{Z}_C$ are aligned with the short axes of the box.

First, compute the inertia tensor of the box in frame $\{C\}$.

Note: When using a frame at the center of mass and along the principal axes, the inertia tensor for the box of uniform density becomes diagonal. It takes the form:

$$ {}^{C}I = \begin{bmatrix} \frac{m}{12}(s_y^2 + s_z^2) & 0 & 0 \\ 0 & \frac{m}{12}(s_x^2 + s_z^2) & 0 \\ 0 & 0 & \frac{m}{12}(s_x^2 + s_y^2) \end{bmatrix} $$

where s_x, s_y and s_z are the dimensions of the box along the $\mathbf{X}_C$, $\mathbf{Y}_C$ and $\mathbf{Z}_C$ axes, respectively. In this case, $s_x = 4$, $s_y = 6$, and $s_z = 2$.

Apply the above formula. We are given that $m = 12$, and from the diagram we see that: $s_x = 4$, $s_y = 6$, and $s_z = 2$. So:

$$\begin{aligned} {}^{C}I &= \begin{bmatrix} \frac{12}{12}(6^2+2^2) & 0 & 0 \\ 0 & \frac{12}{12}(4^2+2^2) & 0 \\ 0 & 0 & \frac{m}{12}(4^2+6^2) \end{bmatrix} \\ &= \begin{bmatrix} 40 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 52 \end{bmatrix} \end{aligned}$$

(c) *Given the transformation matrix from $\{C\}$ to $\{A\}$:*

$${}^{A}_{C}T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

use your formula from part (a) and your inertia tensor from part (b) to compute the inertia tensor of the box in frame $\{A\}$.

We apply the formula from part (a). In this case, from ${}^{A}_{C}T$, we know:

$${}^{A}_{C}R = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{p}_C = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

The first part of the transformation (into the intermediate frame $\{A'\}$) is

$${}^{A'}I = {}^{A}_{C}R\,{}^{C}I\,{}^{A}_{C}R^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 40 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 52 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 30 & 10 & 0 \\ 10 & 30 & 0 \\ 0 & 0 & 52 \end{bmatrix}$$

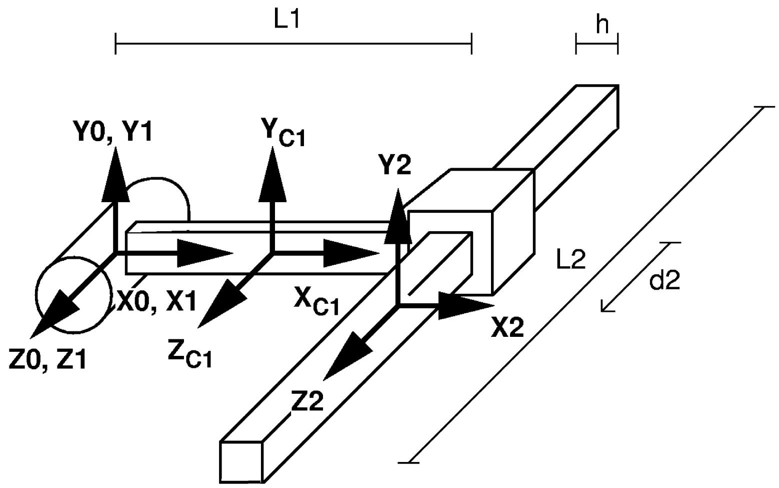
To compute the parallel axis transformation, we need to find the matrix $\left[(\mathbf{p}_C^T \mathbf{p}_C) I_3 - \mathbf{p}_C \mathbf{p}_C^T\right]$:

$$\mathbf{p}_C^T \mathbf{p}_C = 6, \;\; \mathbf{p}_C \mathbf{p}_C^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \; \left[(\mathbf{p}_C^T \mathbf{p}_C) I_3 - \mathbf{p}_C\{\mathbf{p}_C^T\right] = \begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix}$$

We now compute the entire transformation:

$$\begin{aligned} {}^{A}I &= {}^{A}_{C}R\,{}^{C}I\,{}^{A}_{C}R^T + m\left[(\mathbf{p}_C \mathbf{p}_C^T) I_3 - \mathbf{p}_C \mathbf{p}_C^T\right] \\ &= \begin{bmatrix} 30 & 10 & 0 \\ 10 & 30 & 0 \\ 0 & 0 & 52 \end{bmatrix} + 12 \begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix} \\ {}^{A}I &= \begin{bmatrix} 90 & -2 & -24 \\ -2 & 90 & -24 \\ -24 & -24 & 76 \end{bmatrix} \end{aligned}$$

2. *In the rest of this problem set, we will walk through the process of finding the equations of motion for a simple manipulator from the Lagrange formulation. Consider the RP spatial manipulator shown below. The links of this manipulator are modeled as bars of uniform density, having square cross-sections of thickness h, lengths of L_1 and L_2, and total masses of m_1 and m_2, with centers of mass shown. Assume that the joints themselves are massless.*



From the derivation on pp.131-133 of the notes, we know that the equations of motion have the form:

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}] + \mathbf{G}(\mathbf{q}) = \tau$$

where M is the mass matrix, C is the matrix of coefficients for centrifugal forces, B is the matrix of coefficients for Coriolis forces, and $\mathbf{G}$ is the vector of gravity forces.

(a) *For each link i, we have attached a frame $\{C_i\}$ to the center of mass (in this case, frame $\{2\}$ is the same as $\{C_2\}$). Compute kinematics for these frames: that is, calculate the matrices ${}^0_{C_1}T$ and ${}^0_{C_2}T$.*

The transformation ${}^1_{C_1}T$ is just a constant offset of $L_1/2$ along the x axis; the other transformations are found in the regular manner:

$${}^0_{C_1}T = \begin{bmatrix} c_1 & -s_1 & 0 & \frac{1}{2}L_1c_1 \\ s_1 & c_1 & 0 & \frac{1}{2}L_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; {}^0_{C_2}T = \begin{bmatrix} c_1 & -s_1 & 0 & L_1c_1 \\ s_1 & c_1 & 0 & L_1s_1 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For a two-link manipulator, the mass matrix has the form

$$M = m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_1}^T {}^{C_1}I_1 J_{\omega_1} + J_{\omega_2}^T {}^{C_2}I_2 J_{\omega_2}$$

where J_{v_i} is the linear Jacobian of the center of mass of link i, J_{ω_i} is the angular velocity of link i, and ${}^{C_i}I_i$ is the inertia tensor of link i expressed in frame $\{C_i\}$.

(b) *Calculate ${}^0J_{v_1}$ and ${}^0J_{v_2}$.*

These matrices are found directly by differentiating the last columns of ${}^0_{C_i}T$:

$$
{}^0J_{v_1} = \begin{bmatrix} \frac{\partial {}^0\mathbf{p}_{C_1}}{\partial \theta_1} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}L_1 s_1 & 0 \\ \frac{1}{2}L_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad {}^0J_{v_2} = \begin{bmatrix} \frac{\partial {}^0\mathbf{p}_{C_2}}{\partial \theta_1} & \frac{\partial {}^0\mathbf{p}_{C_2}}{\partial d_2} \end{bmatrix} = \begin{bmatrix} -L_1 s_1 & 0 \\ L_1 c_1 & 0 \\ 0 & 1 \end{bmatrix}
$$

(c) *Calculate ${}^{C_1}J_{\omega_1}$ and ${}^{C_2}J_{\omega_2}$.*

$$
{}^{C_1}J_{\omega_1} = \begin{bmatrix} \bar{\epsilon}_1 {}^{C_1}\mathbf{z}_1 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad {}^{C_2}J_{\omega_1} = \begin{bmatrix} \bar{\epsilon}_1 {}^{C_2}\mathbf{z}_1 & \bar{\varepsilon}_2 {}^{C_2}\mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}
$$

(d) *Calculate ${}^{C_1}I_1$ and ${}^{C_2}I_2$ in terms of the masses and dimensions of the links. You can use the same formula that was given for a box of uniform density in Problem 1(b). Be careful which measurements you use along the axes.*

Using the formula from problem 1, we see that the inertia tensor written at the center of mass of a uniform density rectangular solid is

$$
{}^{C}I = \begin{bmatrix} \frac{m}{12}(s_y^2 + s_z^2) & 0 & 0 \\ 0 & \frac{m}{12}(s_x^2 + s_z^2) & 0 \\ 0 & 0 & \frac{m}{12}(s_x^2 + s_y^2) \end{bmatrix}
$$

where s_x, s_y and s_z are the dimensions of the solid along the $\mathbf{x}_C$, $\mathbf{y}_C$ and $\mathbf{z}_C$ axes, respectively. Plugging in the values for our links yields

$$
{}^{C_1}I_1 = \begin{bmatrix} \frac{m_1}{6}h^2 & 0 & 0 \\ 0 & \frac{m_1}{12}(L_1^2 + h^2) & 0 \\ 0 & 0 & \frac{m_1}{12}(L_1^2 + h^2) \end{bmatrix}, \quad {}^{C_2}I_2 = \begin{bmatrix} \frac{m_2}{12}(L_2^2 + h^2) & 0 & 0 \\ 0 & \frac{m_2}{12}(L_2^2 + h^2) & 0 \\ 0 & 0 & \frac{m_2}{6}h^2 \end{bmatrix}
$$

(e) *Calculate the mass matrix, $M(\mathbf{q})$. To make your algebra easier, leave the inertia tensors in symbolic form until the end, i.e.*

$$
{}^{C_1}I_1 = \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}
$$

This just requires a bit of matrix algebra:

$$
{}^0J_{v_1}^T {}^0J_{v_1} = \begin{bmatrix} \frac{L_1^2}{4} & 0 \\ 0 & 0 \end{bmatrix}, \quad {}^0J_{v_2}^T {}^0J_{v_2} = \begin{bmatrix} L_1^2 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
J_{\omega_1}^T {}^{C_1}I_1 J_{\omega_1} = \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix}, \quad J_{\omega_2}^T {}^{C_2}I_2 J_{\omega_2} = \begin{bmatrix} I_{zz2} & 0 \\ 0 & 0 \end{bmatrix}
$$

$$
\begin{aligned}
M &= m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_1}^T {}^{C_1}I_1 J_{\omega_1} + J_{\omega_2}^T {}^{C_2}I_2 J_{\omega_2} \\
&= \begin{bmatrix} \frac{m_1}{4}L_1^2 + m_2(L_1^2) + I_{zz1} + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix} \\
M &= \begin{bmatrix} \frac{m_1}{3}L_1^2 + \frac{m_1}{12}h^2 + m_2 L_1^2 + \frac{m_2}{6}h^2 & 0 \\ 0 & m_2 \end{bmatrix}
\end{aligned}
$$

Now we need to calculate the centrifugal and Coriolis forces. We will derive the form directly.

(f) *Beginning with the equation from p. 136 in the lecture notes,*

$$\mathbf{v}(\mathbf{q},\dot{\mathbf{q}}) = \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_2}\dot{\mathbf{q}} \end{bmatrix},$$

manipulate this equation symbolically into the form

$$\mathbf{v}(\mathbf{q},\dot{\mathbf{q}}) = C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}]$$

where C and B are matrices in terms of the partial derivatives m_{ijk} of the mass matrix. Don't actually substitute in your answer from part (e) into this equation yet: just leave the elements of these matrices in m_{ijk} symbolic form.

$$\begin{aligned}
\mathbf{v}(\mathbf{q},\dot{\mathbf{q}}) &= \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_2}\dot{\mathbf{q}} \end{bmatrix} \\
&= \begin{bmatrix} \dot{m}_{11} & \dot{m}_{12} \\ \dot{m}_{12} & \dot{m}_{22} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} [\dot{q}_1\ \dot{q}_2]\begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \\ [\dot{q}_1\ \dot{q}_2]\begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} m_{111}\dot{q}_1 + m_{112}\dot{q}_2 & m_{121}\dot{q}_1 + m_{122}\dot{q}_2 \\ m_{121}\dot{q}_1 + m_{122}\dot{q}_2 & m_{221}\dot{q}_1 + m_{222}\dot{q}_2 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} [\dot{q}_1\ \dot{q}_2]\begin{bmatrix} m_{111}\dot{q}_1 + m_{121}\dot{q}_2 \\ m_{121}\dot{q}_1 + m_{221}\dot{q}_2 \end{bmatrix} \\ [\dot{q}_1\ \dot{q}_2]\begin{bmatrix} m_{112}\dot{q}_1 + m_{122}\dot{q}_2 \\ m_{122}\dot{q}_1 + m_{222}\dot{q}_2 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} m_{111}\dot{q}_1^2 + m_{112}\dot{q}_1\dot{q}_2 + m_{121}\dot{q}_1\dot{q}_2 + m_{122}\dot{q}_2^2 \\ m_{121}\dot{q}_1^2 + m_{122}\dot{q}_1\dot{q}_2 + m_{221}\dot{q}_1\dot{q}_2 + m_{222}\dot{q}_2^2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} m_{111}\dot{q}_1^2 + 2m_{121}\dot{q}_1\dot{q}_2 + m_{221}\dot{q}_2^2 \\ m_{112}\dot{q}_1^2 + 2m_{122}\dot{q}_1\dot{q}_2 + m_{222}\dot{q}_2^2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}m_{111}\dot{q}_1^2 + m_{122}\dot{q}_2^2 - \frac{1}{2}m_{221}\dot{q}_2^2 + m_{112}\dot{q}_1\dot{q}_2 \\ m_{121}\dot{q}_1^2 - \frac{1}{2}m_{112}\dot{q}_1^2 + \frac{1}{2}m_{222}\dot{q}_2^2 + m_{221}\dot{q}_1\dot{q}_2 \end{bmatrix} \\
\mathbf{v}(\mathbf{q},\dot{\mathbf{q}}) &= \begin{bmatrix} \frac{1}{2}m_{111} & m_{122} - \frac{1}{2}m_{221} \\ m_{121} - \frac{1}{2}m_{112} & \frac{1}{2}m_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} m_{112} \\ m_{221} \end{bmatrix}[\dot{q}_1\dot{q}_2]
\end{aligned}$$

So we have

$$C = \begin{bmatrix} \frac{1}{2}m_{111} & m_{122} - \frac{1}{2}m_{221} \\ m_{121} - \frac{1}{2}m_{112} & \frac{1}{2}m_{222} \end{bmatrix},\ B = \begin{bmatrix} m_{112} \\ m_{221} \end{bmatrix}$$

(g) *Using your answer to part (e), compute the matrices $C(\mathbf{q})$ and $B(\mathbf{q})$ in terms of the masses, dimensions, and configuration $\mathbf{q}$ of the manipulator.*

This wasn't meant to be tricky - the mass matrix is independent of the joints, so

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The last thing that remains is to derive the gravity vector $\mathbf{G}(\mathbf{q})$. This you should be able to figure out for yourself.

(h) *Calculate, ${}^0\mathbf{G}(\mathbf{q})$, the gravity vector in frame $\{0\}$, in terms of the masses, the configuration $\mathbf{q}$, and the gravity constant g (g is positive). Assume that gravity pulls things along the $-\mathbf{z}_0$ direction. Be careful with your signs.*

In terms of a unit gravity vector $\mathbf{g}$, we have

$$\mathbf{G} = -\left[J_{v_1}^T m_1 \mathbf{g} + J_{v_2}^T m_2 \mathbf{g}\right]$$

In frame $\{0\}$, the gravity vector is ${}^0\mathbf{g} = [0\ 0\ -g]^T$, which yields

$$\begin{aligned} {}^0\mathbf{G} &= -\begin{bmatrix} -\frac{1}{2}L_1 s_1 & \frac{1}{2}L_1 c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -m_1 g \end{bmatrix} - \begin{bmatrix} -L_1 s_1 & L_1 c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -m_2 g \end{bmatrix} \\ {}^0\mathbf{G} &= \begin{bmatrix} 0 \\ m_2 g \end{bmatrix} \end{aligned}$$

(i) *As a final step, use your answers to parts (e), (g) and (h) to write out the equations of motion as two great big equations*

$$\begin{aligned} \tau_1 &= f_1(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}) \\ \tau_2 &= f_2(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}) \end{aligned}$$

$$M\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix} + C\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + B\left[\dot{\theta}_1 \dot{d}_2\right] + \mathbf{G} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$\begin{aligned} \tau_1 &= \left(\frac{m_1}{3}L_1^2 + \frac{m_1}{12}h^2 + m_2 L_1^2 + \frac{m_2}{6}h^2\right)\ddot{\theta}_1 \\ \tau_2 &= m_2 \ddot{d}_2 + m_2 g \end{aligned}$$

3. *(30 points) When finding pieces of cubic splines, instead of solving directly for the coefficients a_i, we can think of constructing the cubic by a weighted blending of some known* basis functions. *For example, on the interval $t \in [0,1]$, if we want a cubic segment $\theta(t)$ that moves from θ_0 to θ_1, with starting velocity $\dot{\theta}_0$ and ending velocity $\dot{\theta}_1$, we construct it by summing four known cubics:*

$$\theta(t) = \theta_0 f_0(t) + \theta_1 f_1(t) + \dot{\theta}_0 f_2(t) + \dot{\theta}_1 f_3(t)$$

The functions $f_i(t)$ have special forms that ensure that the conditions at the ends of the interval are met. For example, f_0 has the properties that

$$\begin{aligned} f_0(0) &= 1 \\ f_0(1) &= 0 \\ f_0'(0) &= 0 \\ f_0'(1) &= 0 \end{aligned}$$

These properties ensure that the $\theta_0 f_0(t)$ term contributes θ_0 to the value of $\theta(0)$, but does not contribute at all to the values of $\theta(1)$, $\dot{\theta}(0)$, or $\dot{\theta}(1)$.

(a) *(15 points) Derive the four cubic polynomials f_0, f_1, f_2, and f_3.*

We've already given four properties that must hold for f_0, and we can define similar properties for f_1, f_2 and f_3. Each of these polynomials are cubic, so we can write them as:

$$f_i(t) = a_{i0} + a_{i1}t + a_{i2}t^2 + a_{i3}t^3$$

We can then derive equations for the points that we are interested in, which are the values of the function and its first derivate at 0 and at 1:

$$\begin{aligned} f_i(0) &= a_{i0} & f_i(1) &= a_{i0} + a_{i1} + a_{i2} + a_{i3} \\ f_i'(0) &= a_{i1} & f_i'(1) &= a_{i1} + 2a_{i2} + 3a_{i3} \end{aligned}$$

With these formulas, we can solve for the coefficients a_{ij}, once we're given the values of $f_i(0)$, $f_i(1)$, $f_i'(0)$ and $f_i'(1)$; it is just a linear system of 4 equations and 4 unknowns. These systems have the form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \end{bmatrix} = \begin{bmatrix} f_i(0) \\ f_i(1) \\ f_i'(0) \\ f_i'(1) \end{bmatrix}$$

So, we can solve 4 systems to get our polynomials:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{03} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad f_0(t) = 1 - 3t^2 + 2t^3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad f_1(t) = 3t^2 - 2t^3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad f_2(t) = t - 2t^2 + t^3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_{30} \\ a_{31} \\ a_{32} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad f_3(t) = -t^2 + t^3$$

For your information, these functions f_i are known as the *Hermite basis functions.*

(b) *your polynomials f_i to get a set of four cubic polynomials g_i that satisfy the same conditions over the interval $t \in [0, t_f]$ instead of $t \in [0, 1]$. That is, find four cubic polynomials g_i such that the cubic*

$$\theta(t) = \theta_0 g_0(t) + \theta_f g_1(t) + \dot{\theta}_0 g_2(t) + \dot{\theta}_f g_3(t)$$

satisfies the conditions $\theta(0) = \theta_0$, $\theta(t_f) = \theta_f$, $\dot{\theta}(0) = \dot{\theta}_0$, and $\dot{\theta}(t_f) = \dot{\theta}_f$. Be careful with the derivative terms.

We can transform our basis functions f_i by reparameterizing onto $[0, t_f]$, using $t' = t/t_f$. This substitution works directly for the positional terms (g_0 and g_1), but because the chain rule adds a $1/t_f$ term to the derivative terms, we must augment g_2 and g_3 by scaling it by t_f. So, we get the new equations

$$\begin{array}{rclcrcl} g_0(t) = f_0\left(\frac{t}{t_f}\right) & = & 1 - \frac{3}{t_f^2}t^2 + \frac{2}{t_f^3}t^3 & & g_2(t) = t_f f_2\left(\frac{t}{t_f}\right) & = & t - \frac{2}{t_f}t^2 + \frac{1}{t_f^2}t^3 \\ g_1(t) = f_1\left(\frac{t}{t_f}\right) & = & \frac{3}{t_f^2}t^2 - \frac{2}{t_f^3}t^3 & & g_3(t) = t_f f_3\left(\frac{t}{t_f}\right) & = & -\frac{1}{t_f}t^2 + \frac{1}{t_f^2}t^3 \end{array}$$

(c) *your functions g_i to derive formulas for the coefficients a_i of the cubic $\theta(t)$, such that*

$$\theta(t) = a_0 + a_1 t + a_3 t^2 + a_3 t^3$$

You can check the formulas that you derive against equations 5.20-5.23 in the lecture notes.

More free points – just multiply out the expression for $\theta(t)$ and collect terms:

$$\begin{aligned} \theta(t) &= \theta_0 g_0(t) + \theta_f g_1(t) + \dot{\theta}_0 g_2(t) + \dot{\theta}_f g_3(t) \\ &= \theta_0\left(1 - \frac{3}{t_f^2}t^2 + \frac{2}{t_f^3}t^3\right) + \theta_f\left(\frac{3}{t_f^2}t^2 - \frac{2}{t_f^3}t^3\right) + \dot{\theta}_0\left(t - \frac{2}{t_f}t^2 + \frac{1}{t_f^2}t^3\right) + \dot{\theta}_f\left(-\frac{1}{t_f}t^2 + \frac{1}{t_f^2}t^3\right) \\ &= \theta_0 + \dot{\theta}_0 t + \left[\frac{3}{t_f}(\theta_f - \theta_0) - \frac{1}{t_f}(2\dot{\theta}_0 + \dot{\theta}_f)\right]t^2 + \left[-\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{1}{t_f^3}(\dot{\theta}_0 + \dot{\theta}_f)\right] \end{aligned}$$

So, we have

$$\begin{aligned} a_0 &= \theta_0 \\ a_1 &= \dot{\theta}_0 \\ a_2 &= \frac{3}{t_f^2}(\theta_f - \theta_0) - \frac{1}{t_f}(2\dot{\theta}_0 + \dot{\theta}_f) \\ a_3 &= -\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{1}{t_f^2}(\dot{\theta}_0 + \dot{\theta}_f) \end{aligned}$$