Introduction to Robotics (CS223A) Homework #4 Solution

(Winter 2008/2009)

1. *The mobile robot shown below can be represented as a PPRR manipulator with 4 links, as shown in the schematic.*

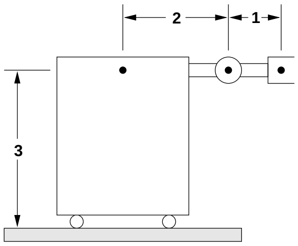


Figure 1: Mobile robot

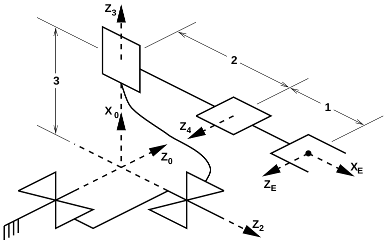


Figure 2: Mobile robot

Luckily, you do not need to compute the forward kinematics, because they are given to you here (with the correction that entry (2,3) of ${}^0_3T$ *is 0.*

$$
{}^0_1T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^0_2T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ -1 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^0_3T = \begin{bmatrix} 0 & 0 & 1 & 3 \\ s_3 & c_3 & 0 & d_2 \\ -c_3 & s_3 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0_4T = \begin{bmatrix} s_4 & c_4 & 0 & 3 \\ s_3c_4 & -s_3s_4 & -c_3 & d_2+2s_3 \\ -c_3c_4 & c_3s_4 & -s_3 & d_1-2c_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^0_ET = \begin{bmatrix} s_4 & c_4 & 0 & 3+s_4 \\ s_3c_4 & -s_3s_4 & -c_3 & d_2+2s_3+s_3c_4 \\ -c_3c_4 & c_3s_4 & -s_3 & d_1-2c_3-c_3c_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(a) *Find the set of all values* (q_1, q_2, q_3, q_4) *such that the end-effector of the robot reaches the point* ${}^0\mathbf{p} = (4, 2, 2)$. *Describe this set in the form of the equations* $f(q_1, q_2) = 0$, $q_3 = g(q_1, q_2)$, *and* $q_4 = c$, *where c is some scalar. What does the curve* $f(q_1, q_2) = 0$ *represent?*

All we need to do is extract the end-effector position vector ${}^0\mathbf{x}_P$ from ${}^0_ET$ and set it equal to $[4\ 2\ 2]^T$, which gives the equations:

$$
\begin{aligned}
3 + s_4 &= 4 && (1) \\
d_2 + s_3(2 + c_4) &= 2 && (2) \\
d_1 - c_3(2 + c_4) &= 2 && (3)
\end{aligned}
$$

From (1), we can immediately determine that $s_4 = 1$, so $\theta_4 = 90°$ (Note this is our equation $q_4 = c$). This means that $c_4 = 0$, so the other equations reduce and rearrange to yield:

$$
\begin{aligned}
-2s_3 &= d_2 - 2 && (4) \\
2c_3 &= d_1 - 2 && (5)
\end{aligned}
$$

Squaring (4) and (5) and adding them together yields $f(q_1, q_2) = 0$:

$$0 = (d_1 - 2)^2 + (d_2 - 2)^2 - 4 \tag{6}$$

Also, we can use (4) and (5) to determine $q_3 = g(q_1, q_2)$:

$$\theta_3 = Atan2(2 - d_2, d_1 - 2) \tag{7}$$

The curve $(d_1 - 2)^2 + (d_2 - 2)^2 - 4 = 0$ is a circle in the (d_1, d_2) plane of radius 2 and center $(2, 2)$. This circle describes the positions that the robot can be in and still reach the center of the circle at $(2, 2)$.

(b) *Derive the basic Jacobian, J_0, that expresses the linear and angular velocity of the end-effector as a function of the joint velocities.*

$$^0J_0(\mathbf{q}) = \begin{bmatrix} 0 & 0 & 0 & c_4 \\ 0 & 1 & c_3(c_4+2) & -s_3 s_4 \\ 1 & 0 & s_3(c_4+2) & c_3 s_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c_3 \\ 0 & 0 & 0 & -s_3 \end{bmatrix}$$

(c) *Assuming that the arm is straight out ($\theta_4 = 0°$), find the joint velocities that will achieve an end-effector angular velocity of $\omega = [1\ 0\ 0]^T$ while keeping the end-effector's position stationary. Describe this set by an equation of the form $\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q})$. Now give a relationship $g(\dot{q_1}, \dot{q_2}) = 0$ that must hold for any configuration $\mathbf{q}$ of the robot which satisfies the problem so far. What does the curve $g(\dot{q_1}, \dot{q_2}) = 0$ represent?*

The velocity vector that we're trying to achieve is $\mathbf{v} = [0\ 0\ 0\ 1\ 0\ 0]^T$ – the first three elements are zero, corresponding to a zero linear velocity. Under the assumption that $\theta_4 = 0$, the Jacobian is:

$$^0J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 3c_3 & 0 \\ 1 & 0 & 3s_3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c_3 \\ 0 & 0 & 0 & -s_3 \end{bmatrix}$$

Since we know the relation $\mathbf{v} = J\dot{\mathbf{q}}$ holds, we have the set of equations:

$$0 = \dot{\theta_4} \tag{8}$$
$$0 = \dot{d_2} + 3c_3\dot{\theta_3} \tag{9}$$
$$0 = \dot{d_1} + 3s_3\dot{\theta_3} \tag{10}$$
$$1 = \dot{\theta_3} \tag{11}$$
$$0 = -c_3\dot{\theta_4} \tag{12}$$
$$0 = -s_3\dot{\theta_4} \tag{13}$$

Already we have $\dot{\theta_3} = 1$ and $\dot{\theta_4} = 0$. Substituting into (9) and (10), we get:

$$\dot{d_2} = -3c_3 \tag{14}$$
$$\dot{d_1} = -3s_3 \tag{15}$$

This gives us the resulting equation

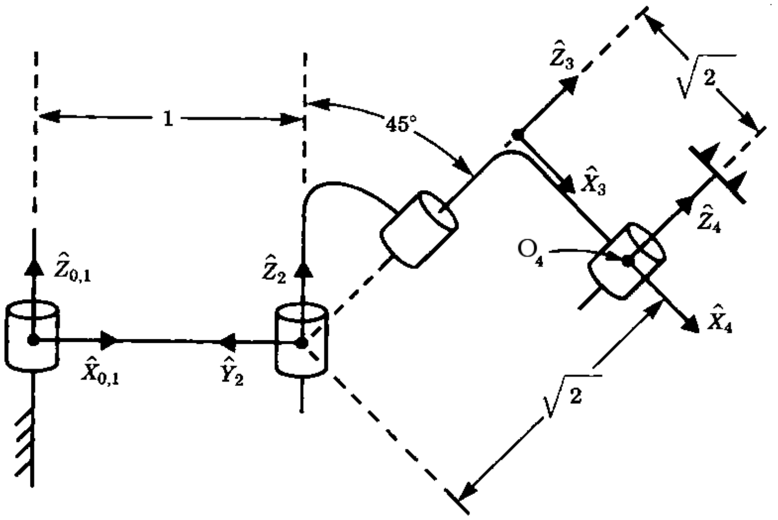
$$\dot{\mathbf{q}} = \begin{bmatrix} -3s_3 \\ -3c_3 \\ 1 \\ 0 \end{bmatrix}$$

By squaring (14) and (15) and adding them together, we can find the relationship $g(\dot{q_1}, \dot{q_2}) = 0$:

$$\dot{d_1}^2 + \dot{d_2}^2 - 9 = 0$$

This relationship says, by the Pythagorean Theorem, that the the velocity of the base of the robot with respect to the floor must be a constant 3.

2. *Consider the following RRRR manipulator (image courtesy J. J. Craig):*



It has the following forward kinematics and rotational Jacobian:

$$^0_4T = \begin{bmatrix} c_{12}c_{34} - \frac{\sqrt{2}}{2}s_{12}s_{34} & -c_{12}s_{34} - \frac{\sqrt{2}}{2}s_{12}c_{34} & \frac{\sqrt{2}}{2}s_{12} & \sqrt{2}c_{12}c_3 - s_{12}(s_3 - 1) + c_1 \\ s_{12}c_{34} + \frac{\sqrt{2}}{2}c_{12}s_{34} & -s_{12}s_{34} + \frac{\sqrt{2}}{2}c_{12}c_{34} & \frac{\sqrt{2}}{2}c_{12} & \sqrt{2}s_{12}c_3 + c_{12}(s_3 - 1) + s_1 \\ \frac{\sqrt{2}}{2}s_{34} & \frac{\sqrt{2}}{2}c_{34} & \frac{\sqrt{2}}{2} & s_3 + 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0J_\omega = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 \\ 1 & 1 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

(a) *Find the basic Jacobian J_o in the $\{0\}$ frame, for the position $\mathbf{q} = [0, 90^0, -90^0, 0]^T$. ($\mathbf{q}$ is the vector of joint variables.)*

$$^0J_v = \begin{bmatrix} \frac{\partial ^0\mathbf{P}_e}{\partial q_1} & \frac{\partial ^0\mathbf{P}_e}{\partial q_2} & \frac{\partial ^0\mathbf{P}_e}{\partial q_3} & \frac{\partial ^0\mathbf{P}_e}{\partial q_4} \end{bmatrix}$$

where ${}^0\mathbf{P}_e$ is from the 4^{th} column of ${}^0_4T$. Thus:

$$
{}^0J_v = \begin{bmatrix} -\sqrt{2}s_{12}c_3 - c_{12}(s_3-1) - s_1 & -\sqrt{2}s_{12}c_3 - c_{12}(s_3-1) & -\sqrt{2}c_{12}s_3 - s_{12}c_3 & 0 \\ \sqrt{2}c_{12}c_3 - s_{12}(s_3-1) + c_1 & \sqrt{2}c_{12}c_3 - s_{12}(s_3-1) & -\sqrt{2}s_{12}s_3 + c_{12}c_3 & 0 \\ 0 & 0 & c_3 & 0 \end{bmatrix}
$$

Plug in $\mathbf{q} = [0, 90^0, -90^0, 0]^T$, and join with ${}^0J_\omega$ (which was directly given to us) to get:

$$
\boxed{{}^0J_o = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 3 & 2 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 1 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right]}
$$

(b) *A general force vector is applied to the origin of frame* $\{4\}$ *and measured in frame* $\{4\}$ *to be* $[0, 6, 0, 7, 0, 8]^T$*. For the position in (a), determine the joint torques that statically balance it.*

We are given a 6×1 force/moment vector $\mathbf{F}_{app}$ which is exerted on the robot. If the arm is statically balancing this, then we know that the robot must be exerting an equal and opposite force/moment vector at the origin of frame $\{4\}$ (we can thank Sir Isaac Newton for that!).

So we know that in the coordinates of frame $\{4\}$, the vector ${}^4\mathbf{F}_4 = -{}^4\mathbf{F}_{app}$ and we want to find the joint torques τ corresponding to ${}^4\mathbf{F}_4$.

Recall that $\tau = J^T F$. To multiply F and J, however, they must be in the same frame. You can transform either the J from frame $\{0\}$ to $\{4\}$, or transform F from frame $\{4\}$ to $\{0\}$. Both give the same answer.

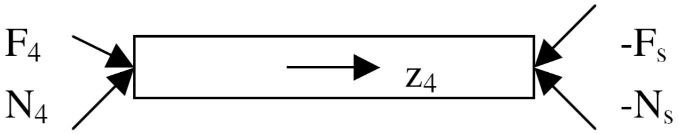
$$
\begin{aligned}
{}^4\mathbf{F}_4 &= -{}^4\mathbf{F}_{app} = -[0, 6, 0, 7, 0, 8]^T \\
{}^0\mathbf{F}_4 &= \begin{bmatrix} {}^0_4R & 0 \\ 0 & {}^0_4R \end{bmatrix} {}^4\mathbf{F}_4 \\
\tau &= {}^0J^{T\,0}\mathbf{F}_4
\end{aligned}
$$

The final answer is:

$$
\boxed{\tau = -\left[18.707, 12.707, 16.485, 8.0\right]^T}
$$

(c) *Consider the same configuration as above. A screw driver is gripped in the end-effector so that its tip is along* $\hat{Z}_4$ *at a distance of* 9 *units of length from the origin of frame* $\{4\}$*. What is the force and torque the screw driver tip applies when the same joint torques that were determined in part (b) are applied?*

Let's look at the free-body diagram of the screw-driver, with the left-end being at origin 0_4 and the screw-driver tip on the right. *NOTE: In this diagram, we consider 3x1 force and moment vectors, so "F" represents the 3x1 linear force, NOT the combined 6x1 vector.*



We must first choose an origin for our computations, and then apply static equilibrium. For this computation, *the choice of origin is arbitrary!* You should get the same answer regardless. Two sensible options are either the origin of frame {4}, or the tip $\{S\}$ of the screw-driver. Let's use the origin of frame {4}. Also, for simplicity we'll express all our vectors using the coordinates of frame {4}.

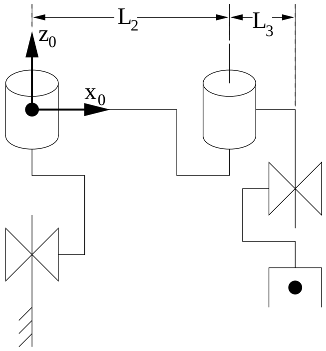
In static equilibrium, we know: $\Sigma\mathbf{F} = 0$ and $\Sigma\mathbf{N} = 0$. These give us:

$$\begin{aligned} \mathbf{F}_4 + (-\mathbf{F}_s) = 0 \quad &\Rightarrow \quad \mathbf{F}_s = \mathbf{F}_4 \\ \mathbf{N}_4 + (-\mathbf{N}_s) + \mathbf{P}_{s4} \times (-\mathbf{F}_s) = 0 \quad &\Rightarrow \quad \mathbf{N}_s = \mathbf{N}_4 +^4 \mathbf{P}_s \times (-\mathbf{F}_s) \end{aligned}$$

The position ${}^4\mathbf{P}_s$ is the position vector from origin {4} to the tip, so we know that: ${}^4\mathbf{P}_s = [0, 0, 9]^T$. Meanwhile, from part (b), we that: $\mathbf{F}_4 = -[0, 6, 0]^T$ and $\mathbf{N}_4 = -[7, 0, 8]^T$. If we first solve for $\mathbf{F}_s$ (using the upper equation), we can then use this value to solve for $\mathbf{N}_s$ in the lower equation. THUS:

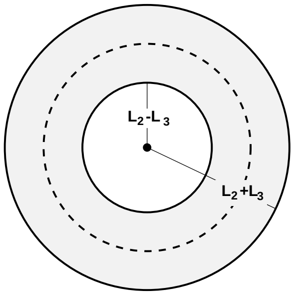
$$\boxed{\begin{aligned} {}^4\mathbf{F}_s &= -[0, 6, 0]^T \\ {}^4\mathbf{N}_s &= -[61, 0, 8]^T \end{aligned}}$$

3. *Consider the PRRP manipulator schematic shown below:*



(a) *Assuming no joint limits, sketch the workspace of this manipulator. Be sure to include dimensions in your drawing. Assume $L_2 > L_3$.*

Since the prismatic joints have no limits, the workspace is an infinite cylinder along the Z_0 direction, whose cross-section is shown in the following figure.



(b) *Describe the (3D) dextrous workspace of this manipulator.*

This manipulator can only point its end-effector downwards, so there are no points for which it can achieve an arbitrary orientation. Even if you consider only the orientation with respect to the (X_0, Y_0) plane (eg. the angle with the X0-axis) there are only two joints to control the position in the plane, leaving no degrees of freedom for controlling the orientation. Therefore, the dexterous workspace is null.

(c) *With no joint limits, if we are considering only the position of the end effector, how many inverse kinematic solutions are there (in general)? Explain briefly.*

If we find a configuration of the joints in this manipulator that places the end-effector at a given position, we can achieve the same position by shortening one prismatic joint and extending the other by any value Δ. This manipulator is redundant in the Z_0 direction, so an infinite number of inverse kinematic solutions exist.

(d) *Imagine that we remove the first prismatic joint, so that the first revolute joint now rotates around the base. Repeat part (c) for such an RRP manipulator.*

If we remove one of the prismatic joints, the manipulator is no longer redundant. For any point (x, y, z), the extension of the prismatic joint is completely determined by z. In the (X_0, Y_0) plane, however, there are two values of the revolute joint angles that will achieve a given (x, y), however: elbow up and elbow down. Therefore, there are two inverse kinematic solutions for a given position.

(e) *Imagine that we further modify the manipulator from part (d) by inserting another revolute joint between the two existing revolute joints, whose axis is oriented in the same direction as the other two. Repeat part (c) for such an RRRP manipulator.*

Compared to part (d), now the manipulator is redundant in the (X_0, Y_0) plane. For a given planar position (x, y), there are three revolute joints for only two position variables (ie. for x and y), thus there are an infinite number of joint angles that will achieve it. This means there are an infinite number of inverse kinematic solutions.