Introduction to Robotics (CS223A) — Homework #3 Solution

(Winter 2008/2009)

1. *Assume that for a given manipulator, frames have been assigned to the links following the standard conventions. Given the transformation matrix from $\{i\}$ to $\{i-1\}$*

$$ {}^{i-1}_{\ \ i}T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} $$

use equation (2.7) to determine the D-H parameters a_{i-1}, α_{i-1}, d_i, θ_i in terms of the t_{jk}'s.

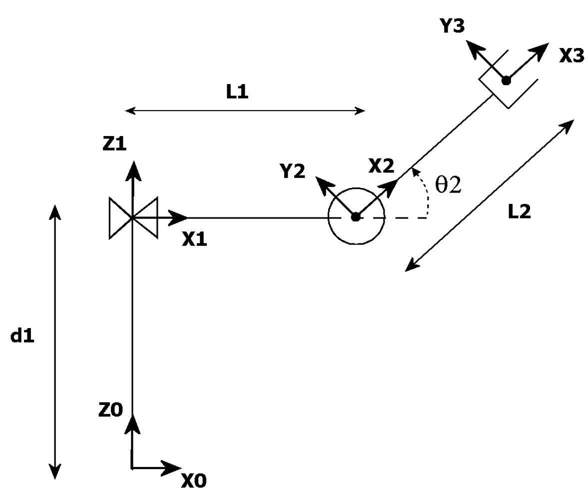
From equation (2.7) we have:

$$ {}^{i-1}_{\ \ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} $$

Thus, matching components, we have

$$ \begin{aligned} a_{i-1} &= t_{14} \\ \alpha_{i-1} &= atan2(s\alpha_{i-1}, c\alpha_{i-1}) \\ &= atan2(-t_{23}, t_{33}) \\ \theta_i &= atan2(s\theta_i, c\theta_i) \\ &= atan2(-t_{12}, t_{11}) \\ d_i &= \begin{cases} t_{24}/t_{23} & \text{if } t_{23} \neq 0 \\ t_{34}/t_{33} & \text{otherwise} \end{cases} \end{aligned} $$

2. *Consider the planar PR manipulator shown here:*



(a) *Find the origin of frame {3} expressed in terms of frame {0}, that is* ${}^0\mathbf{P}_{3org}$.
Tip: you can derive this geometrically, if you want to avoid going through DH parameters.
It's simplest to do this geometrically. Note: I will often refer to the origin of frame {3} as the "tip".

In the base frame {0}, the x-coordinate of the tip is found by projecting the link L_2 onto the $\hat{X}_0$ axis and adding to L_1. Similarly, the z-coordinate is found by projecting L_2 onto the $\hat{Z}_0$ axis and adding to the slider displacement d_1. Thus, in frame {0}:

$${}^0\mathbf{P}_{3org} = \begin{bmatrix} x_{tip} \\ z_{tip} \end{bmatrix} = \begin{bmatrix} L_1 + L_2\cos(\theta_2) \\ d_1 + L_2\sin(\theta_2) \end{bmatrix}$$

(b) *Give the* 2×2 *Jacobian that relates the joint velocities to the linear velocity of* ${}^0\mathbf{P}_{3org}$.

$$J = \begin{bmatrix} \frac{\partial x_{tip}}{\partial d_1} & \frac{\partial x_{tip}}{\partial \theta_2} \\ \frac{\partial z_{tip}}{\partial d_1} & \frac{\partial z_{tip}}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} 0 & -L_2\sin(\theta_2) \\ 1 & L_2\cos(\theta_2) \end{bmatrix}$$

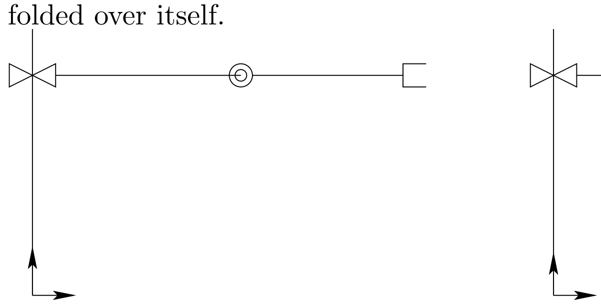
(c) *For what joint values is the manipulator at a singularity? What motion is restricted at this singularity?*
The singularity occurs at a configuration when $det(J) = 0$.

$$\begin{aligned} det(J) &= J_{11}J_{22} - J_{12}J_{21} \\ &= L_2\sin(\theta_2) \end{aligned}$$

So the singularity occurs whenever $\sin(\theta_2) = 0$, which corresponds to:

$$\theta_2 = 0^o \text{ or } \pm 180^o$$

These two situations are portrayed below; either the arm is extended out completely or folded over itself.



In both cases, **the end-effector cannot move instantaneously in the $\hat{X}_0$ direction**, ie. the joints cannot produce a velocity component in the $\hat{X}_0$ direction.

3. *Consider the RRR manipulator shown in Figure 1.*

(a) *Assuming that the joint limits of this manipulator are* $0 \le \theta_2 \le 180°$ *and* $-90° \le \theta_3 \le 90°$*, sketch the workspace of this manipulator. Be sure to include a cross section with dimensions included.*

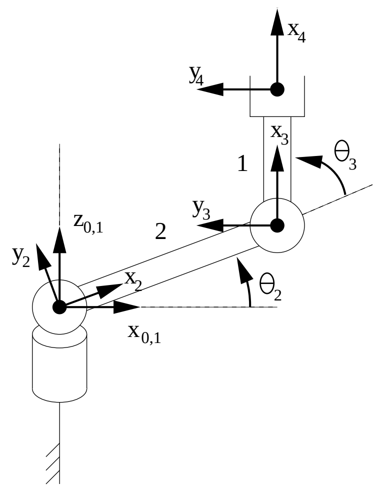


Figure 1: An RRR manipulator.

A good procedure for these workspace problems is to spin the end-effector point around the full travel of the outermost joint, then to take the resulting set of points and spin it around the next joint, etc. For this manipulator, this results in a solid of revolution, whose cross-section is shown below.

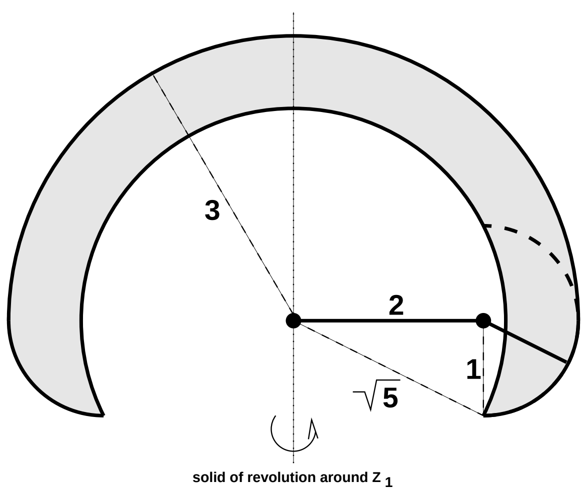


Figure 2: Cross-section of workspace

(b) *Find the DH parameters for this manipulator. Remember to assign the interior frames of this manipulator using the conventions discussed in class.*

i	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	θ_1	0
2	$90°$	0	θ_2	0
3	0	2	θ_3	0

(c) *Derive the forward kinematics, ${}^0_4T$, of this manipulator.* From the D-H equation,

$$
{}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^2_3T = \begin{bmatrix} c_3 & -s_3 & 0 & 2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

We also need the the offset from frame $\{3\}$ to the end-effector frame $\{4\}$, which is a simple translation along x_3:

$$
{}^3_4T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

As usual, we get the transformations by multiplying. We do the multiplications from left to right, as we'll need these sub-results later when computing the Jacobian.

$$
{}^0_2T = {}^0_1T\,{}^1_2T = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 & 0 \\ s_1c_2 & -s_1s_2 & -c_1 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^0_3T = {}^0_2T\,{}^2_3T = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 & 2c_1c_2 \\ s_1c_{23} & -s_1s_{23} & -c_1 & 2s_1c_2 \\ s_{23} & c_{23} & 0 & 2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^0_4T = {}^0_3T\,{}^3_4T = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 & 2c_1c_2 + c_1c_{23} \\ s_1c_{23} & -s_1s_{23} & -c_1 & 2s_1c_2 + s_1c_{23} \\ s_{23} & c_{23} & 0 & 2s_2 + s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(d) *Find the basic Jacobian, J_0, for this manipulator.*

For $\mathbf{x}_P$, we use the position of the end-effector expressed in frame $\{0\}$, which is the last column of ${}^0_4T$:

$$
{}^0\mathbf{x}_P(\mathbf{q}) = \begin{bmatrix} 2c_1c_2 + c_1c_{23} \\ 2s_1c_2 + s_1c_{23} \\ 2s_2 + s_{23} \end{bmatrix}, \quad {}^0J_0 = \begin{bmatrix} \frac{\partial {}^0\mathbf{x}_P}{\partial q_1} & \frac{\partial {}^0\mathbf{x}_P}{\partial q_2} & \frac{\partial {}^0\mathbf{x}_P}{\partial q_3} \\ \bar{\epsilon}_1 {}^0\mathbf{z}_1 & \bar{\epsilon}_2 {}^0\mathbf{z}_2 & \bar{\epsilon}_3 {}^0\mathbf{z}_3 \end{bmatrix}
$$

$$
{}^0J_0 = \begin{bmatrix} -s_1(2c_2 + c_{23}) & -c_1(2s_2 + s_{23}) & -c_1s_{23} \\ c_1(2c_2 + c_{23}) & -s_1(2s_2 + s_{23}) & -s_1s_{23} \\ 0 & 2c_2 + c_{23} & c_{23} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}
$$

(e) *Find ${}^1J_v$, the position Jacobian matrix expressed in frame $\{1\}$.*

This is accomplished by rotating ${}^0J_v$, the top half of ${}^0J_0$, by the matrix ${}^1_0R = {}^0_1R^T$:

$$
\begin{aligned}
{}^1J_v = {}^1_0R\,{}^0J_v &= \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_1(2c_2+c_{23}) & -c_1(2s_2+s_{23}) & -c_1s_{23} \\ c_1(2c_2+c_{23}) & -s_1(2s_2+s_{23}) & -s_1s_{23} \\ 0 & 2c_2+c_{23} & c_{23} \end{bmatrix} \\
&= \begin{bmatrix} 0 & -(2s_2+s_{23}) & -s_{23} \\ 2c_2+c_{23} & 0 & 0 \\ 0 & 2c_2+c_{23} & c_{23} \end{bmatrix}
\end{aligned}
$$

(f) *Use the matrix that you found in (e) to find the singularities (with respect to linear velocity) of this manipulator.*

We need to take ${}^1J_v$ and find the values of $\theta_1, \theta_2, \theta_3$ for which the matrix is singular. We can do this by taking the determinant and setting it to zero:

$$
\begin{aligned}
\det({}^1J_v) &= -(2c_2+c_{23})\left[-(2s_2+s_{23})c_{23} + (2c_2+c_{23})s_{23}\right] \\
&= -(2c_2+c_{23})(-2s_2c_{23} + 2c_2s_{23} - s_{23}c_{23} + c_{23}s_{23}) \\
&= -2(2c_2+c_{23})(s_{23}c_2 - c_{23}s_2) \\
&= -2(2c_2+c_{23})(s_3)
\end{aligned}
$$

Setting the determinant equal to zero, we say that ${}^1J_v$ is singular (and therefore, the manipulator is at a singularity) when $2c_2 + c_{23} = 0$ or $s_3 = 0$.

(g) *For each type of singularity that you found in part (f), explain the physical interpretation of the singularity, by sketching the arm in a singular configuration and describing the resulting limitation on its movement.*

The quantity $2c_2 + c_{23}$ is the distance of the end-effector point to the axis z_1. When this distance is zero, then joint 1 has no effect on the velocity of the end-effector. Since joints 2 and 3 can only effect velocities in their common plane of rotation, the end-effector cannot move in the y_1 direction.

When $s_3 = 0$, the elbow joint is either straight out ($\theta_3 = 0$) or bent back on itself ($\theta_3 = 180°$). In either case it is impossible for the end-effector to move in the x_4 direction.

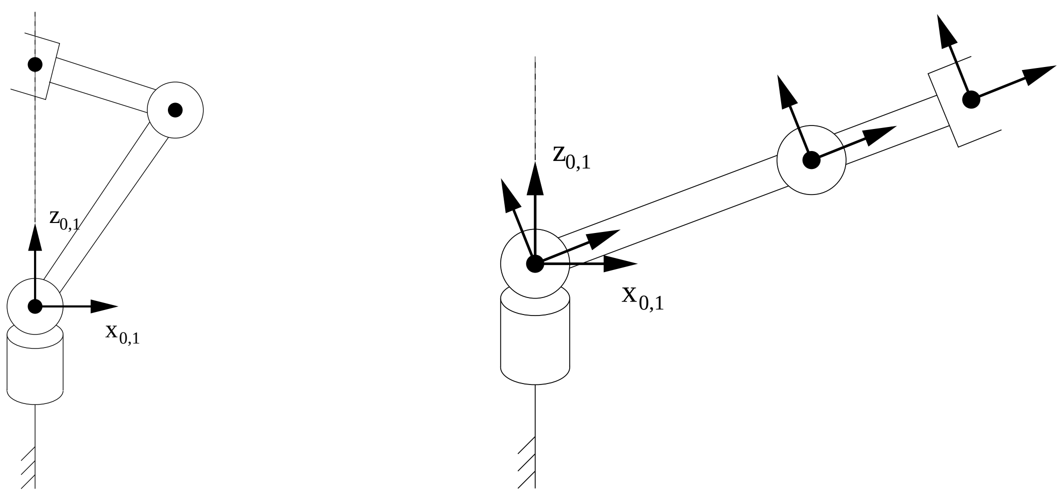


Figure 3: Singular configurations of an RRR manipulator