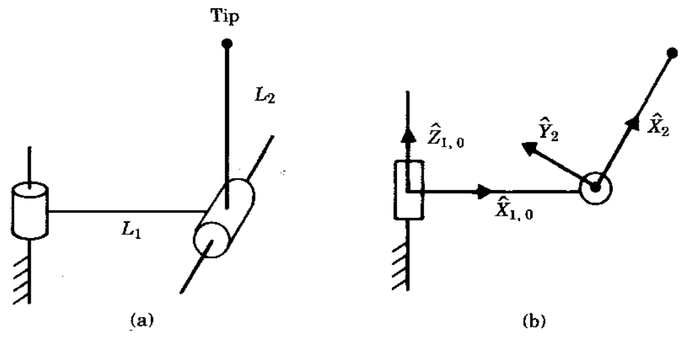
1. **Looking at equation 2.7 of the Lecture Notes, give a geometric interpretation of why $t_{13} = 0$. *Hint: Consider what the third column represents; your answer should be only one or two sentences.***

   The third column of the matrix is the z-axis of frame $\{i\}$ expressed in the coordinates of frame $\{i-1\}$, ie. ${}^{i-1}\hat{Z}_i$. Using DH conventions, the tranformation between $\hat{Z}_i$ and $\hat{Z}_{i-1}$ is only a rotation about the $\hat{X}_{i-1}$ axis, so the x-coordinate of ${}^{i-1}\hat{Z}_i$ will be zero just as it was for ${}^{i-1}\hat{Z}_{i-1}$.

2. **For the 2-link manipulator shown, the link transformations ${}^0_1T$ and ${}^1_2T$ were determined. Their product is:**

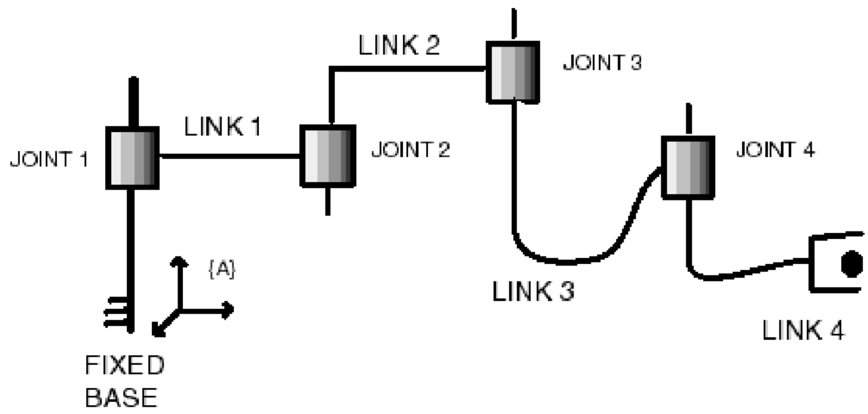
$$
{}^0_2T = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 & l_1c_1 \\ s_1c_2 & -s_1s_2 & -c_1 & l_1s_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

   The frame assingments used are indicated below in the figure. Note that frame 0 is coincident with frame 1 when θ_1 is 0. The length of the second link is l_2. Find an expression for the vector ${}^0P_{tip}$ which locates the tip of the arm relative to the 0 frame (figure courtesy of J. J. Craig).



$$
\begin{bmatrix} {}^0P_{tip} \\ 1 \end{bmatrix} = {}^0_2T \begin{bmatrix} {}^2P_{tip} \\ 1 \end{bmatrix} = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 & c_1l_1 \\ s_1c_2 & -s_1s_2 & -c_1 & s_1l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(c_2l_2 + l_1) \\ s_1(c_2l_2 + l_1) \\ s_2l_2 \\ 1 \end{bmatrix}
$$

3. **The following sketch represents a generic open, serial, kinematic-chain.**



Here each kinematic joint connects two adjacent members. Assume that the relative displacement between adjacent members $i-1$ and i is described by an operator T_i that is a 4x4 matrix whose elements are computed in a coordinate frame $\{A\}$ fixed to the base of the chain. Now, if each member is displaced in sequence, *starting from the free end*, the displacement operator for the resultant total displacement of the free end will be given by $T_1T_2T_3T_4$. (Note: In this problem you are to use only displacements operators, not coordinate transformations)

However, if the displacements are done in the reverse order, ie. *starting at the fixed end*, and moving in the sequence 1, 2, 3, 4, then the operators T_2, T_3 , and T_4 no longer represent the actual displacements.

Determine, in terms of the original T_i:

(a) **The operator for joint 2, when its displacement is done *after* the displacement in joint 1. Let us call this operator T_2'**

The first thing to recognize is that this entire question is based on displacement operators, *not* frame transformations – this concept was discussed Lecture Notes section 1.2.5, and is vital to solving this question.

Now, when joint 1 moves it causes a displacement (given by T_1) which affects the entire manipulator – including the rotation axes of joints 2, 3, and 4. To find the new rotation axis corresponding to joint 2, we use a similarity transform:

$$T_2' = T_1T_2T_1^{-1}$$

(b) **The operator for joint 3 when its displacement follows the displacement in joints 1 and 2 (from part (a)). Let us call this operator T_3'**

Joint 3 gets displaced first by joint 1 (T_1), followed by joint 2 from part (a) (T_2'). So when we do a similarity transform on T_3, we use the combined displacement matrix $T_2'T_1$. This means:

$$T_3' = (T_2'T_1)T_3(T_2'T_1)^{-1}$$

Using the answer from part (a) to substitute in for T_2' gives:

$$\begin{aligned} T_3' &= (T_1T_2)T_3(T_1T_2)^{-1} \\ &= T_1T_2T_3T_2^{-1}T_1^{-1} \end{aligned}$$

(c) **The operator for joint 4 when its displacement follows the displacement in joints 1, 2 and 3 (from part (b)). Let us call this operator T_4'**

The reasoning for this part is the same as before. Joint 4 gets displaced first by T_1, followed by T_2', and T_3'. So when we do the similarity transform, we need to use $T_3'T_2'T_1$. This gives:

$$T_4' = (T_3'T_2'T_1)T_4(T_3'T_2'T_1)^{-1}$$

Plug in the results for T_2' and T_3' from parts (a) and (b) respectively, and then simplify:

$$\begin{aligned} T_4' &= (T_1T_2T_3T_2^{-1}T_1^{-1}T_1T_2T_1^{-1}T_1)T_4(T_1T_2T_3T_2^{-1}T_1^{-1}T_1T_2T_1^{-1}T_1)^{-1} \\ &= (T_1T_2T_3)T_4(T_1T_2T_3)^{-1} \\ &= T_1T_2T_3T_4T_3^{-1}T_2^{-1}T_1^{-1} \end{aligned}$$

(d) **Using your results for parts (a), (b) and (c), show that the resulting displacement operator for the free end is still $T_1T_2T_3T_4$**

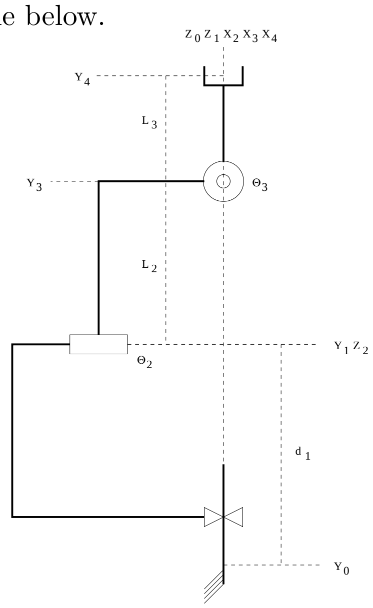
First joint 1 moved, then joint 2 from part (a), then joint 3 from part (b), then joint 4 from part (c). The final displacement of the end effector is given by:

$$T_{final} = T_4'T_3'T_2'T_1$$

Plug in the expressions from the previous parts and simplify:

$$\begin{aligned} T_{final} &= (T_1T_2T_3T_4T_3^{-1}T_2^{-1}T_1^{-1})(T_1T_2T_3T_2^{-1}T_1^{-1})(T_1T_2T_1^{-1})T_1 \\ &= T_1T_2T_3T_4 \end{aligned}$$

4. (a) See figure below.
   (b) See figure below.
   (c) See table below.



i	α_{i-1}	a_{i-1}	d_i	θ_i	conf. shown
1	0	0	d_1	0	$d_1 = d_1$
2	$-90°$	0	0	θ_2	$\theta_2 = -90°$
3	$-90°$	L_2	0	θ_3	$\theta_3 = 0°$
4	0	L_3	0	0	n/a

(d)

$$
{}^0_4T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & L_2 \\ 0 & 0 & 1 & 0 \\ -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} c_2c_3 & -c_2s_3 & -s_2 & c_2(c_3L_3 + L_2) \\ -s_3 & -c_3 & 0 & -s_3L_3 \\ -s_2c_3 & s_2s_3 & -c_2 & -s_2(c_3L_3 + L_2) + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(e)

$$
{}^0P_{4ORG} = \begin{bmatrix} -L_2 \\ L_3 \\ d_1 \end{bmatrix} = \begin{bmatrix} c_2(c_3L_3 + L_2) \\ -s_3L_3 \\ -s_2(c_3L_3 + L_2) + d_1 \end{bmatrix} \Longrightarrow \begin{cases} \theta_2 = \pm k\pi, wherek = 0, 1, 2, \ldots \\ \theta_3 = (-\pi/2 \pm 2k\pi) or (3\pi/2 \pm 2k\pi) \\ d_1 = d_1 \end{cases}
$$

(f) $L_2 = 2L_3 = 0.5m$, $0.5m \le d_1 \le 1.0m$, $-180^\circ \le \theta_2 \le 0^\circ$, and $-90^\circ \le \theta_3 \le 0^\circ$