Introduction to Robotics (CS223A) Homework #1 Solution

(Winter 2008/2009)

1. **A frame $\{B\}$ and a frame $\{A\}$ are initially coincident. Frame $\{B\}$ is rotated about $\hat{Y}_B$ by an angle θ, and then rotated about the new $\hat{Z}_B$ by an angle ϕ. Determine the 3×3 rotation matrix, ${}^A_BR$, which will transform the coordinates of a position vector from ${}^B\mathbf{P}$, its value in frame $\{B\}$, into ${}^A\mathbf{P}$, its value in frame $\{A\}$.**

   Consider the intermediate frame $\{M\}$ which results after the first rotation:

   $$ {}^A_BR = {}^A_MR\,{}^M_BR $$

   Now, the frame transformations from $\{A\}$ to $\{M\}$, and $\{M\}$ to $\{B\}$, are precisely those rotations listed in the question, so we know that ${}^A_MR = R_y(\theta)$ and ${}^M_BR = R_z(\phi)$. Thus:

   $$ {}^A_BR = R_y(\theta)R_z(\phi) $$

   Indeed, this is the Y-Z Euler-angle representation for frame $\{B\}$ w.r.t. frame $\{A\}$.

   Written out:

   $$ \begin{aligned} {}^A_BR &= \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c\theta c\phi & -c\theta s\phi & s\theta \\ s\phi & c\phi & 0 \\ -s\theta c\phi & s\theta s\phi & c\theta \end{bmatrix} \end{aligned} $$

2. **A vector ${}^AP$ is rotated about $\hat{X}_A$ by ϕ degrees and is subsequently rotated about $\hat{Z}_A$ by θ degrees.**

   (a) Give the rotation matrix which accomplishes these rotations in the given order.

   (b) What is the result if $\phi = 30°$ and $\theta = 45°$?

   Since both rotations are being performed with respect to the same frame, the compound rotation can be thought of as an X-Z fixed angle specification. For combining fixed angle rotations, we premultiply. Therefore, the resulting rotation operator is

   $$ R(\phi,\theta) = R_{X\,Z}(\phi,\theta) = R_Z(\theta)R_X(\phi) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} = \begin{bmatrix} c\theta & -s\theta c\phi & s\theta s\phi \\ s\theta & c\theta c\phi & -c\theta s\phi \\ 0 & s\phi & c\phi \end{bmatrix} $$

   And, for $\phi = 30°$ and $\theta = 45°$

   $$ R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} $$

3. **A vector is given by**

$$^{B}P = \begin{bmatrix} 1.0 \\ 5.0 \\ 10.0 \end{bmatrix}.$$

Given

$$^{B}_{A}T = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 3.0 \\ 0.0 & 0.6 & 0.8 & -1.0 \\ 0.0 & -0.8 & 0.6 & 1.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix},$$

compute ^{A}P.

First we find the inverse transform, $^{A}_{B}T$. The given matrix tells us both $^{B}_{A}R$ and $^{B}P_{AORG}$.

Since $^{A}_{B}R = ^{B}_{A}R^{T}$ and $^{A}P_{BORG} = -(^{B}_{A}R^{T})(^{B}P_{AORG})$, we compute: $^{A}_{B}T = \begin{bmatrix} 1.0 & 0.0 & 0.0 & -3.0 \\ 0.0 & 0.6 & -0.8 & 1.4 \\ 0.0 & 0.8 & 0.6 & 0.2 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}.$

Thus, $^{A}P =^{A}_{B} T^{B}P = \begin{bmatrix} 1.0 & 0.0 & 0.0 & -3.0 \\ 0.0 & 0.6 & -0.8 & 1.4 \\ 0.0 & 0.8 & 0.6 & 0.2 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{bmatrix} 1.0 \\ 5.0 \\ 10.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -2.0 \\ -3.6 \\ 10.2 \end{bmatrix}$

Notice I'm switching freely between homogenous and regular 3D coordinates.

4. **Given the following 3×3 matrix:**

$$\begin{bmatrix} 0.7905 & -0.3864 & 0.4752 \\ 0.6046 & 0.3686 & -0.7061 \\ 0.0977 & 0.8455 & 0.5250 \end{bmatrix}$$

(a) **Verify that, within practical numerical limits, it does in fact represent a rigid body rotation or a rotational Cartesian coordinate transformation.**

This can be done by proving that the matrix has orthonormal columns, ie. each column vector is a unit vector, and orthogonal to the other columns. The same principle applies to the rows.

There are two ways to check this, and they amount to the exact same set of computations: either (i) check if $R^{T}R = I$, or (ii) check that each column vector has unit length and that its inner product with any other column gives zero.

Here is the first option in Matlab™ :

```
% question 4(a)
R = [0.7905 -0.3864 0.4752
0.6046 0.3686 -0.7061
0.0977 0.8455 0.5250];
R'*R
```

The output is:

```
1.0000          0.0000          0.0000
0.0000          1.0000          0.0000
0.0000          0.0000          1.0000
```

(b) **Determine a set of Euler parameters defined by this matrix.**

Apply equations 1.61, 1.62 of the Lecture Notes:

```
% question 4(b)
e4 = 0.5*sqrt(1 + R(1,1) + R(2,2) + R(3,3))
e1 = ( R(3,2) - R(2,3) ) / ( 4*e4 )
e2 = ( R(1,3) - R(3,1) ) / ( 4*e4 )
e2 = ( R(2,1) - R(1,2) ) / ( 4*e4 )
```

The answers are: $e4 = 0.8192$, $e1 = 0.4735$, $e2 = 0.1152$, and $e3 = 0.3024$.

(c) **Determine a unit vector describing the axis of rotation, and the angle (in degrees) of the rotation.**

Apply equations 1.52, 1.53 of the Lecture Notes:

```
% question 4(c)
rot_angle_rad = acos( (R(1,1) + R(2,2) + R(3,3) - 1)/2 );
rot_angle_deg = rot_angle_rad*180/pi
temp_axis = [R(3,2) - R(2,3)
R(1,3) - R(3,1)
R(2,1) - R(1,2)];
rot_axis = ( 1 / (2*sin(rot_angle_rad)) ) * temp_axis
```

The answers are: $angle = 70^o$, and $axis = \begin{bmatrix} 0.8256 & 0.2009 & 0.5273 \end{bmatrix}^T$.