Introduction to Robotics (CS223A) Homework #5

(Winter 2008/2009) Due: **Monday, March 2**

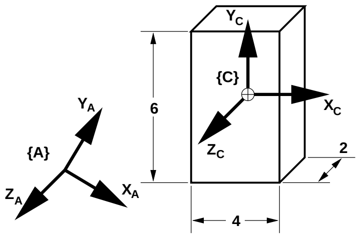
YOU HAVE BEEN GIVEN 1.5 WEEKS FOR THIS PROBLEM SET. IT IS SLIGHTLY LONGER THAN USUAL SO PLEASE DON'T LEAVE IT TO THE LAST MINUTE!

- Please include a little grid like this at the top of your first page of homework:

1	2	3	Total

1. (a) Derive a formula that transforms an inertia tensor given in some frame $\{C\}$ into a new frame $\{A\}$. The frame $\{A\}$ can differ from frame $\{C\}$ by both translation and rotation. You may assume that frame $\{C\}$ is located at the center of mass.

   (b) Consider, for example, the uniform density box shown below. It has mass $m = 12kg$, and dimensions $6 \times 4 \times 2$.



   Frame $\{C\}$ lies at the center of mass of the box, and the coordinate axes are lined up with the principal axes of the box. In other words, $\mathbf{Y}_C$ is aligned with the long axis of the box, and $\mathbf{X}_C$ and $\mathbf{Z}_C$ are aligned with the short axes of the box.

   First, compute the inertia tensor of the box in frame $\{C\}$.

   Note: When using a frame at the center of mass and along the principal axes, the inertia tensor for the box of uniform density becomes diagonal. It takes the form:

$$
{}^{C}I = \begin{bmatrix} \frac{m}{12}(s_y^2 + s_z^2) & 0 & 0 \\ 0 & \frac{m}{12}(s_x^2 + s_z^2) & 0 \\ 0 & 0 & \frac{m}{12}(s_x^2 + s_y^2) \end{bmatrix}
$$

   where s_x, s_y and s_z are the dimensions of the box along the $\mathbf{X}_C$, $\mathbf{Y}_C$ and $\mathbf{Z}_C$ axes, respectively. In this case, $s_x = 4$, $s_y = 6$, and $s_z = 2$.

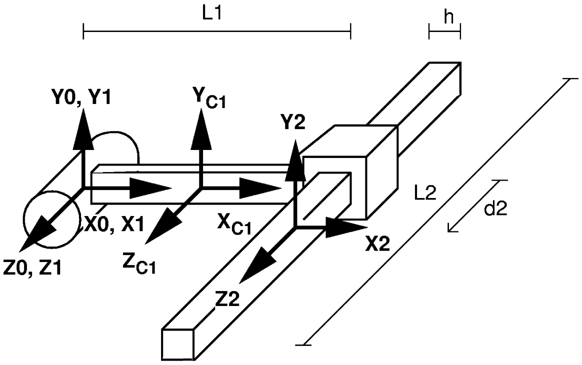
   (c) Given the transformation matrix from $\{C\}$ to $\{A\}$:

$$
{}^{A}_{C}T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

   use your formula from part (a) and your inertia tensor from part (b) to compute the inertia tensor of the box in frame $\{A\}$.

2. In the rest of this problem set, we will walk through the process of finding the equations of motion for a simple manipulator using the Lagrange formulation. Consider the RP spatial manipulator shown below. The links of this manipulator are modeled as bars of uniform density, having square cross-sections of thickness h, lengths of L_1 and L_2, and total masses of m_1 and m_2. The centers of mass are at the midpoints of the respective members, and are marked by the origins of the coordinate systems $\{c1\}$ and $\{2\}$. Assume that the joints themselves are massless.

   Note: The distance $d2$ shown in the diagram is the DH parameter which measures distance from link 1 to the origin of $\{2\}$, in the Z_2 direction



   From the derivation in the Lecture Notes, we know that the equations of motion have the form:

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q})\dot{\mathbf{q}}^2 + B(\mathbf{q})\left[\dot{\mathbf{q}}\dot{\mathbf{q}}\right] + \mathbf{G}(\mathbf{q}) = \tau$$

   where M is the mass matrix, C is the matrix of coefficients for centrifugal forces, B is the matrix of coefficients for Coriolis forces, and $\mathbf{G}$ is the vector of gravity forces.

   (a) For each link i, we have attached a frame $\{C_i\}$ to the center of mass (in this case, the DH frame $\{2\}$ is the same as $\{C_2\}$).
   Compute kinematics for these frames: that is, calculate the matrices ${}^{0}_{C_1}T$ and ${}^{0}_{C_2}T$.

   For a two-link manipulator, the mass matrix has the form

$$M = m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_1}^T {}^{C_1}I_1 J_{\omega_1} + J_{\omega_2}^T {}^{C_2}I_2 J_{\omega_2}$$

   where J_{v_i} is the linear Jacobian of the center of mass of link i, J_{ω_i} is the angular velocity Jacobian of link i, and ${}^{C_i}I_i$ is the inertia tensor of link i expressed in frame $\{C_i\}$.

   (b) Calculate ${}^{0}J_{v_1}$ and ${}^{0}J_{v_2}$.

   (c) Calculate ${}^{C_1}J_{\omega_1}$ and ${}^{C_2}J_{\omega_2}$.

   (d) Calculate ${}^{C_1}I_1$ and ${}^{C_2}I_2$ in terms of the masses and dimensions of the links. You can use the same formula that was given for a box of uniform density in Problem 2(b). Be careful which measurements you use along the axes.

(e) Calculate the mass matrix, $M(\mathbf{q})$. To make your algebra easier, leave the inertia tensors in symbolic form until the end, i.e.

$$^{C_1}I_1 = \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}$$

Now we need to calculate the centrifugal and Coriolis forces. We will derive the form directly.

(f) Beginning with the equation in the Lecture Notes,

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_2}\dot{\mathbf{q}} \end{bmatrix},$$

manipulate this equation symbolically into the form

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}]$$

where C and B are matrices in terms of the partial derivatives m_{ijk} of the mass matrix. Don't actually substitute in your answer from part (e) into this equation yet: just leave the elements of these matrices in m_{ijk} symbolic form.

(g) Using your answer to part (e), compute the matrices $C(\mathbf{q})$ and $B(\mathbf{q})$ in terms of the masses, dimensions, and configuration $\mathbf{q}$ of the manipulator.

The last thing that remains is to derive the gravity vector $\mathbf{G}(\mathbf{q})$.

(h) Calculate, $^0\mathbf{G}(\mathbf{q})$, the gravity vector in frame $\{0\}$, in terms of the masses, the configuration $\mathbf{q}$, and the gravity constant g (g is positive). Assume that gravity pulls things along the $-\mathbf{Z}_0$ direction. Be careful with your signs.

(i) As a final step, use your answers to parts (e), (h) and (i) to write out the equations of motion as two great big equations:

$$\begin{aligned} \tau_1 &= f_1(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}) \\ \tau_2 &= f_2(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}) \end{aligned}$$

3. When finding pieces of cubic splines, instead of solving directly for the coefficients a_i, we can think of constructing the cubic by a weighted blending of some known *basis functions.* For example, on the interval $t \in [0, 1]$, if we want a cubic segment $\theta(t)$ that moves from θ_0 to θ_1, with starting velocity $\dot{\theta}_0$ and ending velocity $\dot{\theta}_1$, we construct it by summing four known cubics:

$$\theta(t) = \theta_0 f_0(t) + \theta_1 f_1(t) + \dot{\theta}_0 f_2(t) + \dot{\theta}_1 f_3(t)$$

The functions $f_i(t)$ have special forms that ensure that the conditions at the ends of the interval are met. For example, f_0 has the properties that

$$\begin{aligned} f_0(0) &= 1 \\ f_0(1) &= 0 \\ f_0'(0) &= 0 \\ f_0'(1) &= 0 \end{aligned}$$

These properties ensure that the $\theta_0 f_0(t)$ term contributes θ_0 to the value of $\theta(0)$, but does not contribute at all to the values of $\theta(1)$, $\dot{\theta}(0)$, or $\dot{\theta}(1)$.

(a) Derive the four cubic polynomials f_0, f_1, f_2, and f_3.

(b) Modify your polynomials f_i to get a set of four cubic polynomials g_i that satisfy the same conditions over the interval $t \in [0, t_f]$ instead of $t \in [0, 1]$. That is, find four cubic polynomials g_i such that the cubic

$$\theta(t) = \theta_0 g_0(t) + \theta_f g_1(t) + \dot{\theta}_0 g_2(t) + \dot{\theta}_f g_3(t)$$

satisfies the conditions $\theta(0) = \theta_0$, $\theta(t_f) = \theta_f$, $\dot{\theta}(0) = \dot{\theta}_0$, and $\dot{\theta}(t_f) = \dot{\theta}_f$. Be careful with the derivative terms.

(c) Use your functions g_i to derive formulas for the coefficients a_i of the cubic $\theta(t)$, such that

$$\theta(t) = a_0 + a_1 t + a_3 t^2 + a_3 t^3$$

You can check the formulas that you derive against equations 5.20-5.23 in the lecture notes.