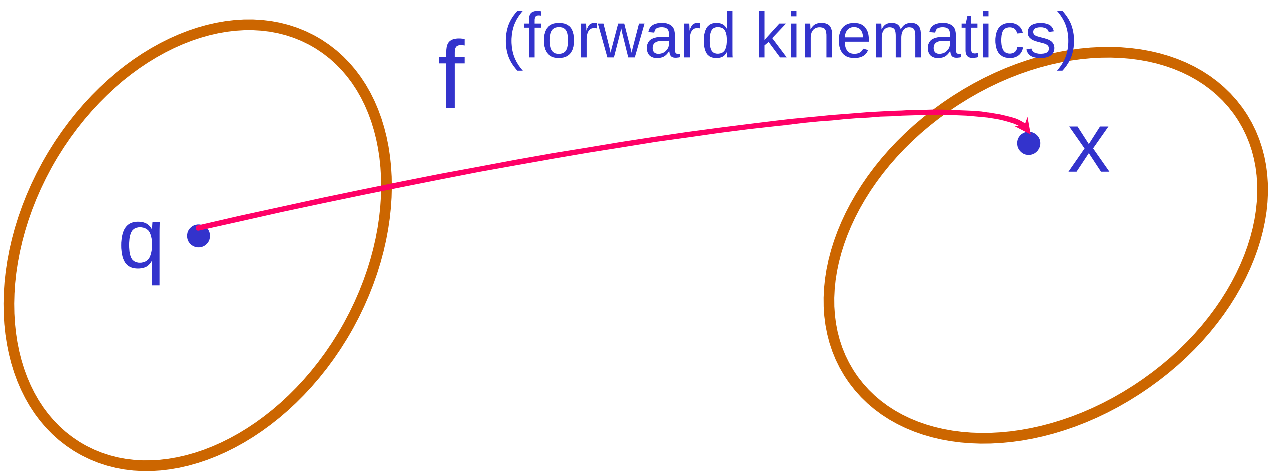
Inverse Kinematics

Direct Kinematics



Joint Space
(dimensions n)

Task Space
(dimensions m)

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{bmatrix}$$

$$\mathbf{x} = f(\mathbf{q})$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{bmatrix}$$

Joint Coordinates

Revolute Joints θ_i

Prismatic Joints d_i

$$q_i = \bar{\varepsilon}_i \theta_i + \varepsilon_i d_i$$

$$\varepsilon_i = \begin{cases} 0 & \text{revolute joint} \\ 1 & \text{prismatic joint} \end{cases}$$

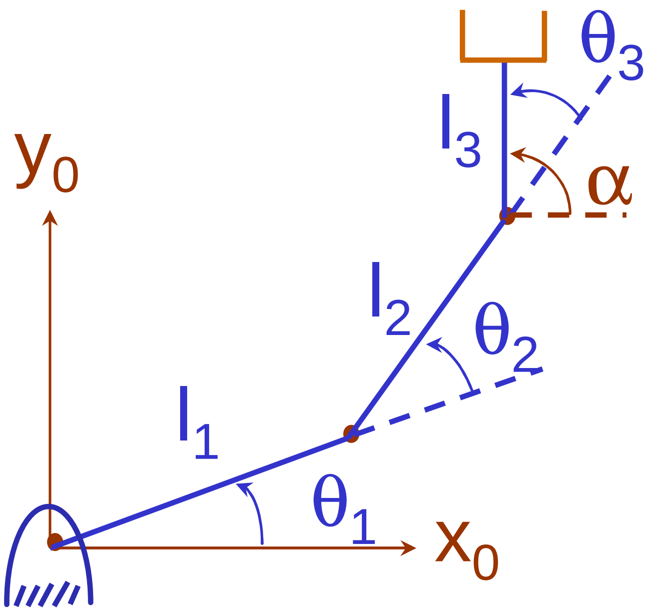
$$\bar{\varepsilon}_i \equiv 1 - \varepsilon_i$$

<u>Direct Kinematics</u>

Given $\mathbf{q} = (q_1 \quad q_2 \quad \cdots \quad q_n)^T$

${}^0_nT = {}^0_nT(\mathbf{q})$ or $\mathbf{x} = f(\mathbf{q})$ (Geometric Model)

Inverse Kinematics



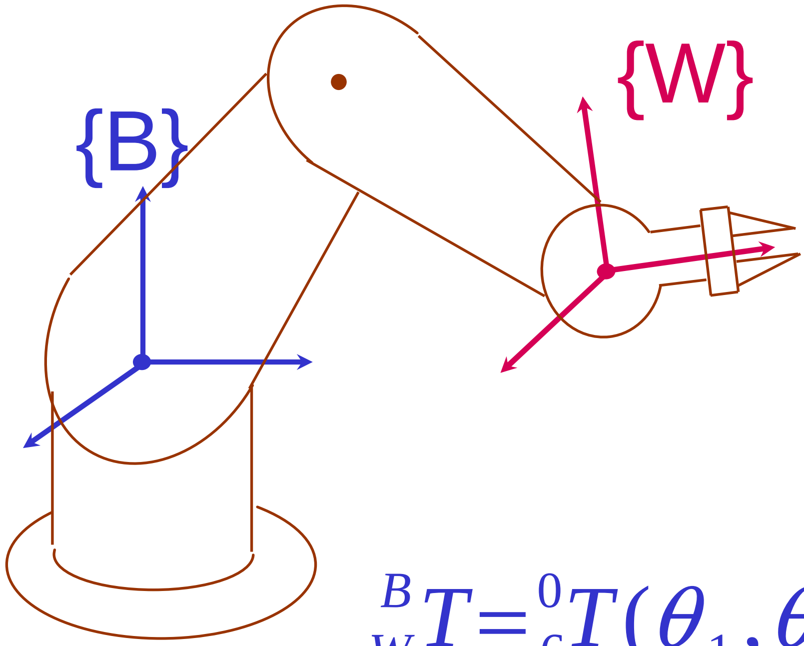
$$\mathbf{x} = \begin{bmatrix} x \\ y \\ \alpha \end{bmatrix} = f(\mathbf{q})$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ \alpha \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ \theta_1 + \theta_2 + \theta_3 \end{bmatrix}$$

$$\cos(\theta_1 + \theta_2) = c_{12}$$

Given **q** ⟶ a unique **x**

Inverse Kinematics



$$ {}^{B}_{W}T = {}^{0}_{6}T(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) $$

or $$ X = \begin{bmatrix} X_P \\ X_R \end{bmatrix} = f(\Theta) $$

Inverse Problem

Given (${}^{B}_{W}T$ or X) find Θ

Inverse Kinematics

Finding

$$\Theta = f^{-1}(X)$$

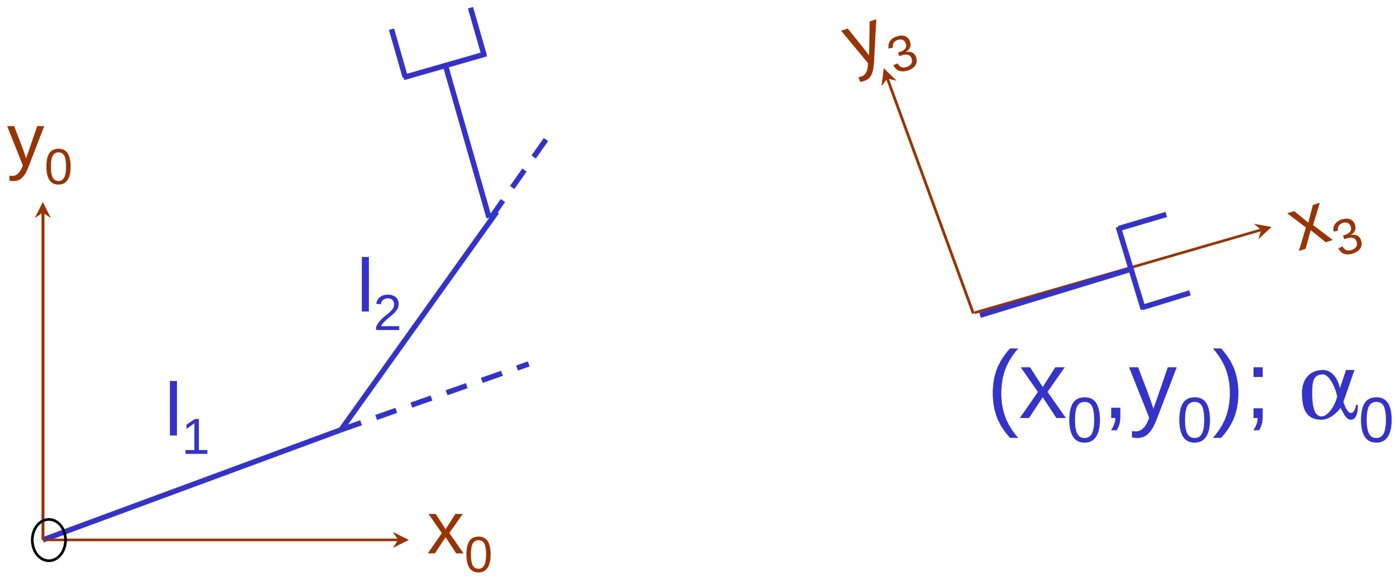
or

Solving

$${}^{0}_{6}T(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = {}^{B}_{W}T$$

$$\left(\begin{array}{c} \text{12 equations} \\ \text{6 unknowns} \end{array} \right)$$

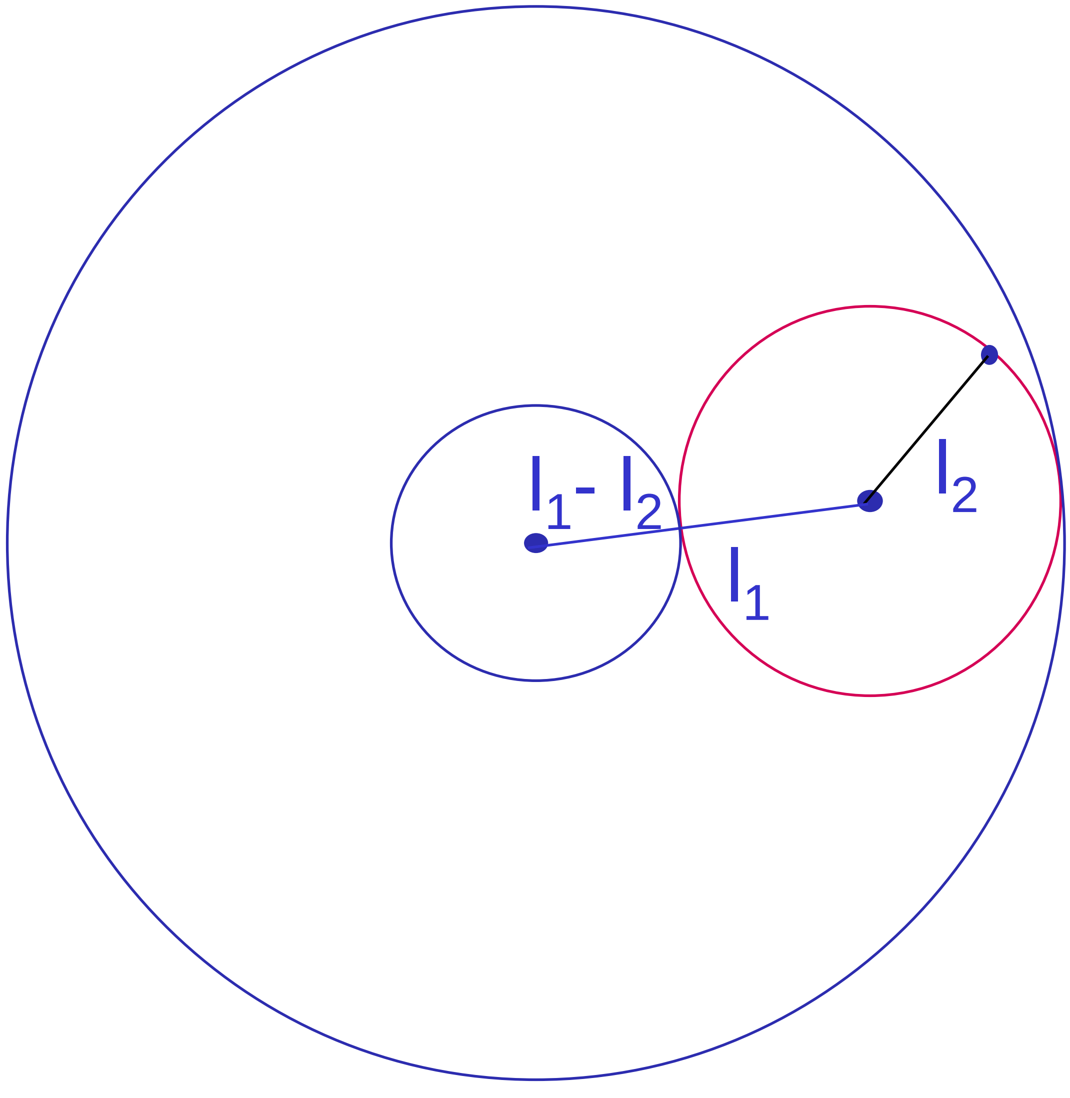
Existence of Solutions



$$ {}^0_3T = \begin{pmatrix} c_{123} & -s_{123} & 0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c\alpha_0 & -s\alpha_0 & 0 & x_0 \\ s\alpha_0 & c\alpha_0 & 0 & y_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

solution if

$$ (l_1 - l_2)^2 \le x_0^2 + y_0^2 \le (l_1 + l_2)^2 $$

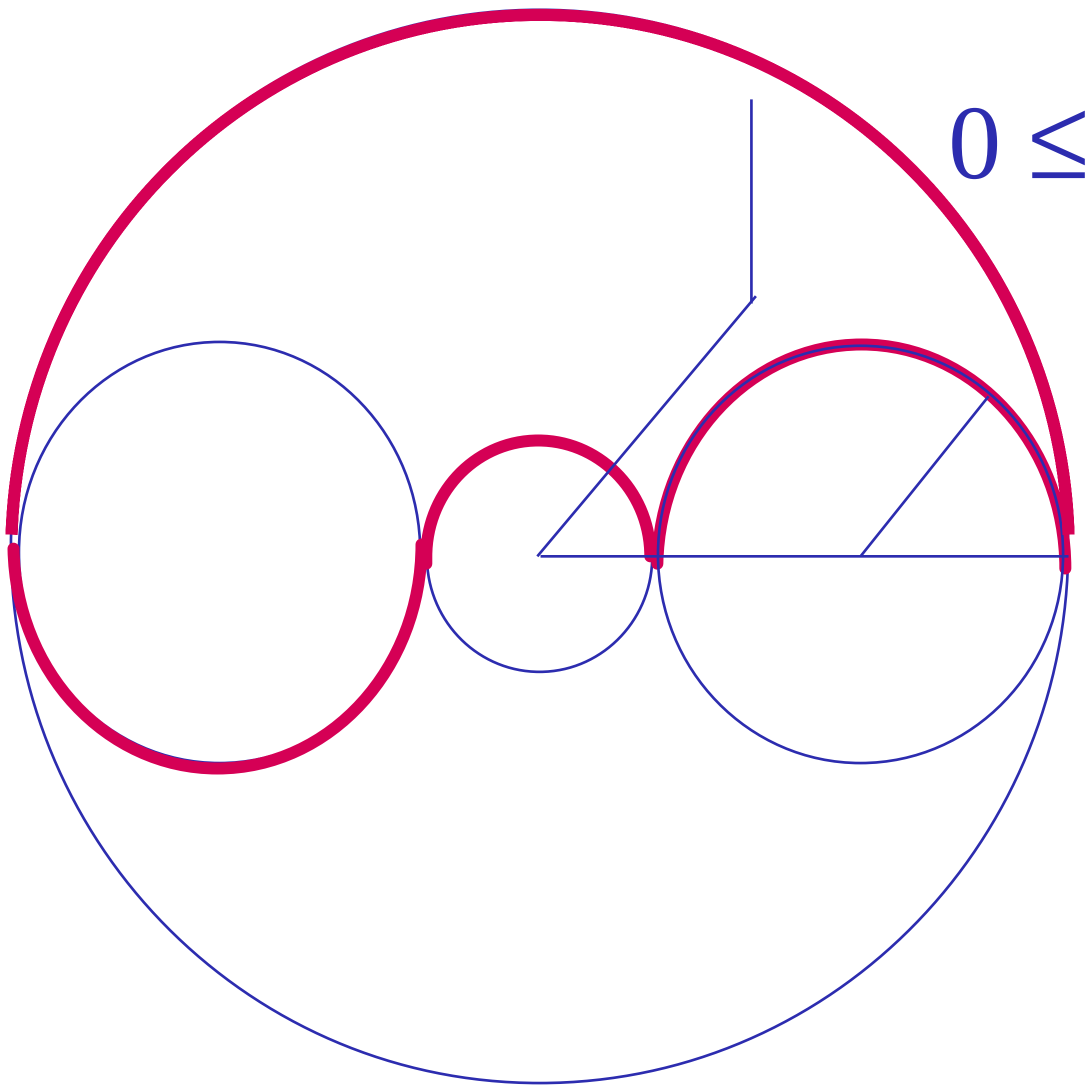


$l_1 - l_2$
l_2
l_1

Joint Limits

$$0 \leq \theta_1 \leq 180^o$$

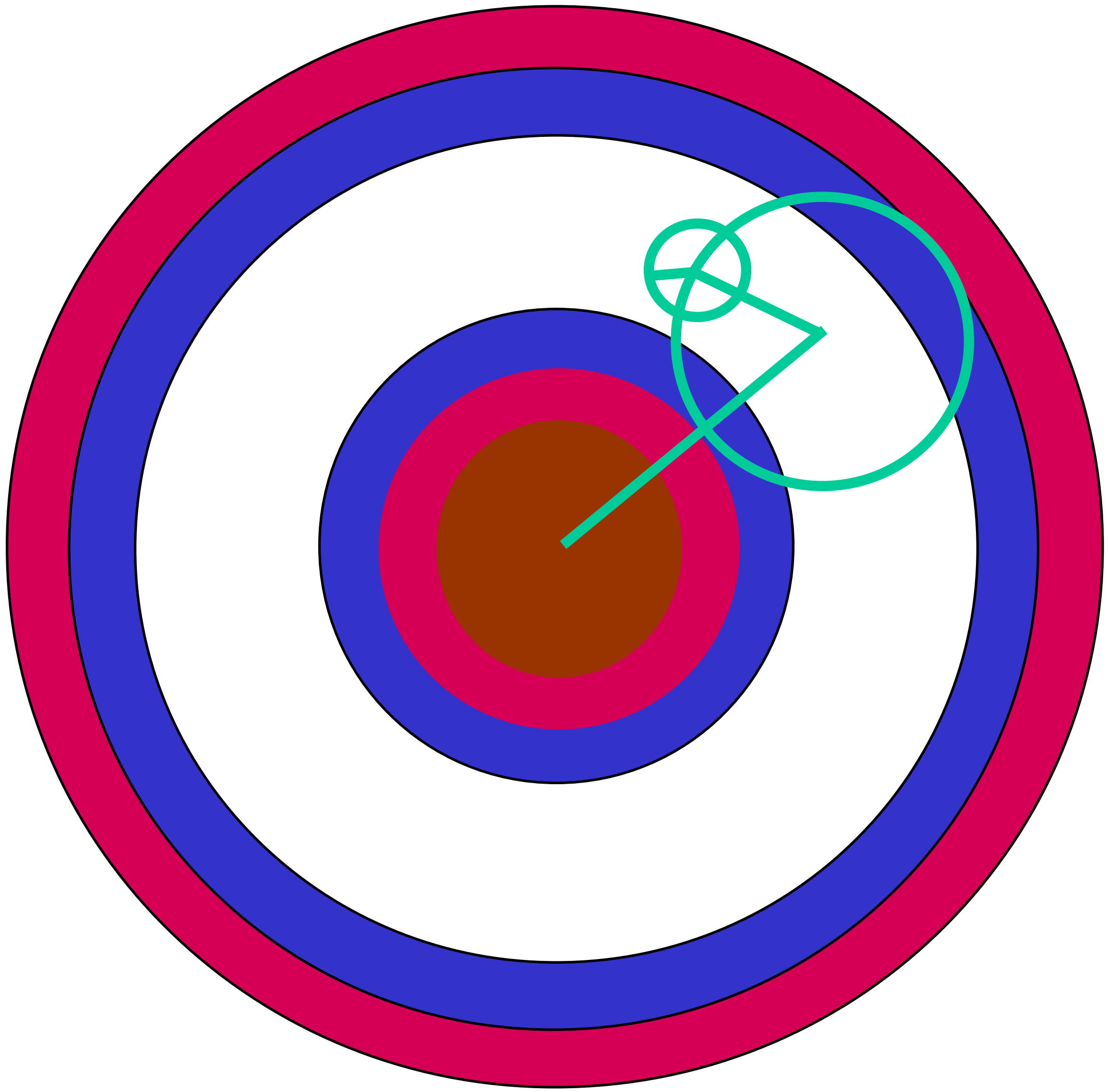
$$0 \leq \theta_2 \leq 180^o$$



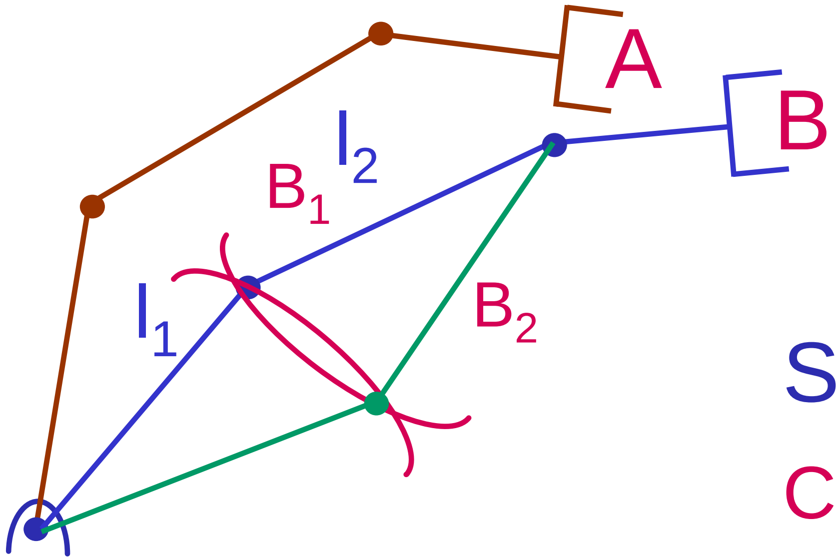
Workspace

- Reachable Workspace
- Dextrous Workspace

Dextrous Workspace



Multiplicity of Solutions



Selection of a solution

Criterion: Joint distance

$$C_1 = \left\| \Theta_{(B1)} - \Theta_{(A)} \right\|$$

$$C_2 = \left\| \Theta_{(B2)} - \Theta_{(A)} \right\|$$

Weighted Joint distance

moving smaller joints

Number of Solutions

It depends on

- Number of Joints
- Link Parameters

e.g. 6-revolute-joint manipulator

if all $a_i \neq 0$ Number solutions $\leq$ 16

if $a_1 = a_3 = a_5 = 0$ Number solutions $\leq$ 4

- Range of Motion

General Mechanism with 6 d.o.f.

Number of solutions $\leq$ 16

Main Results

General 6R open-chain 16 solutions
General 5RP open-chain 16 solutions
General 4R2P open-chain 8 solutions
General 3R3P open-chain 2 solutions

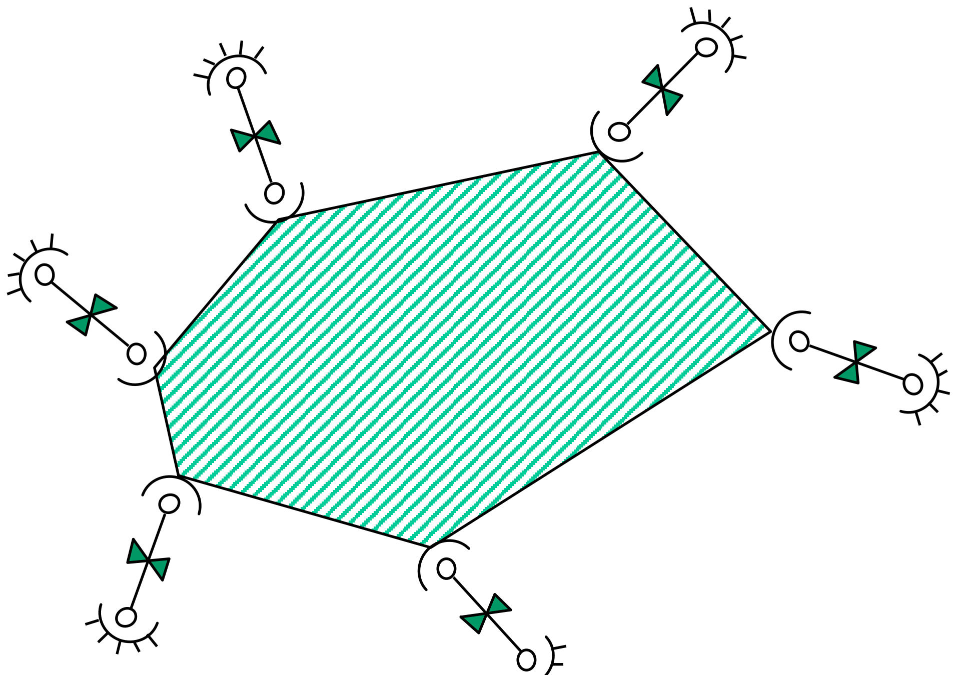
Special conditions in the structure [such as intersecting or parallel axes] cause the general number of solutions to reduce. There exist open-chain manipulators with 16, 14, 12, 10, 8, 6, 4, 2 solutions.

For a given set of 6 lengths of the legs

General in-parallel structure has

40 configurations

By specializing structure the number of configurations can be reduced



PUMA 560

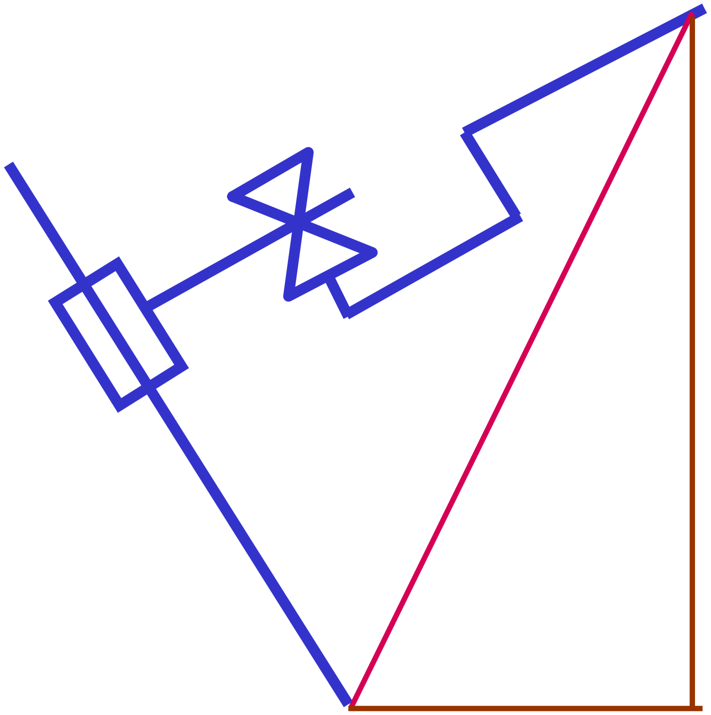
8 Solutions

$$\theta_4 \longrightarrow \theta_4 + 180^o$$

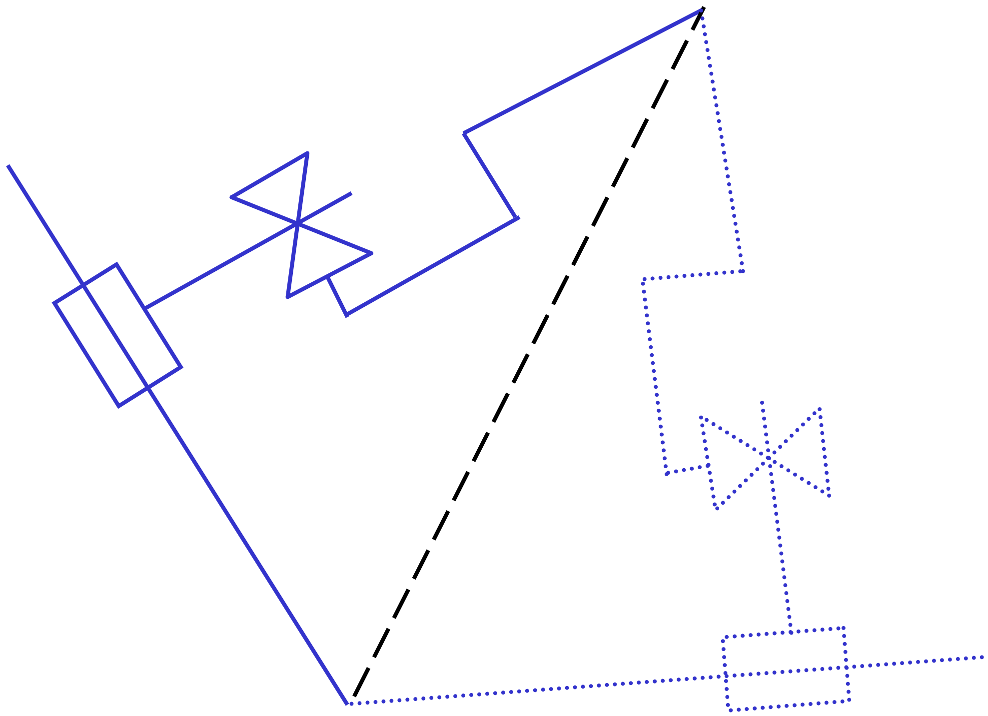
$$\theta_5 \longrightarrow -\theta_5$$

$$\theta_6 \longrightarrow \theta_6 + 180^o$$

Stanford Scheinman Arm



Stanford Scheinman Arm



Solvability

A manipulator is solvable if ALL the sets of solutions can be determined.

6 d.o.f. open-chain mechanisms are "now" solvable.

(the general solution is a numerical one)

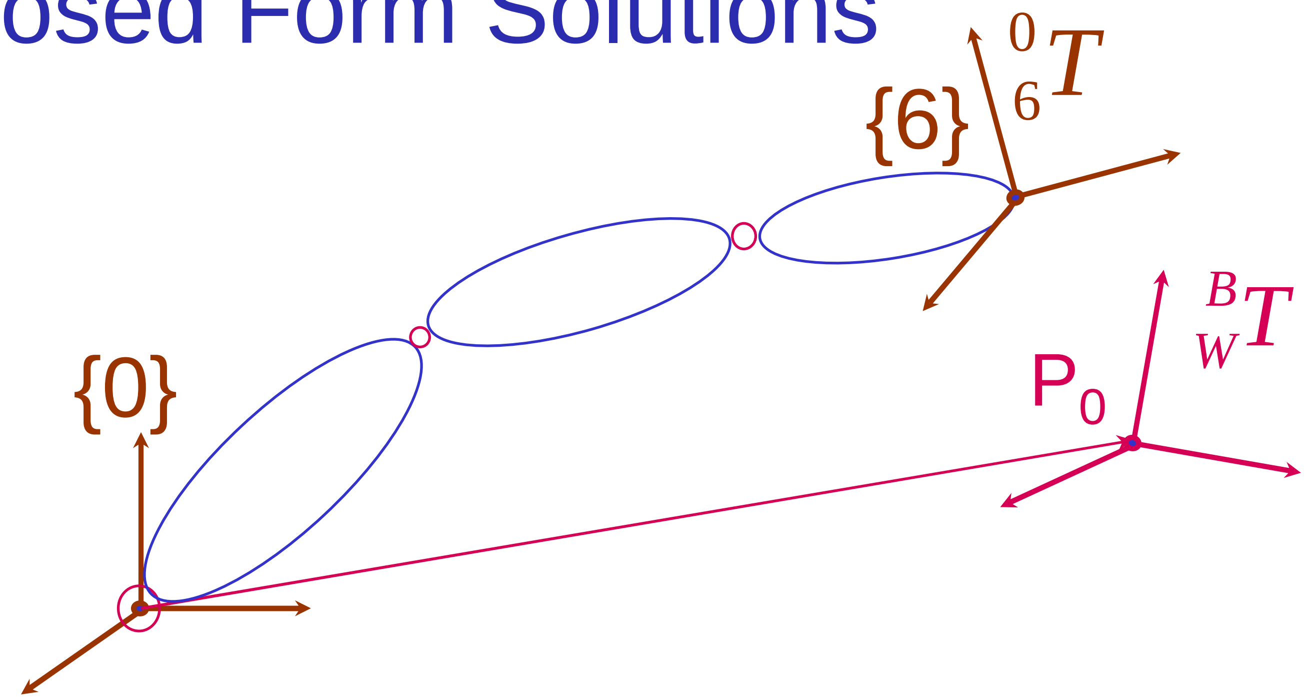
Closed Form Solutions

Analytical Solutions - Exist for a large class of mechanisms.

Sufficient Condition

3 intersecting neighboring axes
(most industrial robots)

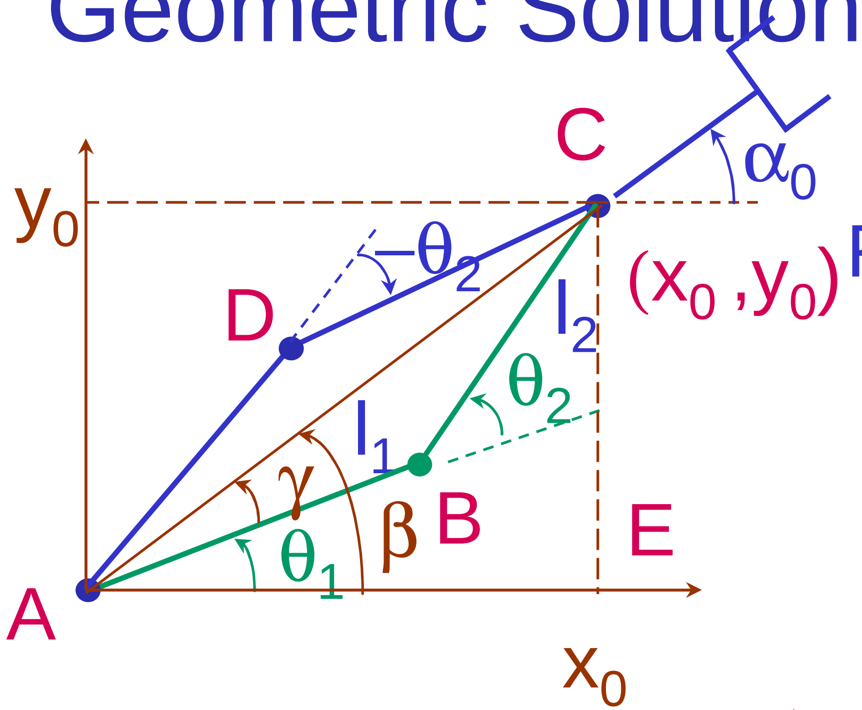
Closed Form Solutions



$$ {}^0_6T(\theta_1, \theta_2, \ldots, \theta_6) = {}^B_WT $$

Solutions:
- Algebraic
- Geometric

Geometric Solutions



First θ_2:

$$l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta_2 = x_0^2 + y_0^2$$

$$\cos\theta_2 = \frac{(x_0^2 + y_0^2) - (l_1^2 + l_2^2)}{2 l_1 l_2}$$

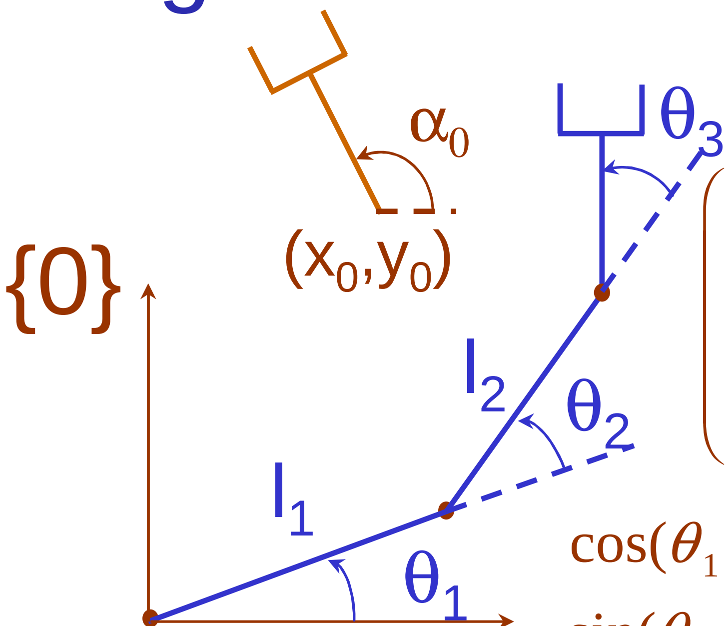
=> θ_2 and $-\theta_2$

θ_1: $l_2^2 = l_1^2 + (x_0^2 + y_0^2) - 2 l_1 \sqrt{x_0^2 + y_0^2} \cos\gamma$

$$\cos\gamma = \frac{x_0^2 + y_0^2 + l_1^2 - l_2^2}{2 l_1 \sqrt{x_0^2 + y_0^2}} \quad \text{and} \quad \tan\beta = \frac{y_0}{x_0}$$

θ_3: $\theta_1 = \beta \pm \gamma$ $\qquad \theta_3 = \alpha_0 - (\theta_1 + \theta_2)$

Algebraic Solutions



$$ {}^{0}_{3}T \equiv {}^{B}_{W}T $$

$$\begin{pmatrix} c_{123} & -s_{123} & 0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c\alpha_0 & -s\alpha_0 & 0 & x_0 \\ s\alpha_0 & c\alpha_0 & 0 & y_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left.\begin{array}{l} \cos(\theta_1 + \theta_2 + \theta_3) = \cos\alpha_0 \\ \sin(\theta_1 + \theta_2 + \theta_3) = \sin\alpha_0 \end{array}\right\} \Rightarrow \theta_1 + \theta_2 + \theta_3 = \alpha_0$$

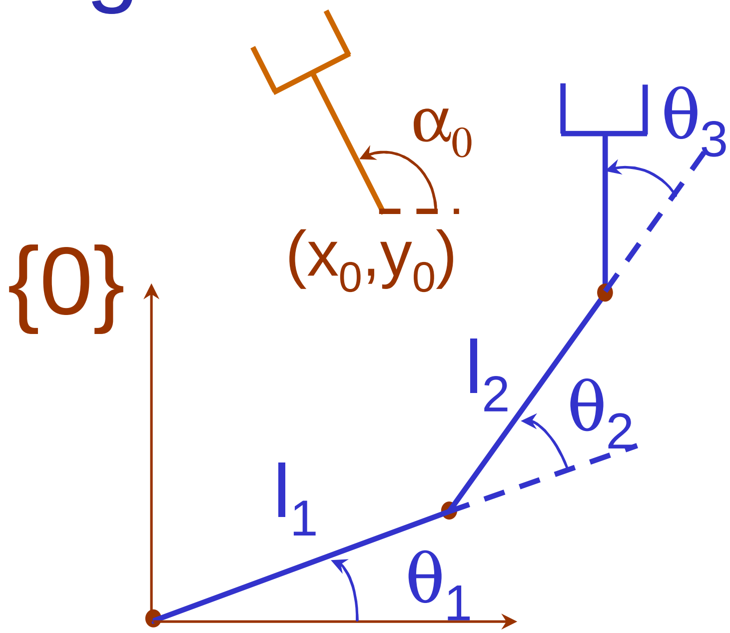
For θ_1 and θ_2:
$$l_1c_1 + l_2c_{12} = x_0$$
$$l_1s_1 + l_2s_{12} = y_0$$

Solution if (x_0, y_0) is in the workspace

$$-1 \le \cos\theta_2 = \frac{(x_0^2 + y_0^2) - (l_1^2 + l_2^2)}{2l_1l_2} \le 1$$

$$\Rightarrow \theta_2 = A\tan 2(\pm\sqrt{1-\cos^2\theta_2}, \cos\theta_2)$$

Algebraic Solutions



$$l_1 c_1 + l_2 c_{12} = x_0$$
$$l_1 s_1 + l_2 s_{12} = y_0$$

For θ_1:

$$\left.\begin{aligned}(l_1 + l_2 c_2)c_1 - (l_2 s_2)s_1 &= x_0 \\ (l_1 + l_2 c_2)s_1 + (l_2 s_2)c_1 &= y_0\end{aligned}\right\}$$

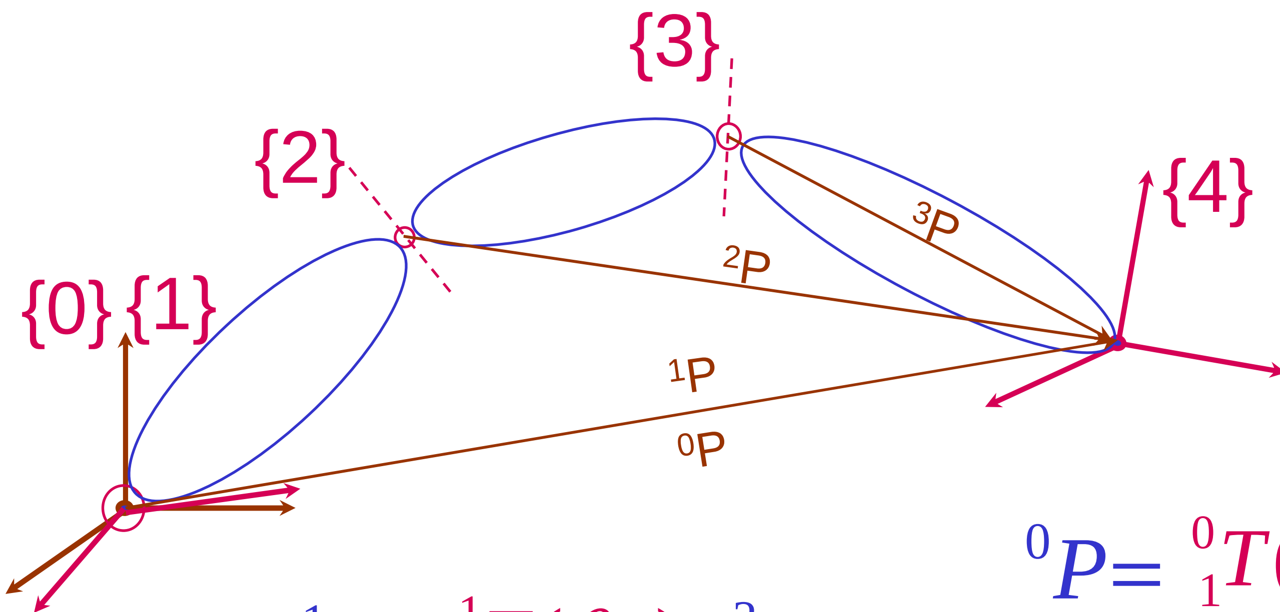
$$\Rightarrow \quad \left.\begin{aligned}k_1 c_1 - k_2 s_1 &= x_0 \\ k_1 s_1 + k_2 c_1 &= y_0\end{aligned}\right\}$$

$$(k_1, k_2) \xrightarrow[\tan\gamma = k_2 / k_1]{r = \sqrt{k_1^2 + k_2^2}} \begin{pmatrix} k_1 = r.\cos\gamma \\ k_2 = r.\sin\gamma \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} x_0 &= r.\cos(\theta_1 + \gamma) \\ y_0 &= r.\sin(\theta_1 + \gamma) \end{aligned}$$

$$\Rightarrow \quad \theta_1 = A\tan 2(y_0, x_0) - A\tan 2(k_2, k_1)$$

Pieper's Solution



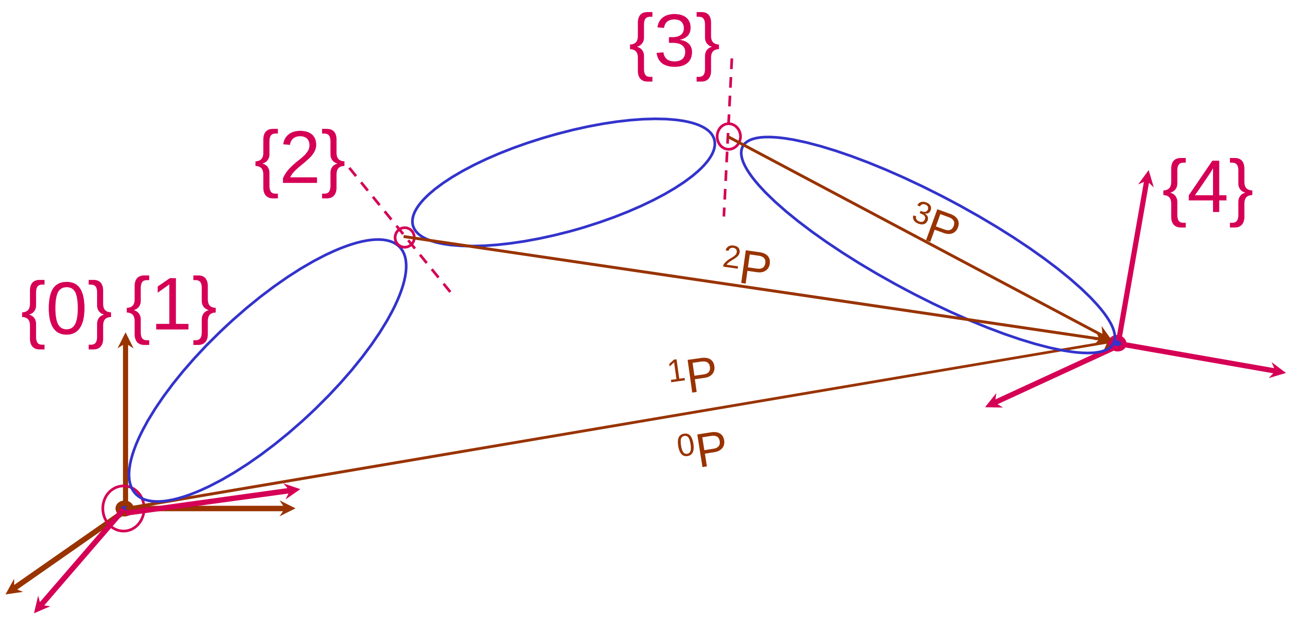
$$ {}^{0}P = {}^{0}_{1}T(\theta_1)\,.\,{}^{1}P $$

$$ {}^{1}P = {}^{1}_{2}T(\theta_2)\,.\,{}^{2}P $$

$$ {}^{1}P = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ 1 \end{bmatrix}; g_i = g_i(c_2, s_2, f_i) $$

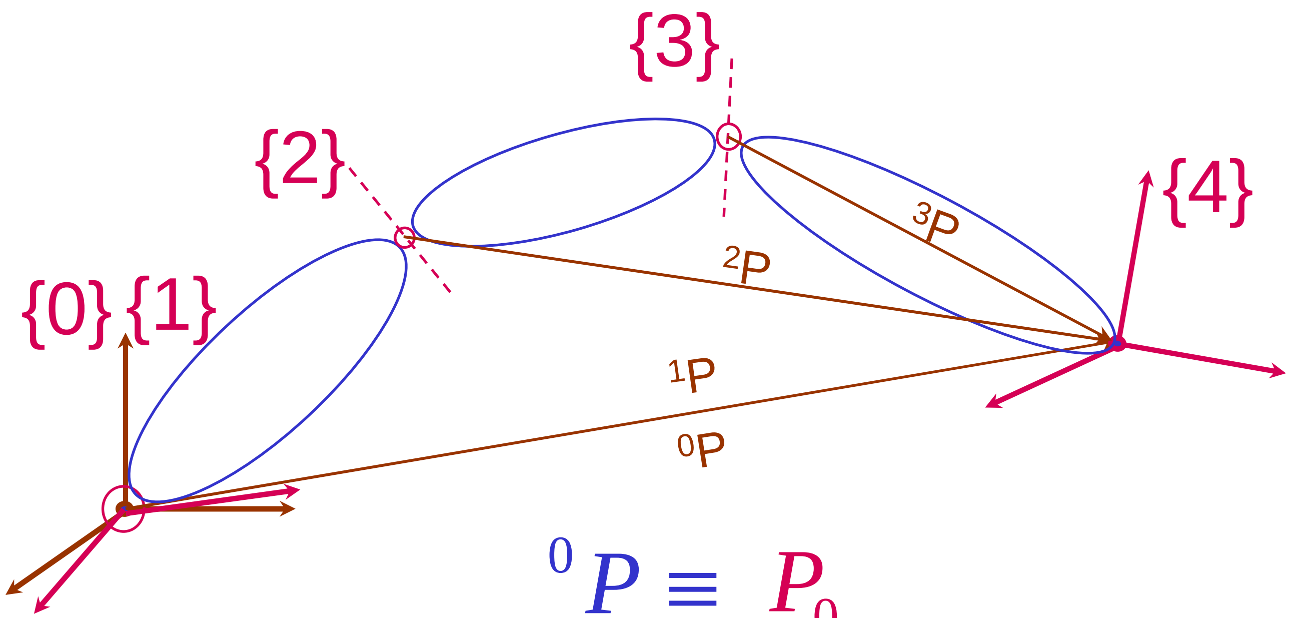
$$ {}^{0}P = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix} $$

Pieper's Solution



$$^3P = \begin{bmatrix} a_3 \\ -s\alpha_3.d_4 \\ c\alpha_3.d_4 \\ 1 \end{bmatrix} \qquad {}^2P = {}^2_3T(\theta_3).{}^3P \qquad {}^2P = \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}$$

Pieper's Solution



$$^0P \equiv P_0$$

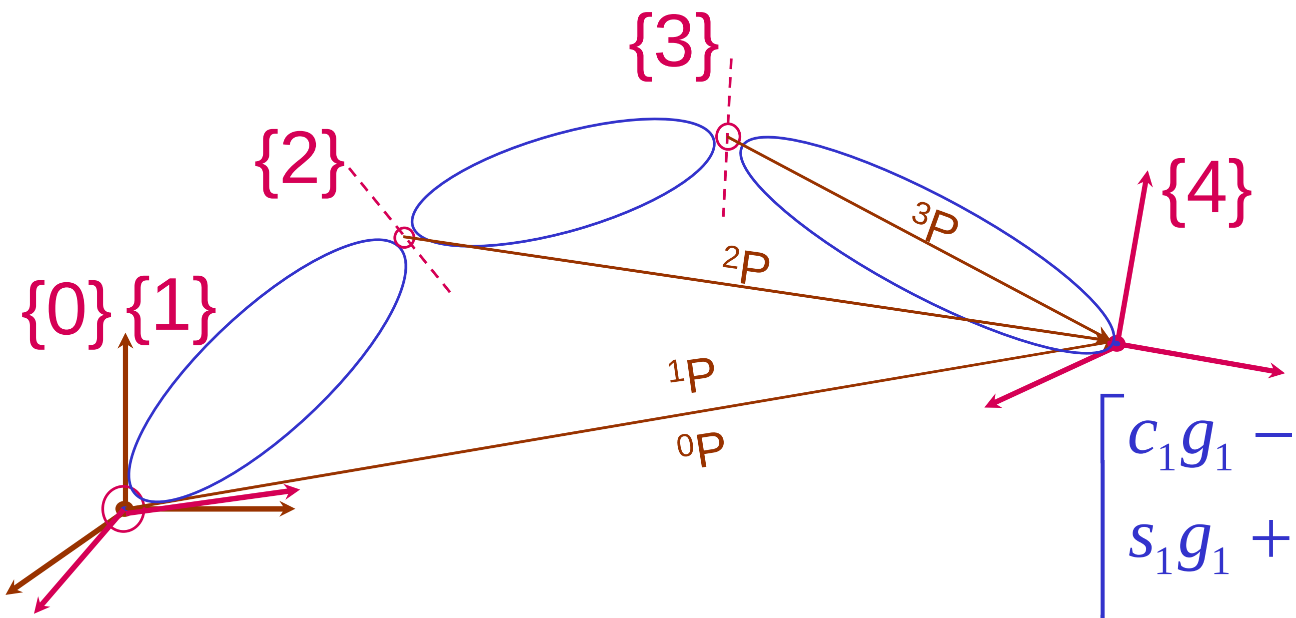
$$\begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

For θ_1:
$$\left.\begin{aligned} c_1 g_1 - s_1 g_2 &= x_0 \\ s_1 g_1 + c_1 g_2 &= y_0 \end{aligned}\right\} \theta_1$$

if g_1 and g_2 are known

$$\theta_1 = \text{Atan2}(y_0, x_0) - \text{Atan2}(g_2, g_1)$$

Pieper's Solution



$$\begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

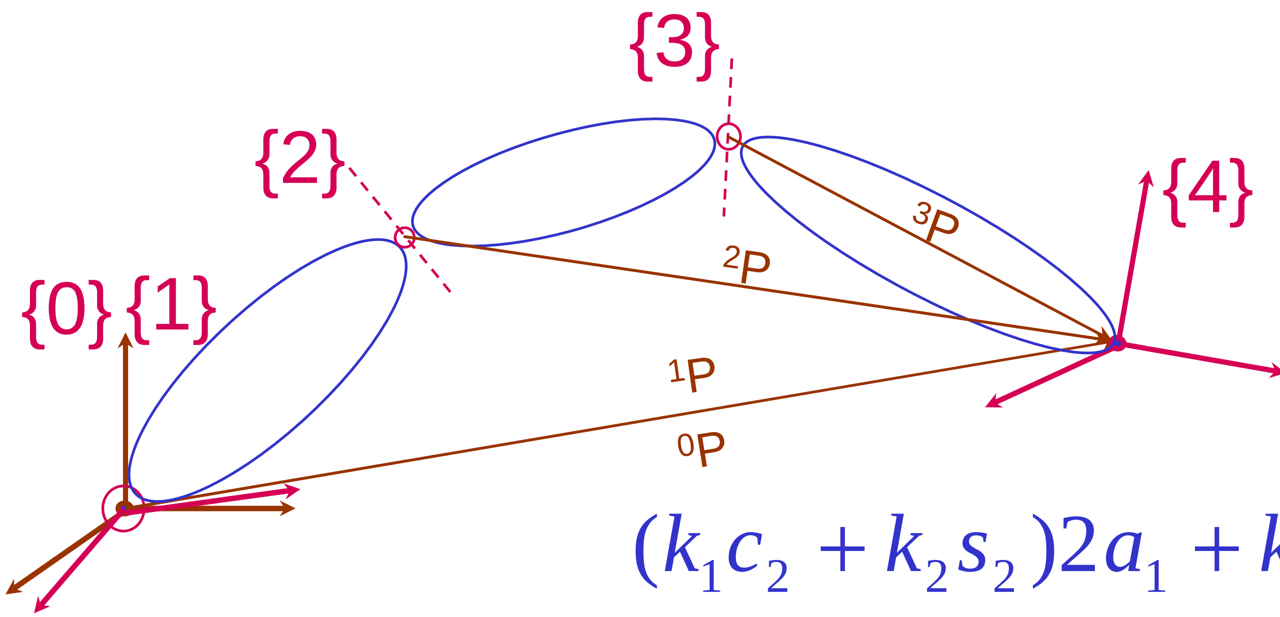
For θ_2: $$\begin{cases} g_1^2 + g_2^2 + g_3^2 = x_0^2 + y_0^2 + z_0^2 = r_0^2 \\ g_3 = z_0 \end{cases}$$

$$(k_1 c_2 + k_2 s_2) 2 a_1 + k_3 = r_0^2$$

$$g_i = g_i(c_2, s_2, f_1, f_2, f_3) \quad (k_1 s_2 - k_2 c_2) s\alpha_1 + k_4 = z_0$$

$$k_i = k_i(f_1, f_2, f_3) \rightarrow \theta_2$$ if k_i are known

Pieper's Solution



$$(k_1 c_2 + k_2 s_2)2a_1 + k_3 = r_0^2$$

$$(k_1 s_2 - k_2 c_2)s\alpha_1 + k_4 = z_0$$

For θ_3:

$$(r_0^2 - k_3)^2 . s^2\alpha_1 + (Z_0 - k_4)^2 .4. a_1^2 = 4. a_1^2 . s^2\alpha_1 (k_1^2 + k_2^2)$$

$$k_i = k_i(f_i(c_3, s_3))$$

Transcendental Equations

Reduction to Polynomial

$$u = \tan\frac{\theta}{2} \Rightarrow \begin{cases} \cos\theta = \dfrac{1-u^2}{1+u^2} \\ \sin\theta = \dfrac{2u}{1+u^2} \end{cases}$$

For θ_3: $$k_i = k_i(u, u^2)$$

$$A.u^4 + B.u^3 + C.u^2 + D.u + E = 0$$

$$\text{with } u = \tan\frac{\theta_3}{2}$$

For θ_4, θ_5, and θ_6

$$ {}^{0}_{6}R(\Theta) \equiv R_0 $$

$$ {}^{0}_{6}R(\Theta) = {}^{0}_{1}R(\theta_1) \cdot {}^{1}_{2}R(\theta_2) \cdot {}^{2}_{3}R(\theta_3) \cdot \underline{{}^{3}_{4}R(\theta_4)} \cdot {}^{4}_{5}R(\theta_5) \cdot {}^{5}_{6}R(\theta_6) $$

$$ {}^{3}_{4}R(\theta_4) = {}^{3}_{4}R\big|_{\theta_4=0} \cdot R_Z(\theta_4) $$

$$ \underbrace{[\,{}^{0}_{4}R\big|_{\theta_4=0}(\theta_1,\theta_2,\theta_3)]}_{\text{is known}} \cdot \underbrace{[R_Z(\theta_4) \cdot {}^{4}_{6}R(\theta_5,\theta_6)]}_{R} = R_0 $$

$$ R(\theta_4,\theta_5,\theta_6) = R_0' $$

Euler Angle Solution

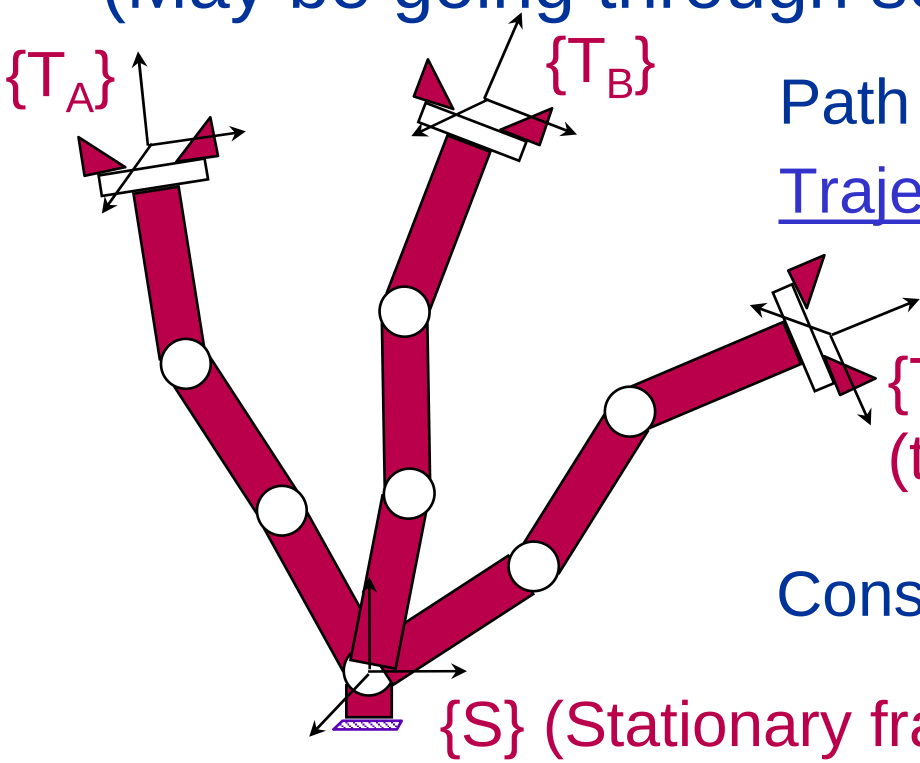
Trajectory Generation

Trajectory Generation

Basic Problem:

Move the manipulator arm from some initial position $\{T_A\}$ to some desired final position $\{T_C\}$.

(May be going through some via point $\{T_B\}$)



Path points : Initial, final and via points

Trajectory : Time history of position, velocity and acceleration for each DOF

Constraints: Spatial, time, smoothness

Solution Spaces :

Joint space

- Easy to go through via points
  (Solve inverse kinematics at all path points and plan)
- No problems with singularities
- Less calculations
- Can not follow straight line

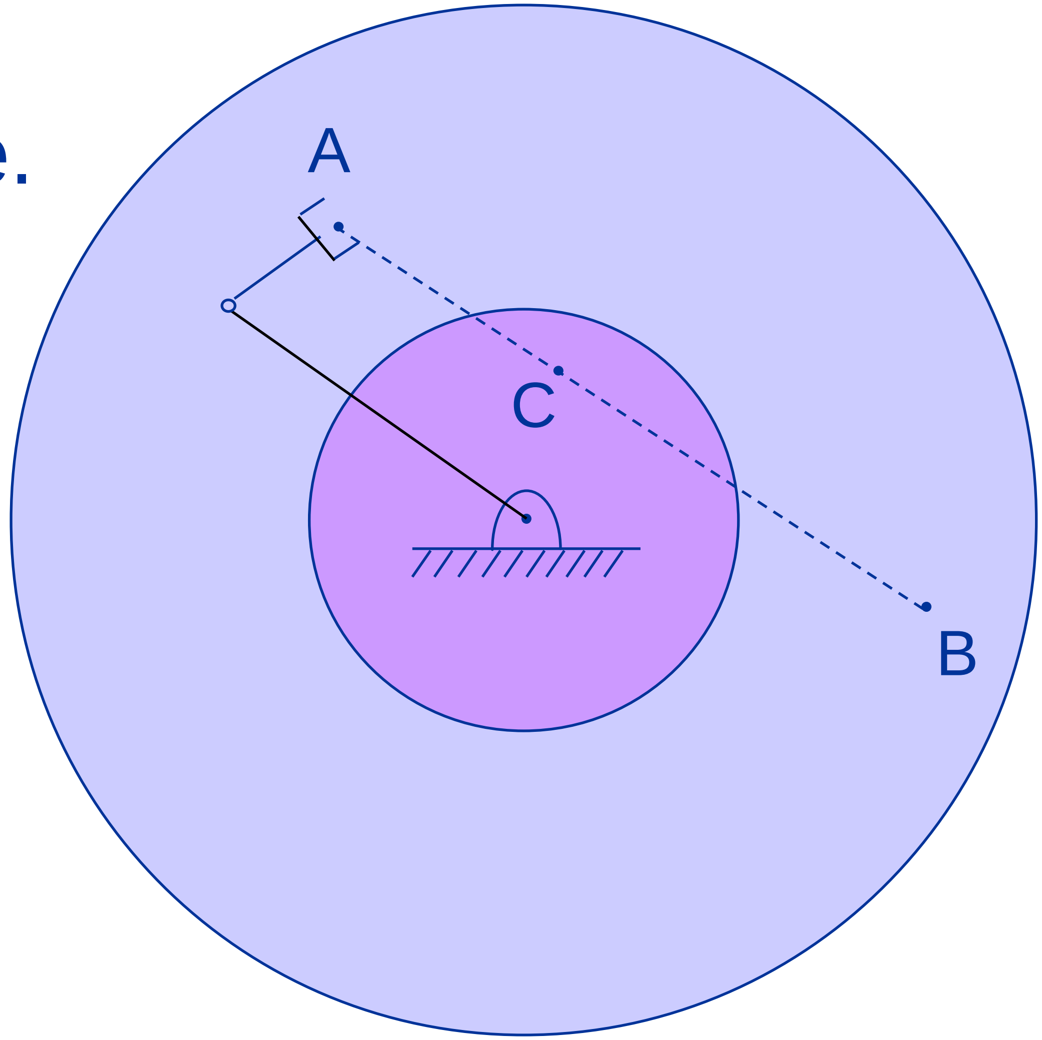
Cartesian space

- We can track a shape
  (for orientation : equivalent axes, Euler angles,…
- More expensive at run time
  (after the path is calculated need joint angles
  in a lot of points)
- Discontinuity problems

Cartesian planning difficulties :

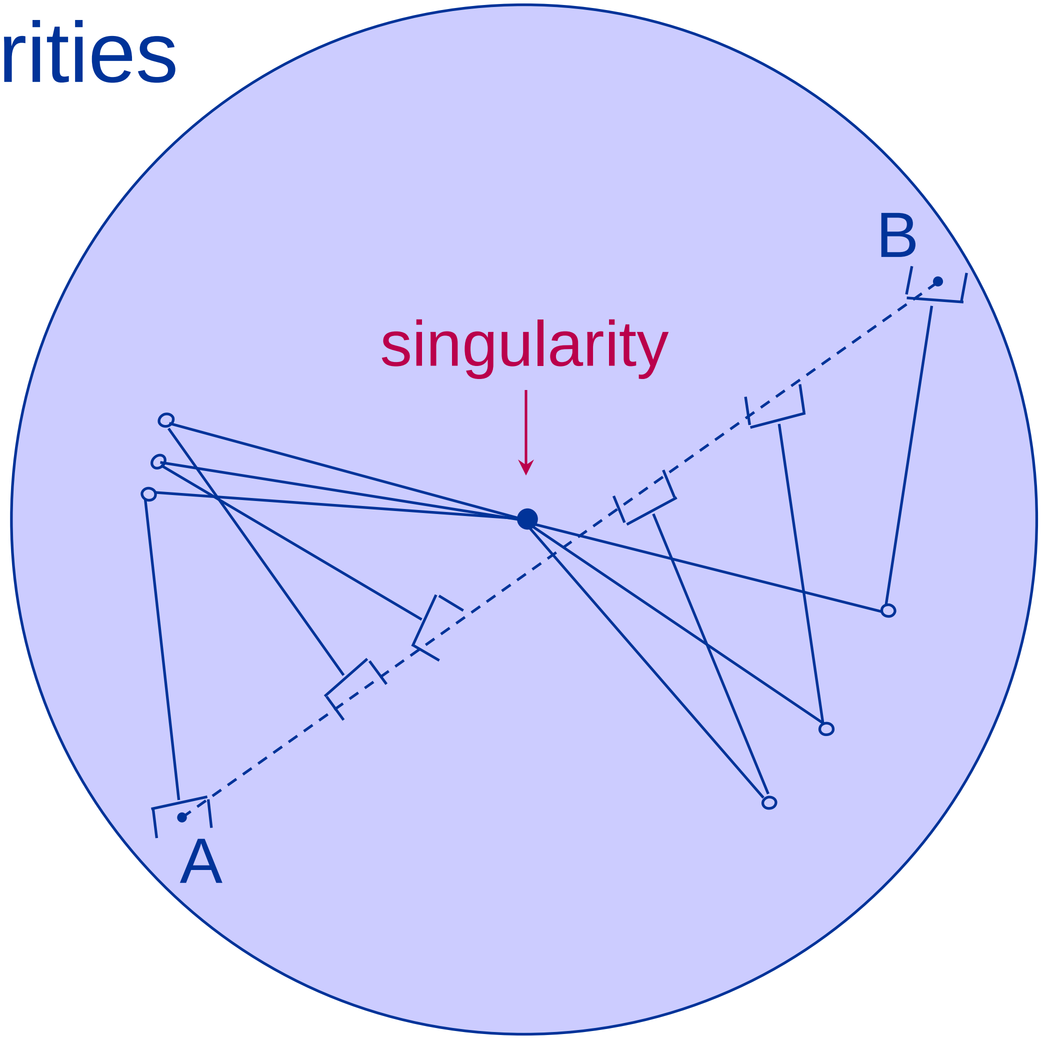
Initial and Goal Points are reachable.

Intermediate points (C) unreachable.



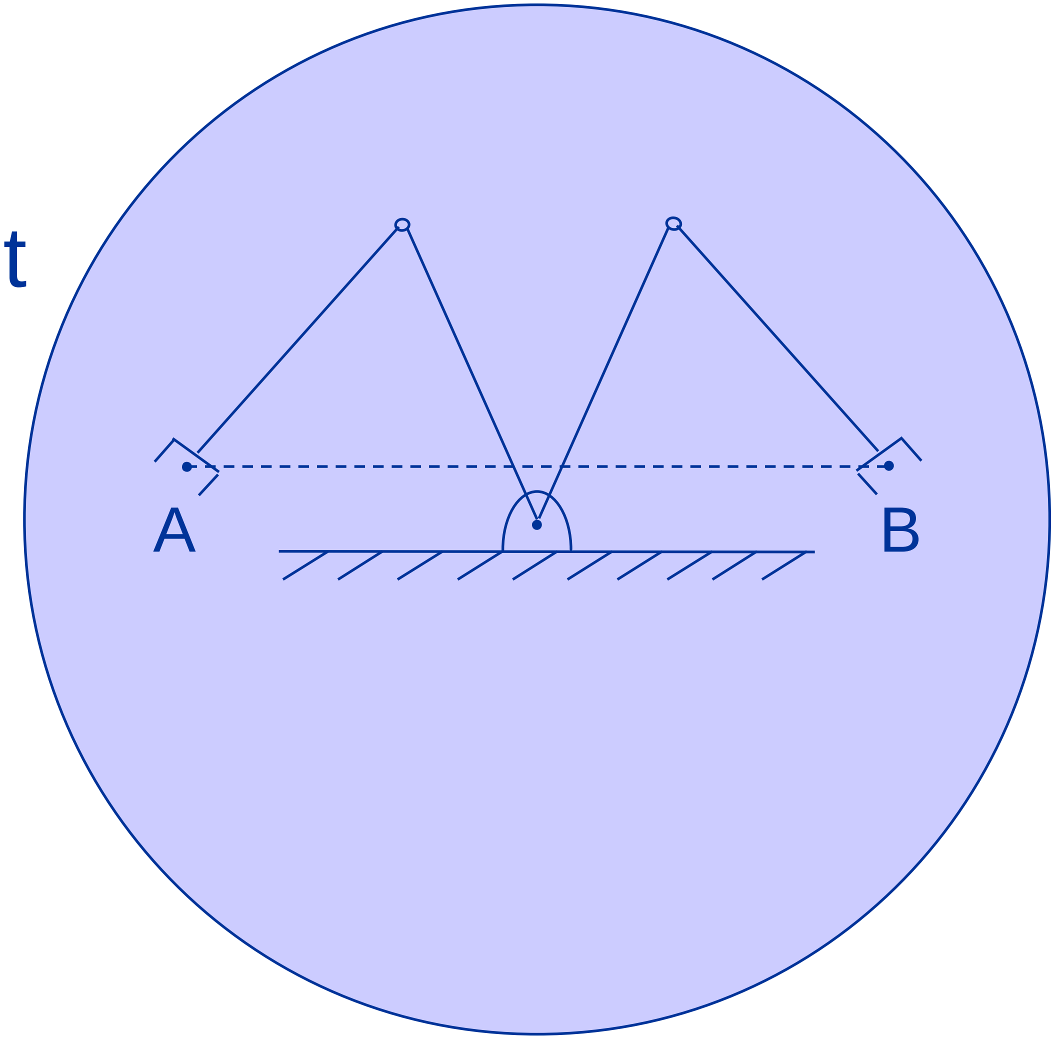
Cartesian planning difficulties :

Approaching singularities
some joint velocities
go to ∞
causing deviation
from the path



Cartesian planning difficulties :

Start point (A) and goal point (B) are reachable in different joint space solutions (The middle points are reachable from below.)



Actual planning in <u>any</u> space:

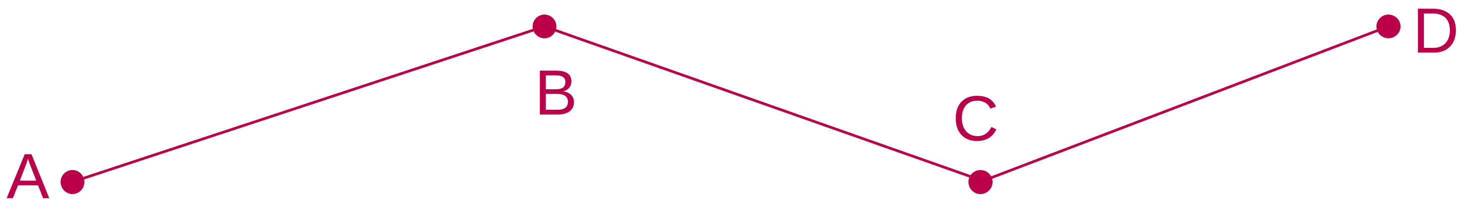
Assume one <u>generic</u> variable <u>u</u>

(can be x, y, z, orientation - α, β, γ)

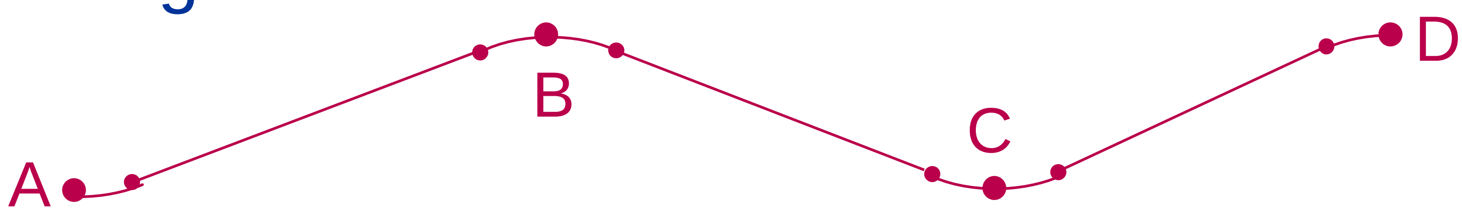
joint variables direction cosines

<u>Candidate curves</u> :

straight line (discontinuous velocity at path points)



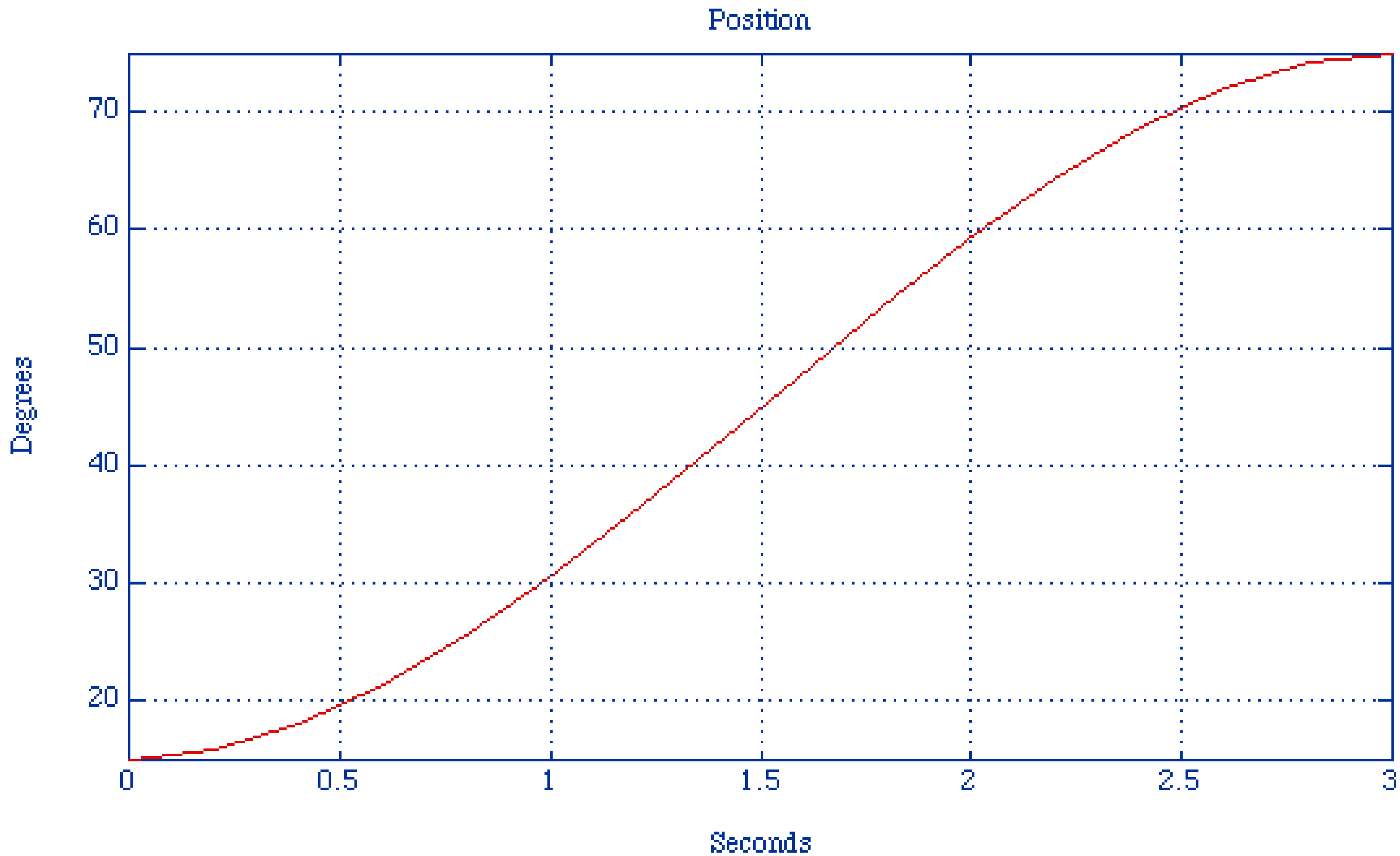
straight line with blends



cubic polynomials (splines)

higher order polynomials (quintic,…) or other curves

Single Cubic Polynomial

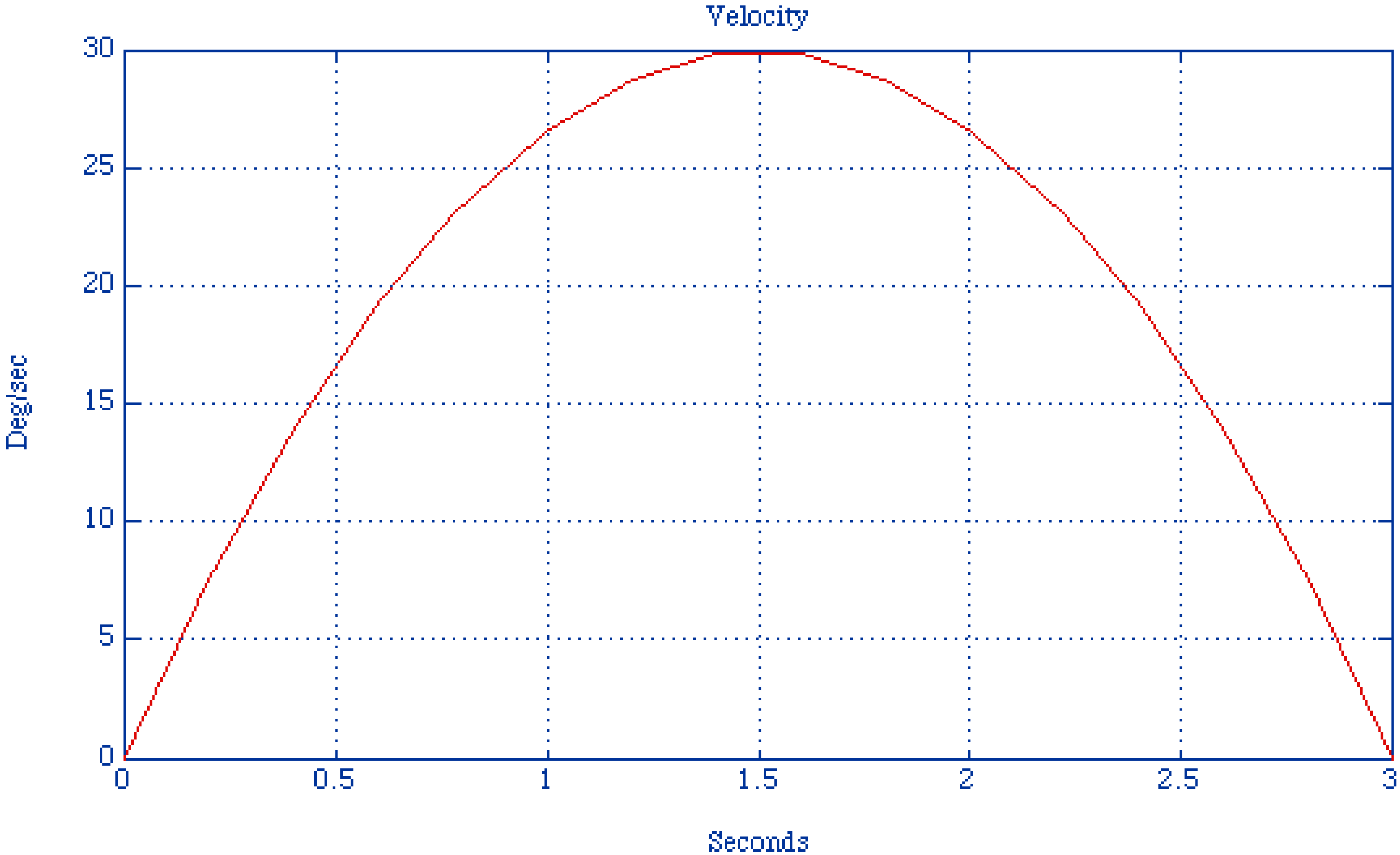


$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Initial Conditions:

$$\theta(0) = \theta_0 \ ; \quad \theta(t_f) = \theta_f$$

Single Cubic Polynomial



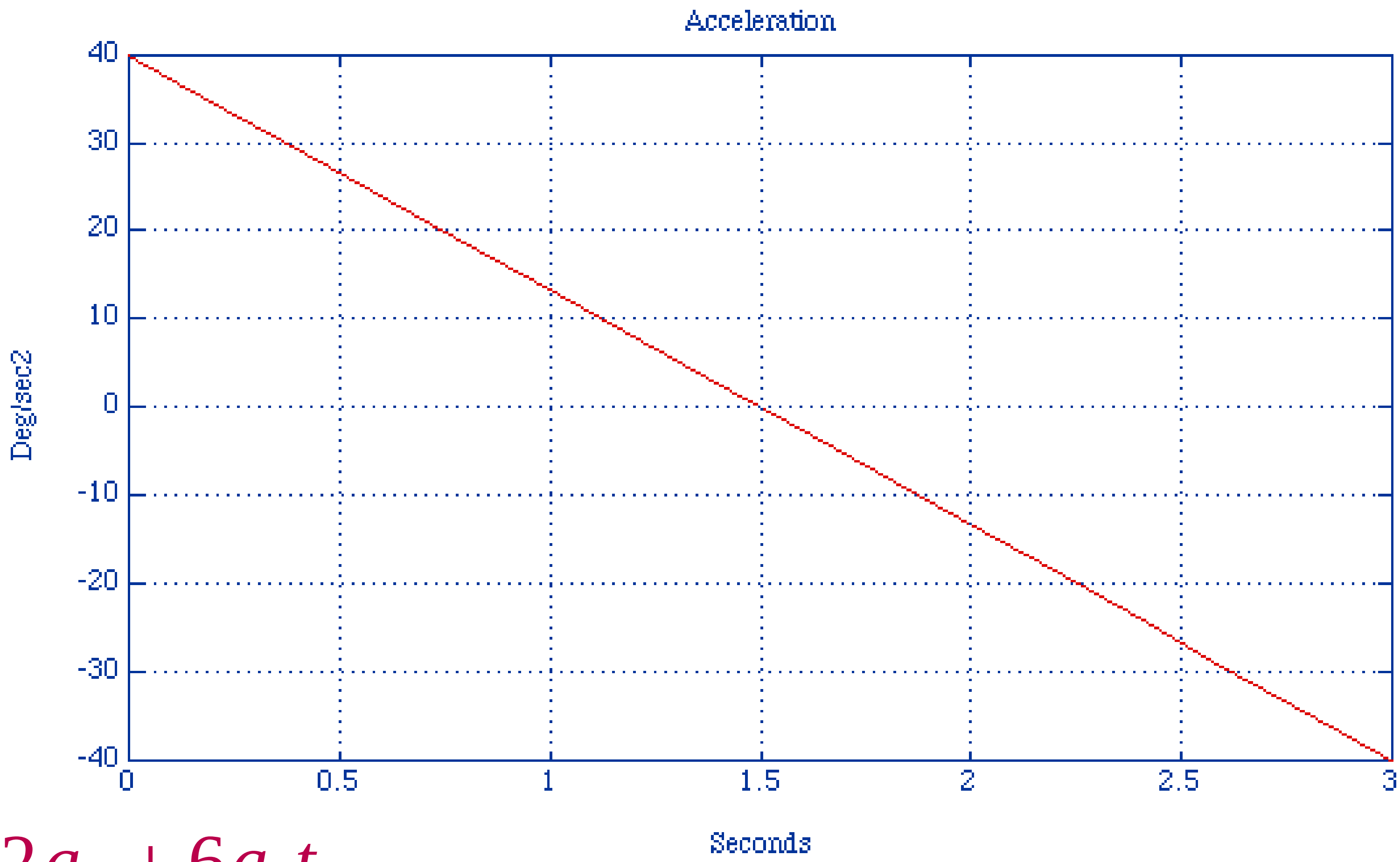
$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

Initial Conditions:

$$\dot{\theta}(0) = 0 \;;\quad \dot{\theta}(t_f) = 0$$

Starts and ends at rest

Single Cubic Polynomial



$$\ddot{\theta}(t) = 2a_2 + 6a_3 t$$

$$\dddot{\theta}(t) = 6a_3 \text{ (constant)}$$

Solution : $\theta(t) = \theta_0 + \frac{3}{t_f^2}\left(\theta_f - \theta_0\right)t^2 + \left(-\frac{2}{t_f^3}\right)\left(\theta_f - \theta_0\right)t^3$

Cubic Polynomials with via points

- If we come to rest at each point
  use formula from previous slide
- For continuous motion (no stops)
  need velocities at intermediate points:

$$\dot{\theta}(0) = \dot{\theta}_0$$

$$\dot{\theta}(t_f) = \dot{\theta}_f$$

Initial Conditions

Solution : $a_0 = \theta_0$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0) - \frac{2}{t_f}\dot{\theta}_0 - \frac{1}{t_f}\dot{\theta}_f$$

$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{1}{t_f^2}(\dot{\theta}_f + \dot{\theta}_0)$$

How to find $\dot{\theta}_0, \dot{\theta}_f$,…(velocities at via points)

- if we know Cartesian linear and angular velocities

$$\rightarrow \text{ use } J^{-1}: \quad \boxed{\dot{\theta} = J^{-1}\begin{pmatrix} \mathbf{v} \\ \omega \end{pmatrix}}$$

- the system chooses reasonable velocities using
  <u>heuristics</u> (average of 2 sides etc.)

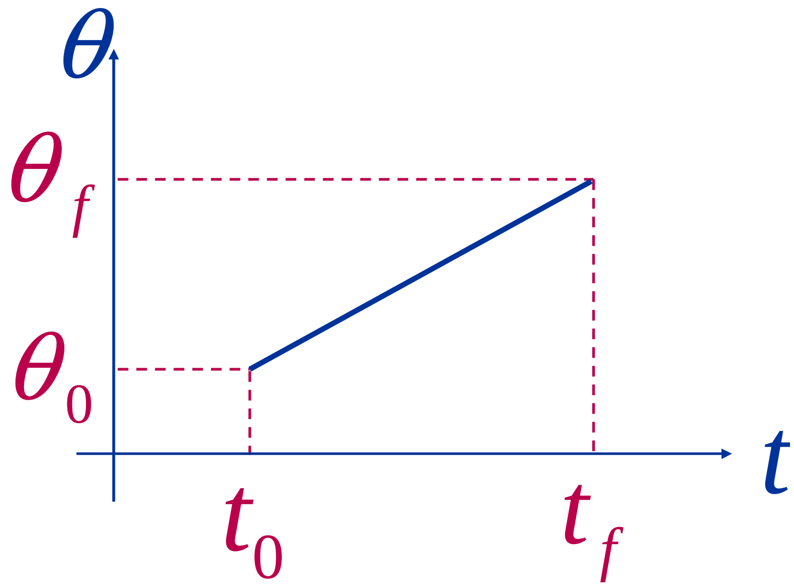
- the system chooses them for continuous

  velocity $\dot{\theta}_1(t_f) = \dot{\theta}_2(0)$ and

  acceleration $\ddot{\theta}_1(t_f) = \ddot{\theta}_2(0)$

Linear interpolation:

Straight line



$$\theta(t) = a_0 + a_1 t$$

2 conditions : $\theta(t_0) = \theta_0$

$$\theta(t_f) = \theta_f$$

Discontinuous velocity - can not be controlled

Linear interpolation:

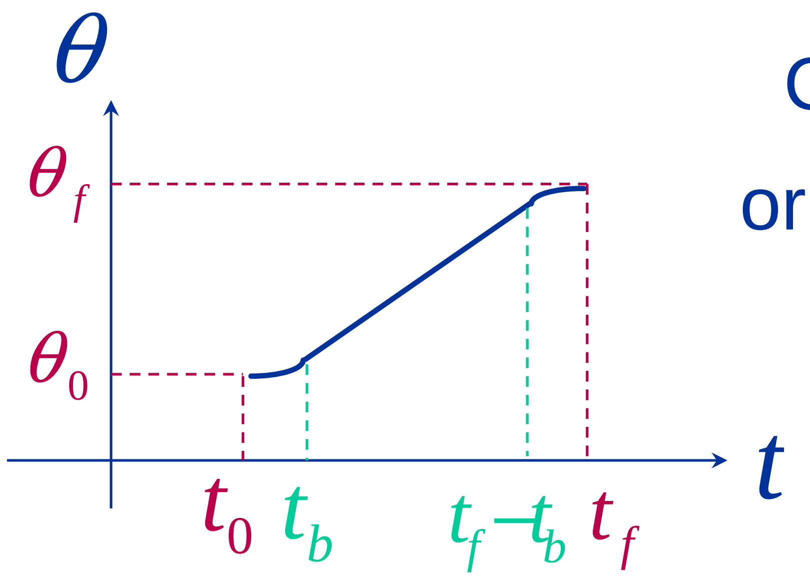
Parabolic blend

$$\theta(t) = \frac{1}{2}at^2$$ at blend regions

Linear velocity $\dot{\theta}(t) = at$

Constant acceleration $\ddot{\theta}(t) = a$

or $$\theta(t) = \frac{1}{2}\ddot{\theta}t^2$$ at blend regions



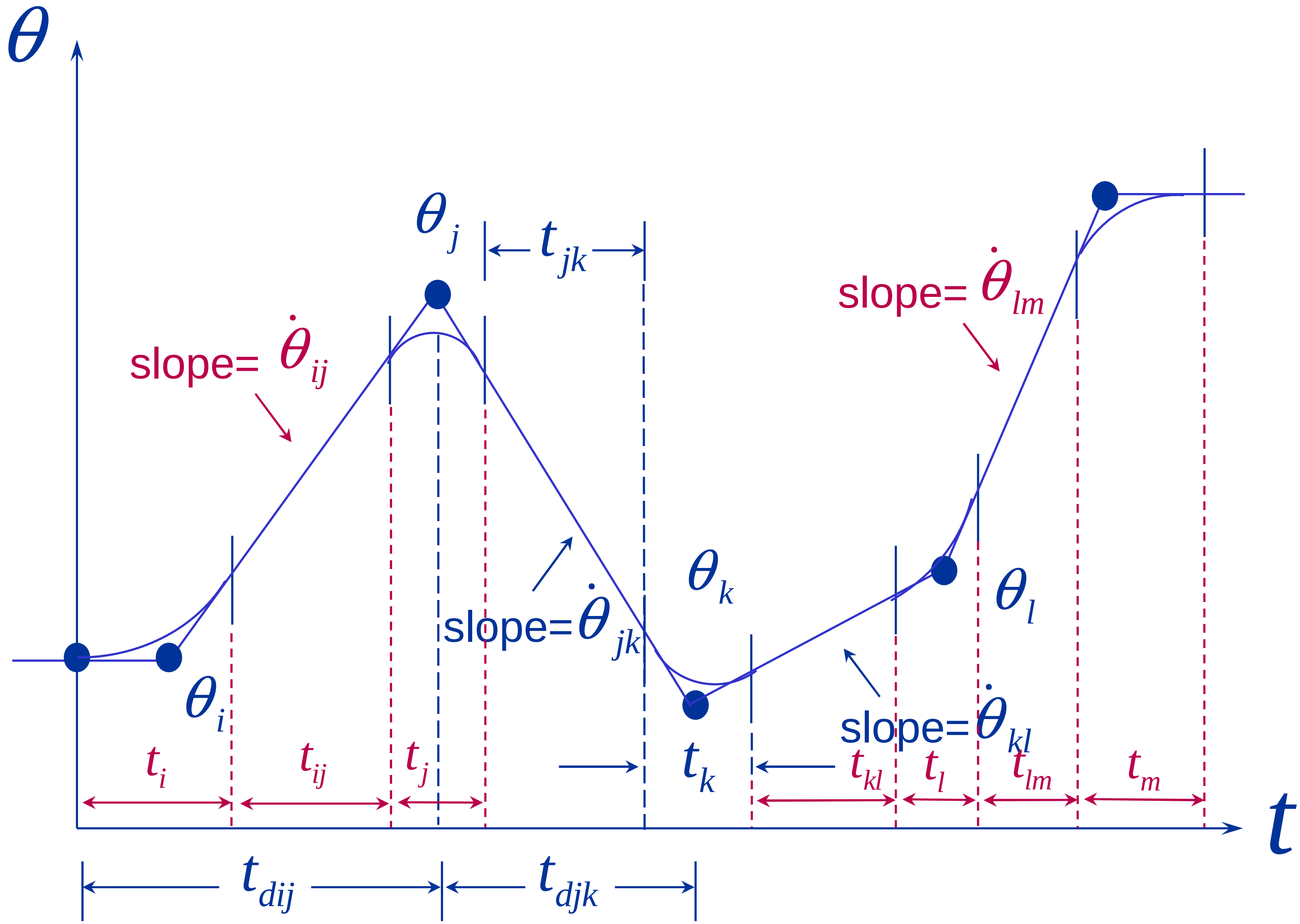
From continuous velocity:

$$t_b = \frac{t}{2} - \frac{\sqrt{\ddot{\theta}t^2 - 4\ddot{\theta}(\theta_f - \theta_0)}}{2\ddot{\theta}}$$

where $t = t_f - t_0$

desired duration of motion

Linear Interpolation
with blends for several segments



Given:

- positions u_i, u_j, u_k, u_l, u_m
- desired time durations $t_{dij}, t_{djk}, t_{dkl}, t_{dlm}$
- the magnitudes of the accelerations: $|\ddot{u}_i|, |\ddot{u}_j|, |\ddot{u}_k|, |\ddot{u}_l|$

Compute:

- blends times t_i, t_j, t_k, t_l, t_m
- straight segment times $t_{ij}, t_{jk}, t_{kl}, t_{lm}$
- slopes (velocities) $\dot{u}_{ij}, \dot{u}_{jk}, \dot{u}_{kl}, \dot{u}_{lm}$
- signed accelerations

Formulas (7.24), (7.26) and (7.28)

System usually calculates or uses default values for accelerations. The system can also calculate desired time durations based on default velocities.

First segment

$$\ddot{u}_1 = sign(u_2 - u_1)|\ddot{u}_1|$$

$$t_1 = t_{d12} - \sqrt{t_{d12}^2 - \frac{2(u_2 - u_1)}{\ddot{u}_1}}$$

$$\dot{u}_{12} = \frac{u_2 - u_1}{t_{d12} - \frac{1}{2}t_1}$$

$$t_{12} = t_{d12} - t_1 - \frac{1}{2}t_2$$

Inside segments

$$\dot{u}_{jk} = \frac{u_k - u_j}{t_{djk}}$$

$$\ddot{u}_k = sign(\dot{u}_{kl} - \dot{u}_{jk})\left|\ddot{u}_k\right|$$

$$t_k = \frac{\dot{u}_{kl} - \dot{u}_{jk}}{\ddot{u}_k}$$

$$t_{jk} = t_{djk} - \frac{1}{2}t_j - \frac{1}{2}t_k$$

Last segment

$$\ddot{u}_n = sign(u_{n-1} - u_n)\left|\ddot{u}_n\right|$$

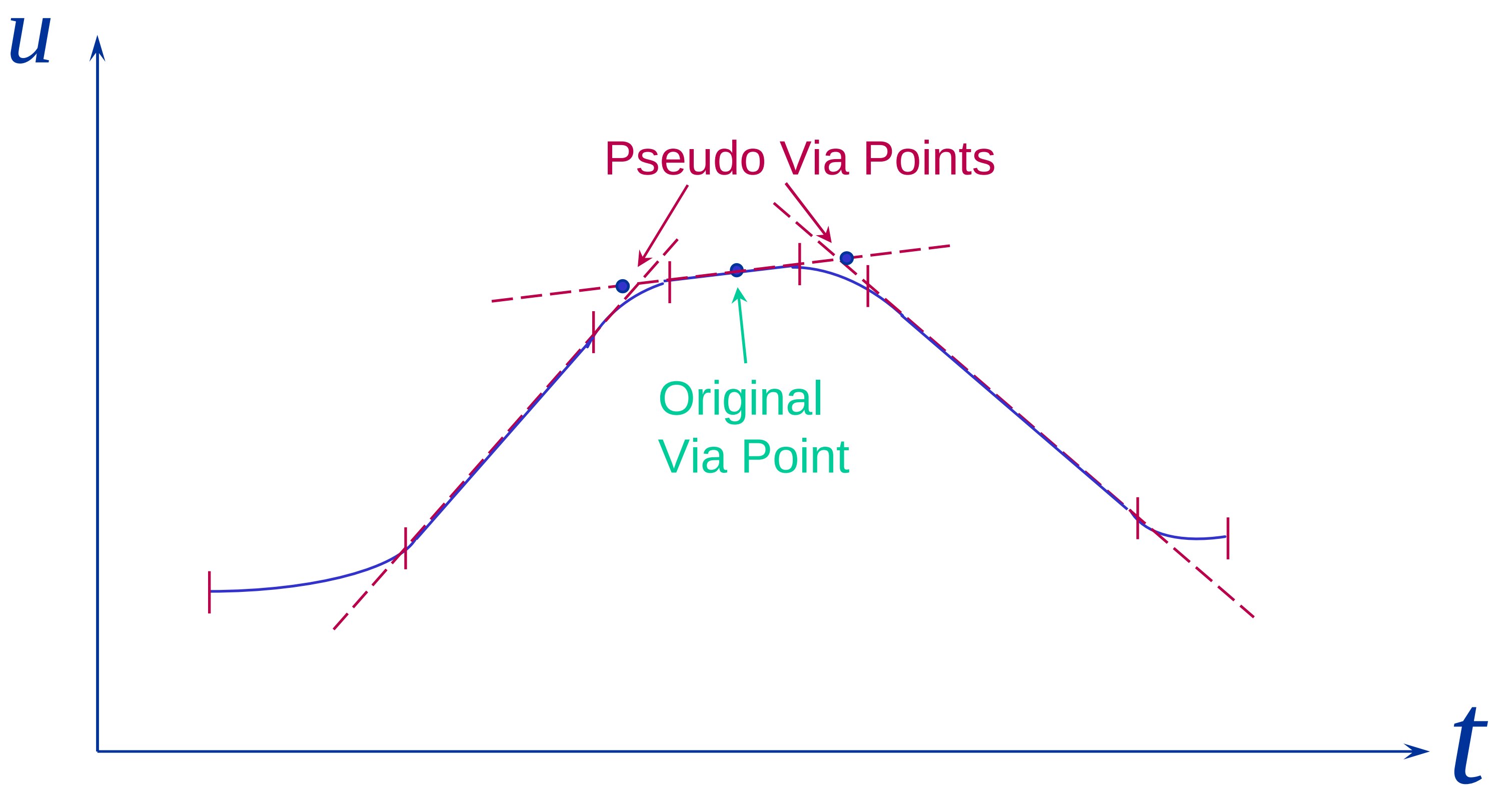
$$t_n = t_{d(n-1)n} - \sqrt{t_{d(n-1)n}^2 - \frac{2(u_n - u_{n-1})}{\ddot{u}_n}}$$

$$\dot{u}_{(n-1)n} = \frac{u_n - u_{n-1}}{t_{d(n-1)n} - \frac{1}{2}t_n}$$

$$t_{(n-1)n} = t_{d(n-1)n} - t_n - \frac{1}{2}t_{n-1}$$

To go through the actual via points:

- Introduce "Pseudo Via Points"



- Use sufficiently high acceleration
- If we want to stop there, simply repeat the via point

Higher Order Polynomials

- For example if given:

$$\text{6 conditions}\begin{cases}\text{position} & (\text{initial } u_0,\ \text{final } u_f) \\ \text{velocity} & (\dot{u}_0, \dot{u}_f) \\ \text{acceleration} & (\ddot{u}_0, \ddot{u}_f)\end{cases}$$

Use quintic: $u(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$

and find a_i (i=0 to 5)   (formulas (7.18) in the book)

Use different functions (exponential, trigonometric,…)

Run Time Path Generation

- trajectory in terms of $\Theta, \dot{\Theta}, \ddot{\Theta}$ fed to the control system
- Path generator computes at path update rate
- In joint space directly:
  - cubic splines -- change set of coefficients at the end of each segment
  - linear with parabolic blends -- check on each update if you are in linear or blend portion and use appropriate formulas for **u**
- In Cartesian space:
  - calculate Cartesian position and orientation at each update point using same formulas
  - convert into joint space using inverse Jacobian and derivatives

  or

  find equivalent frame representation and use inverse kinematics function to find $\Theta, \dot{\Theta}, \ddot{\Theta}$

Trajectory Planning with Obstacles

- Path planning for the whole manipulator
  - Local vs. Global Motion Planning
    - Gross motion planning for relatively uncluttered environments
    - Fine motion planning for the end-effector frame
  - Configuration space (C-space) approach
- Planning for a point robot
  - graph representation of the free space, quadtree
  - Artificial Potential Field method
- Multiple robots, moving robots and/or obstacles