Kinematic Singularity

The Effector Locality loses the ability to move in a direction or to rotate about a direction - singular direction

$$J = \begin{pmatrix} J_1 & J_2 & \cdots & J_n \end{pmatrix}$$

$$\det\left(J\right) = 0$$

$$\det\left({}^{i}J\right) = \det\left({}^{j}J\right)$$

Kinematic Singularity

$${}^{B}J = \begin{pmatrix} {}^{B}_{A}R & 0 \\ 0 & {}^{B}_{A}R \end{pmatrix} {}^{A}J$$

$$\det[{}^{B}J] \equiv \det[{}^{A}J]$$

$$\boxed{\det\left({}^{i}J\right) = \det\left({}^{j}J\right)}$$

Singular Configurations

$$\det[J(q)] = 0$$

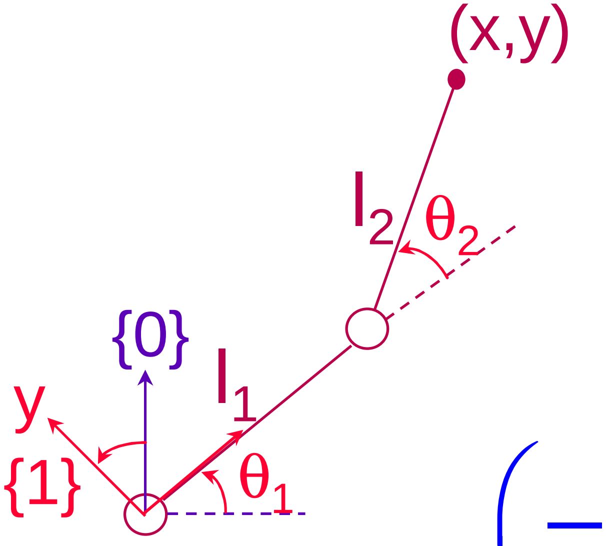
$\Longrightarrow$ Singular Configurations

$$\det[J(q)] = S_1(q) S_2(q) \ldots S_s(q) = 0$$

⇨

$$\boxed{\begin{array}{l} S_1(q) = 0 \\ S_2(q) = 0 \\ \quad \vdots \\ S_s(q) = 0 \end{array}}$$

Example (Kinematic Singularities)



$$x = l_1 C1 + l_2 C12$$

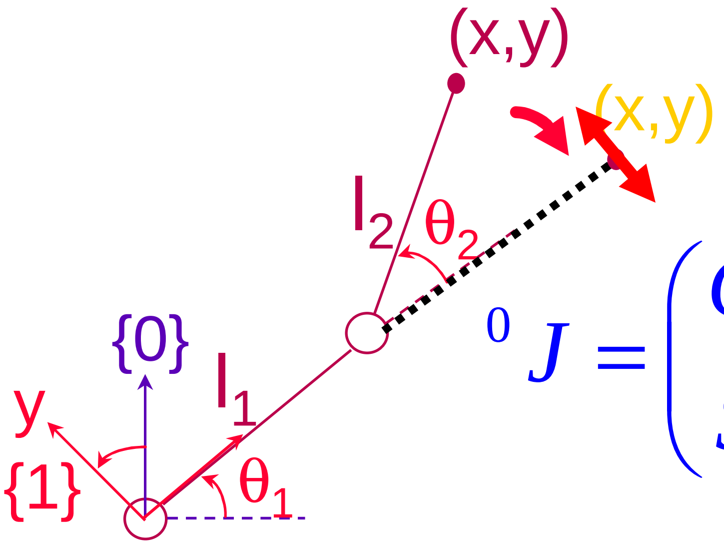
$$y = l_1 S1 + l_2 S12$$

$$J = \begin{pmatrix} -\left(l_1 S1 + l_2 S12\right) & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \end{pmatrix}$$

$$\det\left(J\right) = l_1 l_2 S2$$

Singularity at $q_2 = k\pi$

Example (Kinematic Singularities)



$$^1J = {}^1_0R\ {}^0J$$

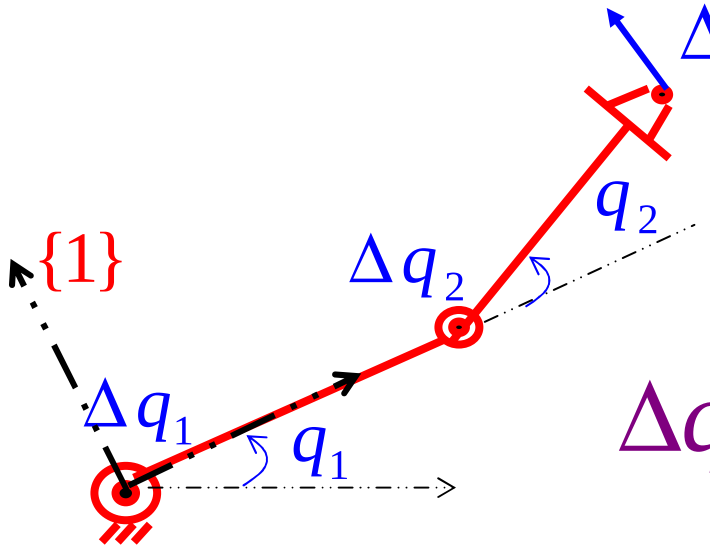
$$^0J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix} \begin{pmatrix} -l_2S2 & -l_2S2 \\ l_1 + l_2C2 & l_2C2 \end{pmatrix}$$

At Singularity

$$^1J = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}$$

$$\begin{cases} {}^1\delta x = 0 \\ {}^1\delta y = (l_1 + l_2)\delta\theta_1 + l_2\delta\theta_2 \end{cases}$$

Small Displacements $\Delta q, \Delta X$



$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

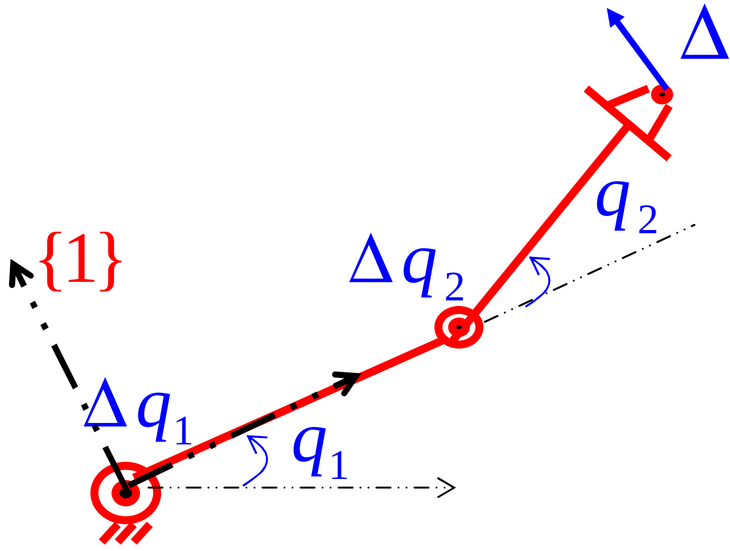
$$\Delta q = J^{-1} \Delta X$$

small θ_2

$$J_{(1)}^{-1} \cong \begin{pmatrix} \dfrac{1}{l_1\theta_2} & \dfrac{1}{l_1} \\ -\dfrac{l_1 + l_2}{l_1 l_2 \theta_2} & -\dfrac{1}{l_1} \end{pmatrix}$$

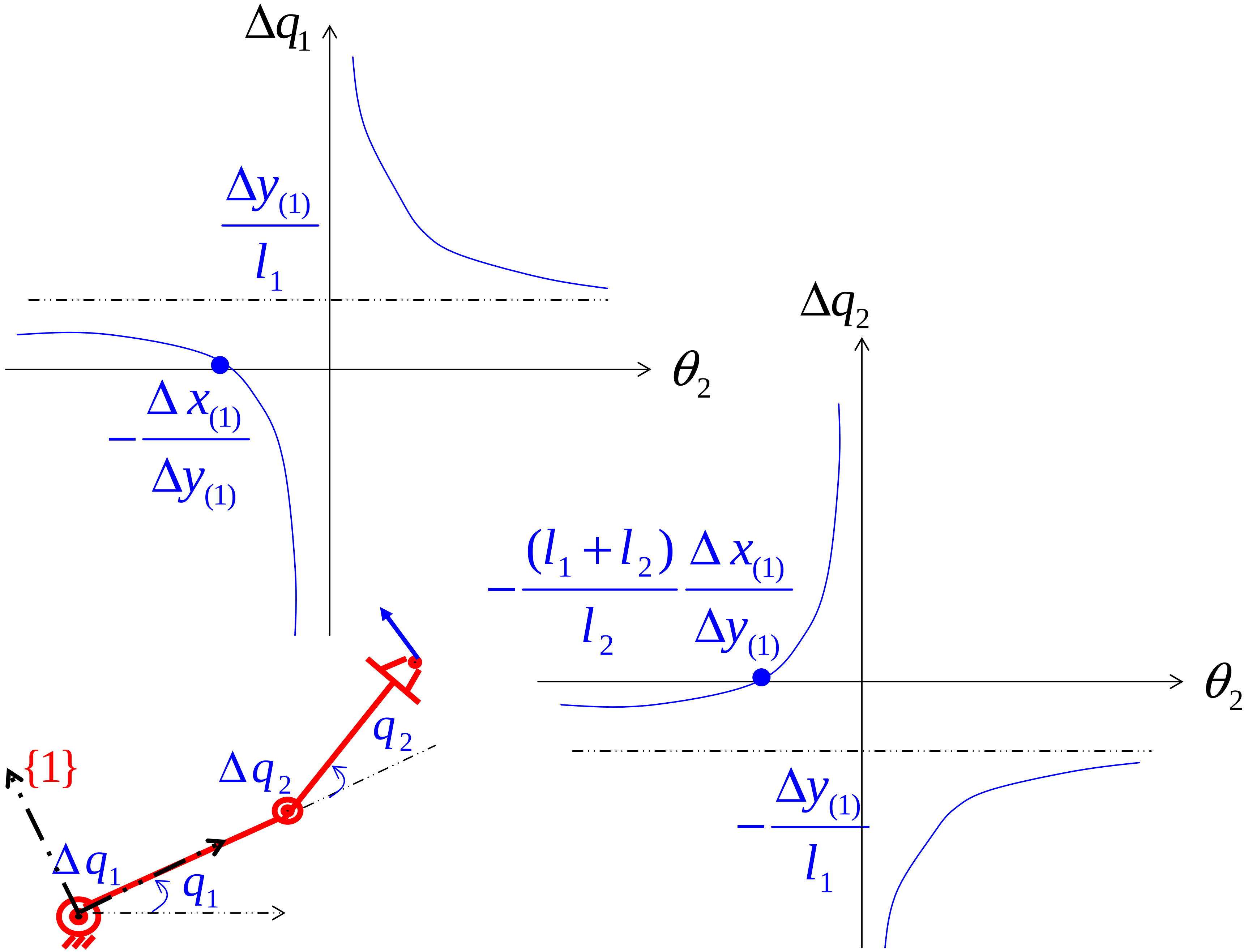
Small Displacements $\Delta q, \Delta X$

$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

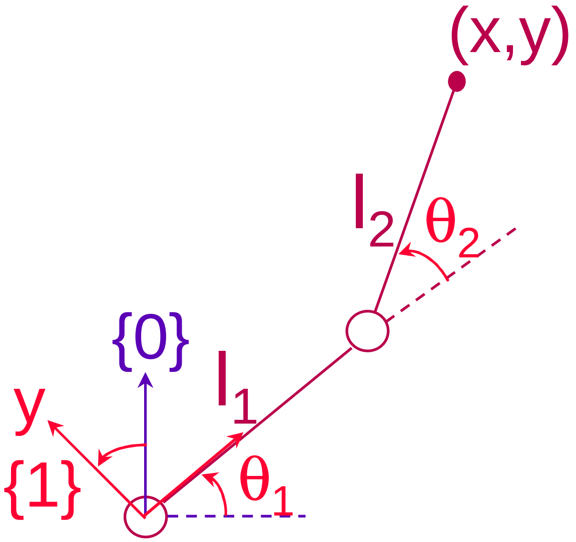


$$\Delta q_1 = \frac{\Delta x_{(1)}}{l_1} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$

$$\Delta q_2 = \frac{\left(l_1 + l_2\right) \Delta x_{(1)}}{l_1 l_2} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$


Δq_1
$\frac{\Delta y_{(1)}}{l_1}$
θ_2
$-\frac{\Delta x_{(1)}}{\Delta y_{(1)}}$
Δq_2
$-\frac{(l_1+l_2)}{l_2}\frac{\Delta x_{(1)}}{\Delta y_{(1)}}$
θ_2
$-\frac{\Delta y_{(1)}}{l_1}$
$\{1\}$
Δq_1
q_1
Δq_2
q_2

Kinematic Singularities (reduced matrix)



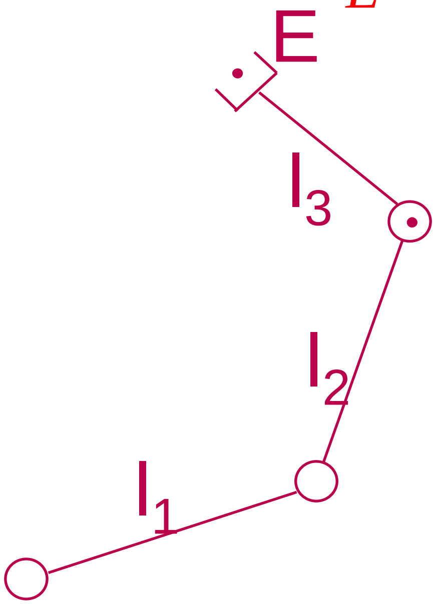
$$J=\begin{pmatrix} -\left(l_1S1+l_2S12\right) & -l_2S12 \\ l_1C1+l_2C12 & l_2C12 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\det\left(J\right)=l_1l_2S2$$

$$J=\begin{pmatrix} -\left(l_1S1+l_2S12\right) & -l_2S12 \\ l_1C1+l_2C12 & l_2C12 \end{pmatrix}$$

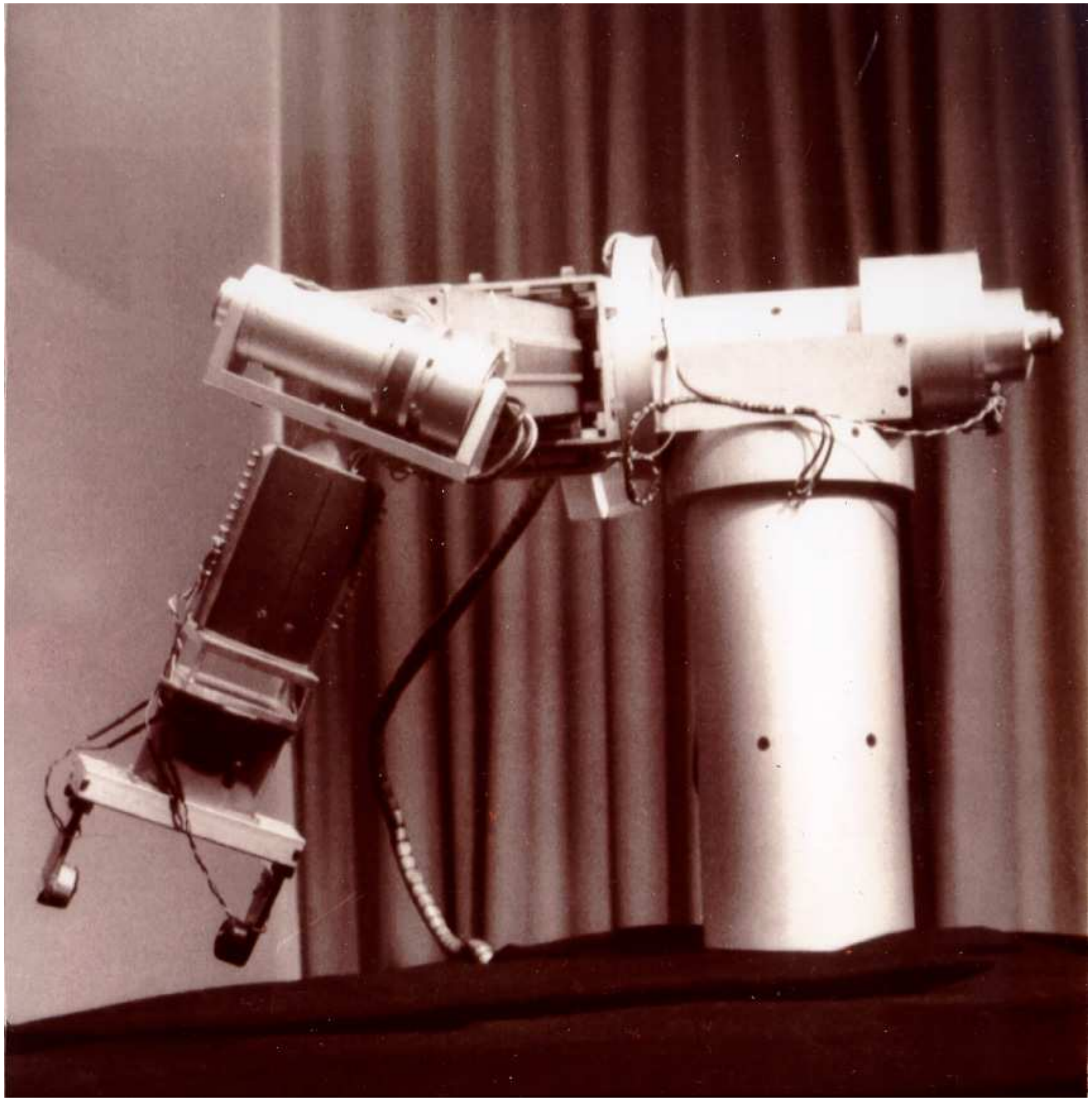
Singularity at $q_2=k\pi$

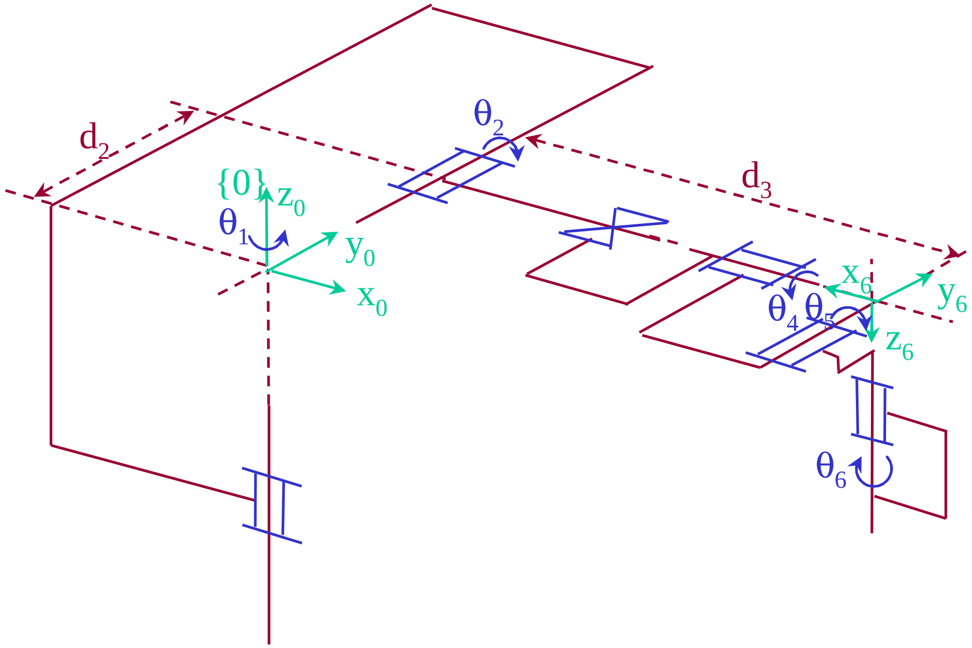
$$ {}^{0}J_E = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 1 & 1 & 1 \end{pmatrix} $$



$$ {}^{0}J_E = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} $$

Stanford Scheinman Arm





i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

$$
{}^{i-1}_{1}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i \, c\alpha_{i-1} & c\theta_i \, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} \, d_i \\ s\theta_i \, s\alpha_{i-1} & c\theta_i \, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} \, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Forward Kinematics: $${}^{0}_{N}T = {}^{0}_{1}T \; {}^{1}_{2}T \; \dots \; {}^{N-1}_{N}T$$

Stanford Scheinman Arm Jacobian

$$
{}^{0}J = \begin{pmatrix} \dfrac{\partial {}^{0}x_P}{\partial q_1} & \dfrac{\partial {}^{0}x_P}{\partial q_2} & \dfrac{\partial {}^{0}x_P}{\partial q_3} & 0 & 0 & 0 \\ {}^{0}Z_1 & {}^{0}Z_2 & 0 & {}^{0}Z_4 & {}^{0}Z_5 & {}^{0}Z_6 \end{pmatrix}
$$

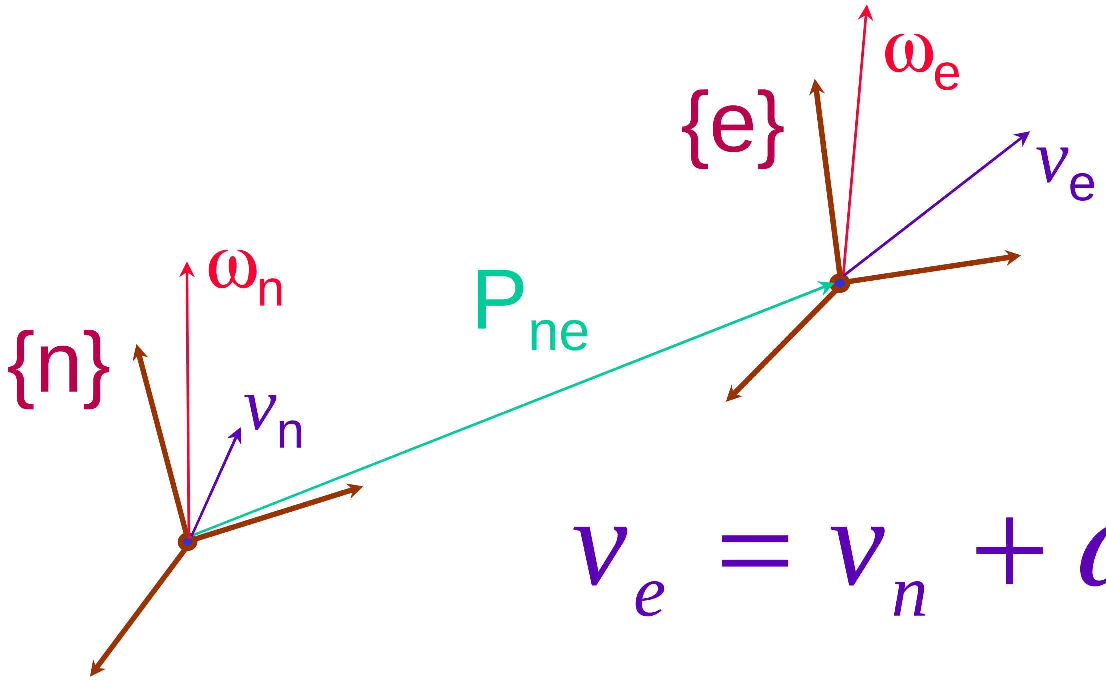
$$
\begin{bmatrix}
-c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\
-s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\
0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\
0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\
0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\
1 & 0 & 0 & c_2 & s_2 s_4 & -s_2 c_4 s_5 + c_5 c_2
\end{bmatrix}
$$

Stanford Scheinman Arm Jacobian

$$\theta_5 = k\pi$$

$$J = \begin{bmatrix} -c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\ -s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\ 0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 s_2 \\ 0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 s_2 \\ 1 & 0 & 0 & c_2 & s_2 s_4 & c_2 \end{bmatrix}$$

Jacobian at the End-Effector



$$v_e = v_n + \omega_n \times P_{ne}$$

$$\begin{cases} v_e = v_n - P_{ne} \times \omega_n \\ \omega_e = \omega_n \end{cases}$$

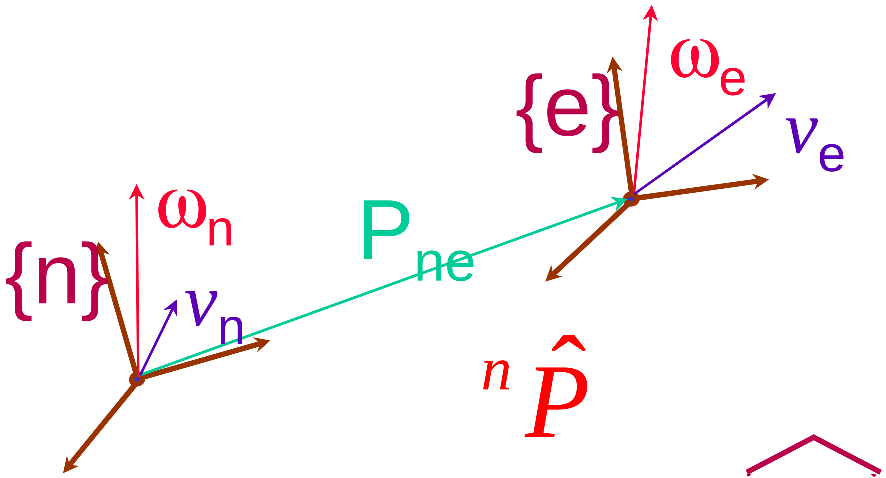
$$\begin{cases} v_e = v_n - P_{ne} \times \omega_n \\ \omega_e = \omega_n \end{cases}$$

$$\begin{pmatrix} v_e \\ \omega_e \end{pmatrix} = \left(\begin{array}{c:c} \mathrm{I} & -\hat{P}_{ne} \\ \hdashline \mathrm{O} & \mathrm{I} \end{array}\right) \begin{pmatrix} v_n \\ \omega_n \end{pmatrix}$$

$$J_e \dot{q} = \left(\begin{array}{c:c} \mathrm{I} & -\hat{P}_{ne} \\ \hdashline \mathrm{O} & \mathrm{I} \end{array}\right) J_n \dot{q}$$

$$J_e = \left(\begin{array}{c:c} \mathrm{I} & -\hat{P}_{ne} \\ \hdashline \mathrm{O} & \mathrm{I} \end{array}\right) J_n$$

Cross Product Operator (in diff. frames)



$${}^0J_e = \begin{pmatrix} I & -{}^0\hat{P}_{ne} \\ 0 & I \end{pmatrix} {}^0J_n$$

$${}^n\hat{P}$$

$${}^0\hat{P} \neq {}^0_nR\;{}^n\hat{P}; \qquad \widehat{{}^0P} = \widehat{({}^0_nR.{}^nP)} \neq {}^0_nR.\widehat{{}^nP}$$

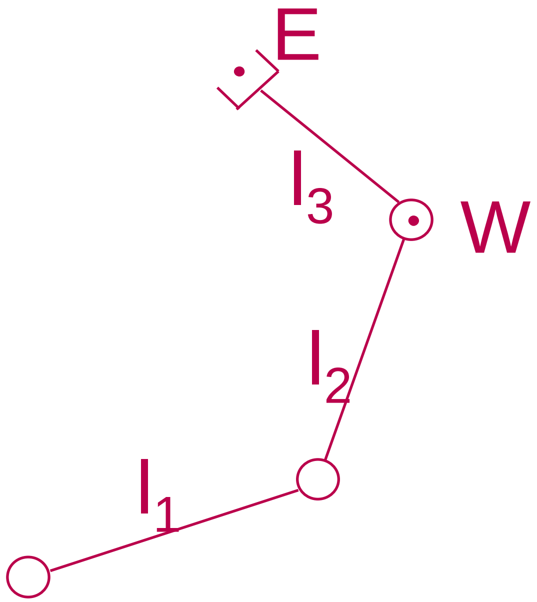
$${}^0P\times{}^0\omega = {}^0_nR.({}^nP\times{}^n\omega)$$

$${}^0\hat{P}.{}^0\omega = {}^0_nR.({}^n\hat{P}.{}^n\omega) = {}^0_nR.({}^n\hat{P}.{}^0_nR^T.{}^0\omega)$$

$$\boxed{{}^0\hat{P} = {}^0_nR\;{}^n\hat{P}\;{}^0_nR^T}$$

$$ {}^{i}J = \begin{pmatrix} {}^{i}_{j}R & 0 \\ 0 & {}^{i}_{j}R \end{pmatrix} {}^{j}J $$

$$ {}^{0}J_{e} = \begin{pmatrix} {}^{0}_{n}R & -{}^{0}_{n}R\, {}^{n}\hat{P}_{ne}\, {}^{0}_{n}R^{T} \\ 0 & {}^{0}_{n}R \end{pmatrix} {}^{n}J_{n} $$



Wrist Point

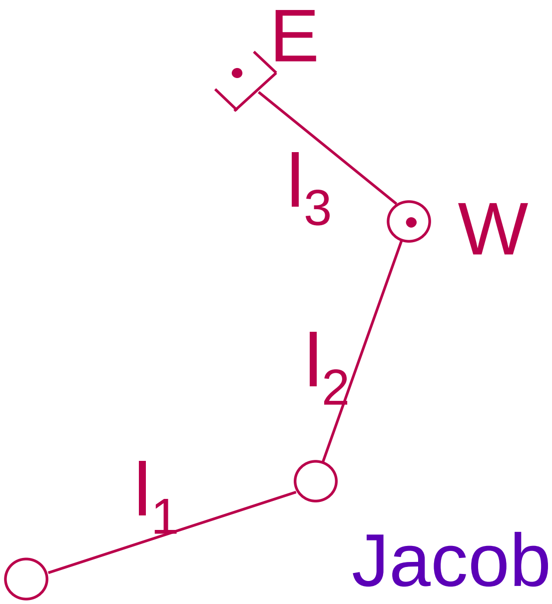
$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

End-Effector Point

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$



Wrist Point

$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

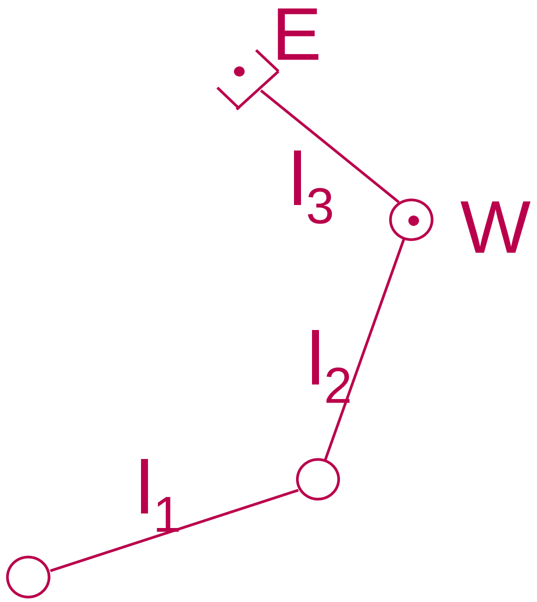
End-Effector Point

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

Jacobian (W)

$$J_W = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} ;\ {}^0 J_E = \begin{pmatrix} I & -{}^0\hat{P}_{WE} \\ 0 & I \end{pmatrix} {}^0 J_W$$



Wrist Point

$$x = l_1 c_1 + l_2 c_{12}$$

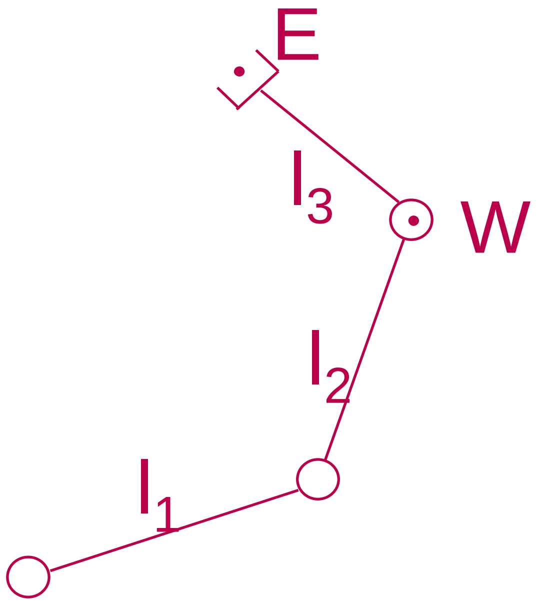
$$y = l_1 s_1 + l_2 s_{12}$$

End-Effector Point

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

$$J_W = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad {}^0 J_E = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$



Wrist Point

$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

End-Effector Point

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

$$J_W = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$${}^0J_E = \begin{pmatrix} I & -{}^0\hat{P}_{WE} \\ 0 & I \end{pmatrix} {}^0J_W$$

$${}^0P_{WE} = \begin{bmatrix} l_3 c_{123} \\ l_3 s_{123} \\ 0 \end{bmatrix} \Rightarrow {}^0\hat{P}_{WE} = \begin{pmatrix} 0 & 0 & l_3 s_{123} \\ 0 & 0 & -l_3 c_{123} \\ -l_3 s_{123} & l_3 c_{123} & 0 \end{pmatrix}$$

Resolved Motion Rate Control (Whitney 72)

$$\delta x = J(\theta)\delta\theta$$

Outside singularities

$$\delta\theta = J^{-1}(\theta)\delta x$$

Arm at Configuration θ

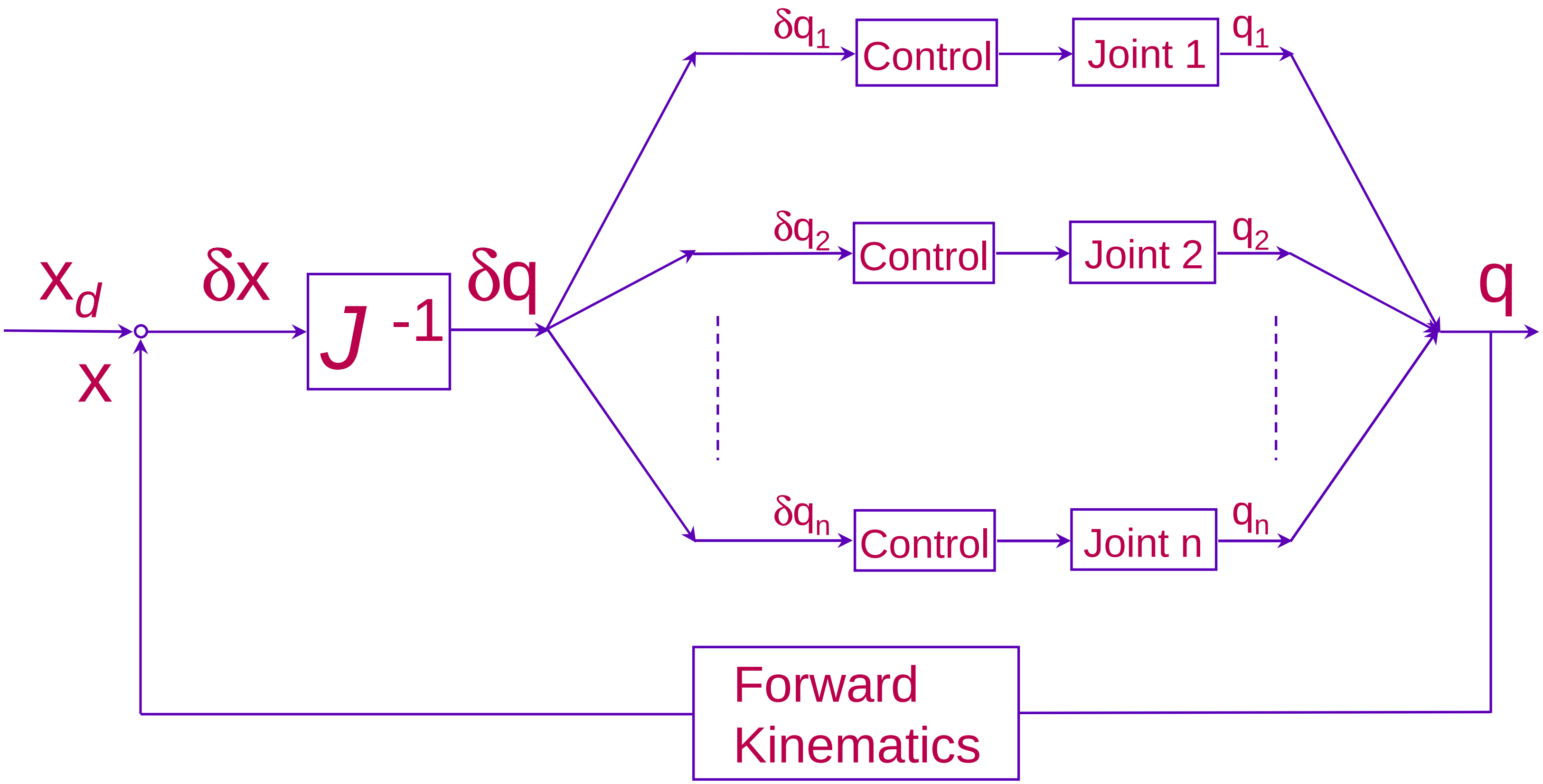
$$x = f(\theta)$$

$$\delta x = x_d - x$$

$$\delta\theta = J^{-1}\delta x$$

$$\boxed{\theta^{+} = \theta + \delta\theta}$$

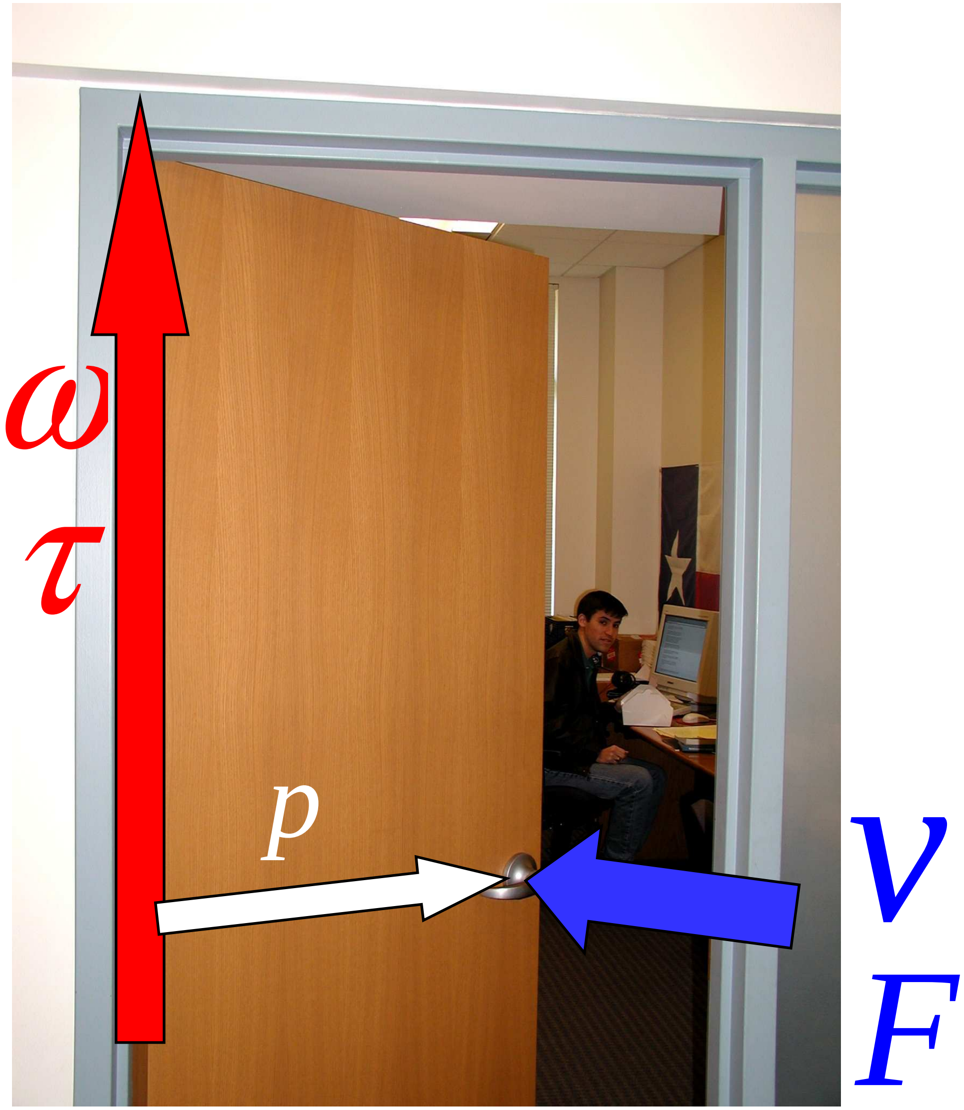
Resolved Motion Rate Control



Jacobian

- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces

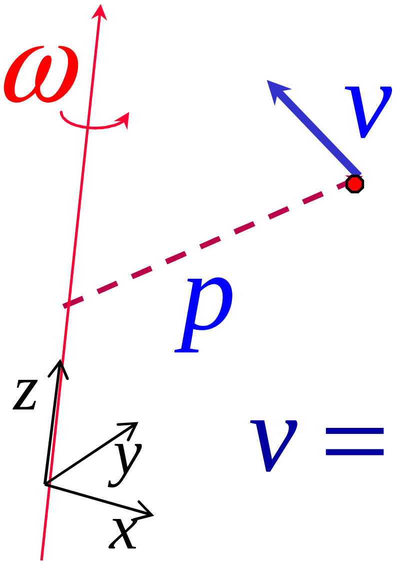
Angular/Linear – Velocities/Forces



$$v = \omega \times p$$

$$\tau = p \times F$$

Angular/Linear – Velocities/Forces

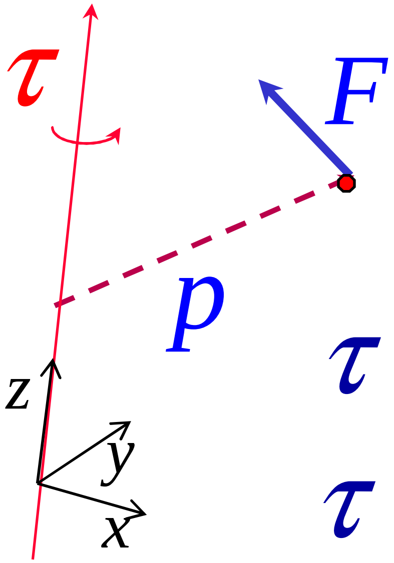


$$v = \omega \times p$$

$$v = -\hat{p}\ \omega$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -p_y \\ p_x \end{pmatrix} \dot{\theta}$$

$$\boxed{v = J\ \dot{\theta}}$$



$$\tau = p \times F$$

$$\tau = \hat{p}\ F$$

$$\tau = (-\hat{p})^T\ F$$

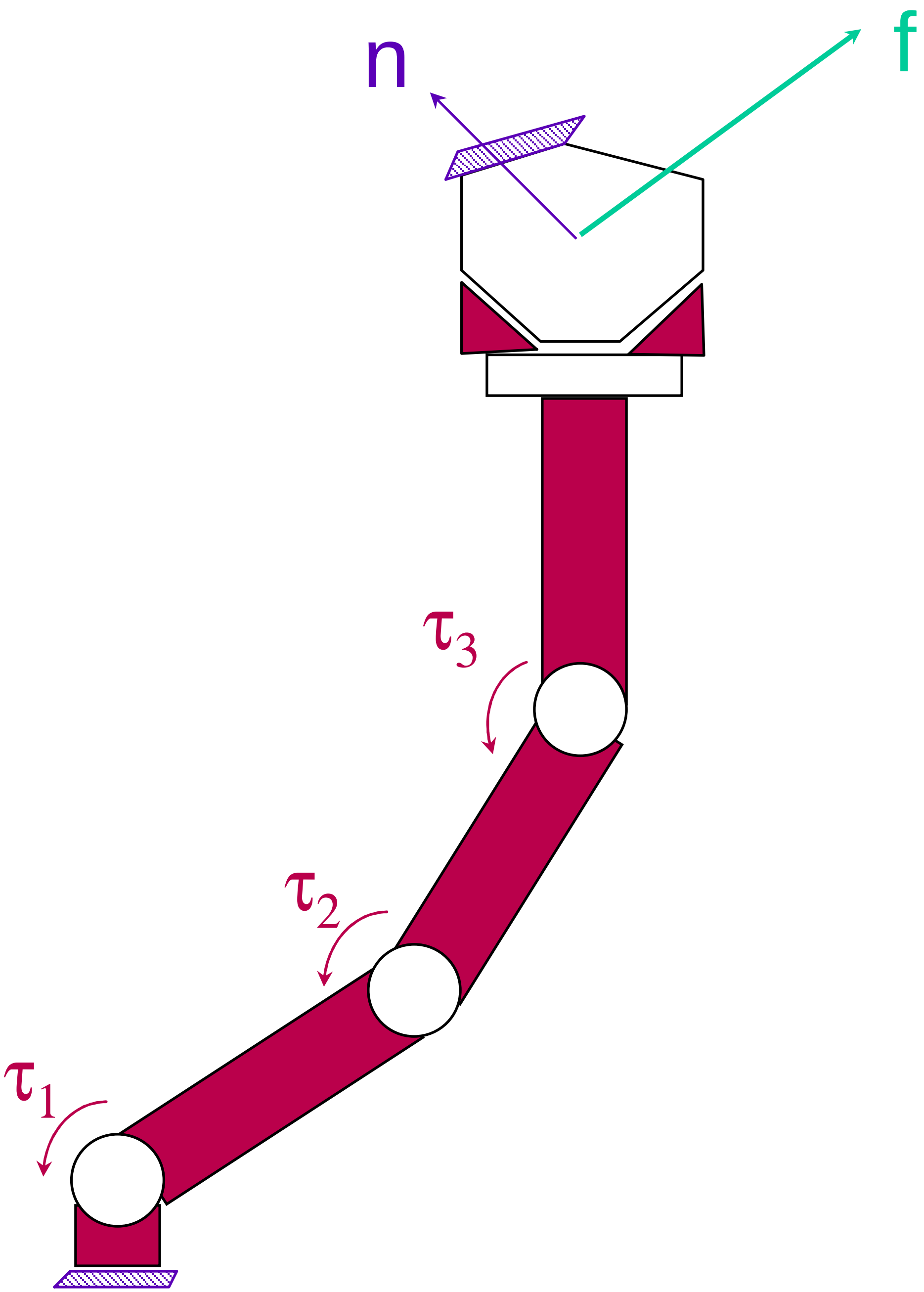
$$\tau = \begin{pmatrix} -p_y & p_x \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

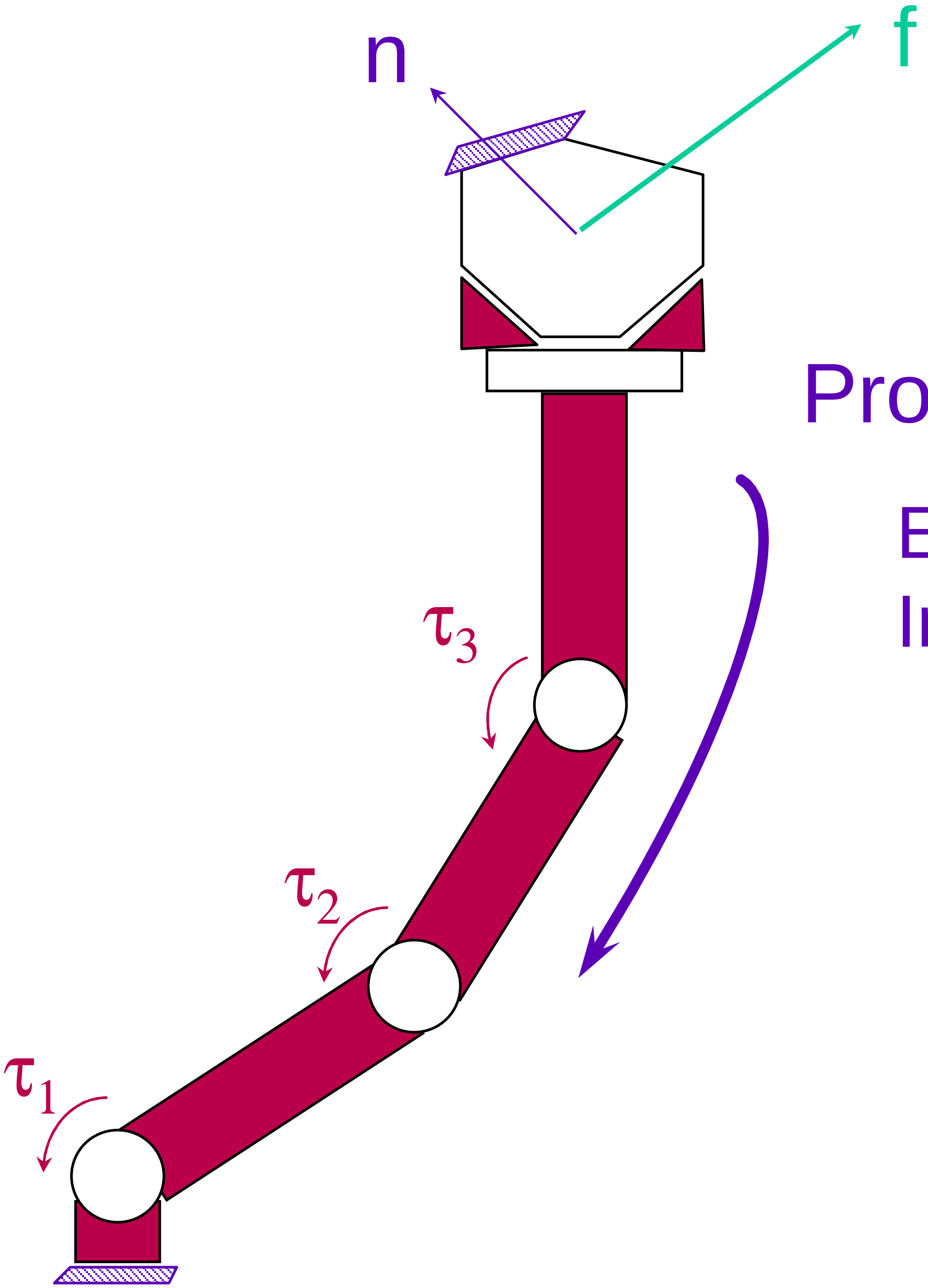
$$\boxed{\tau = J^T F}$$

Velocity/Force Duality

$$\boxed{\dot{x} = J\,\dot{\theta}}$$

$$\boxed{\tau = J^T F}$$


n
f
τ3
τ2
τ1



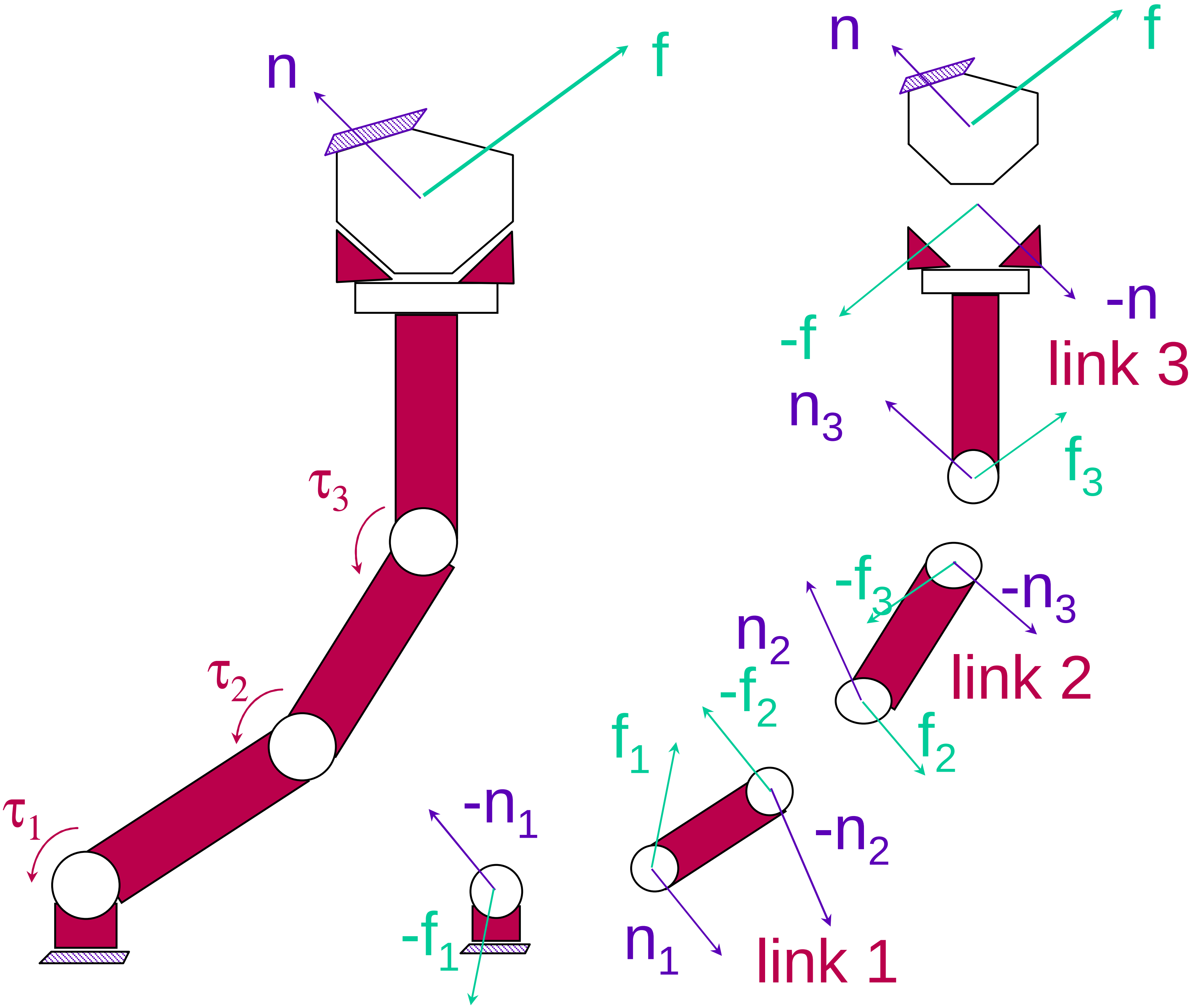
Propagation

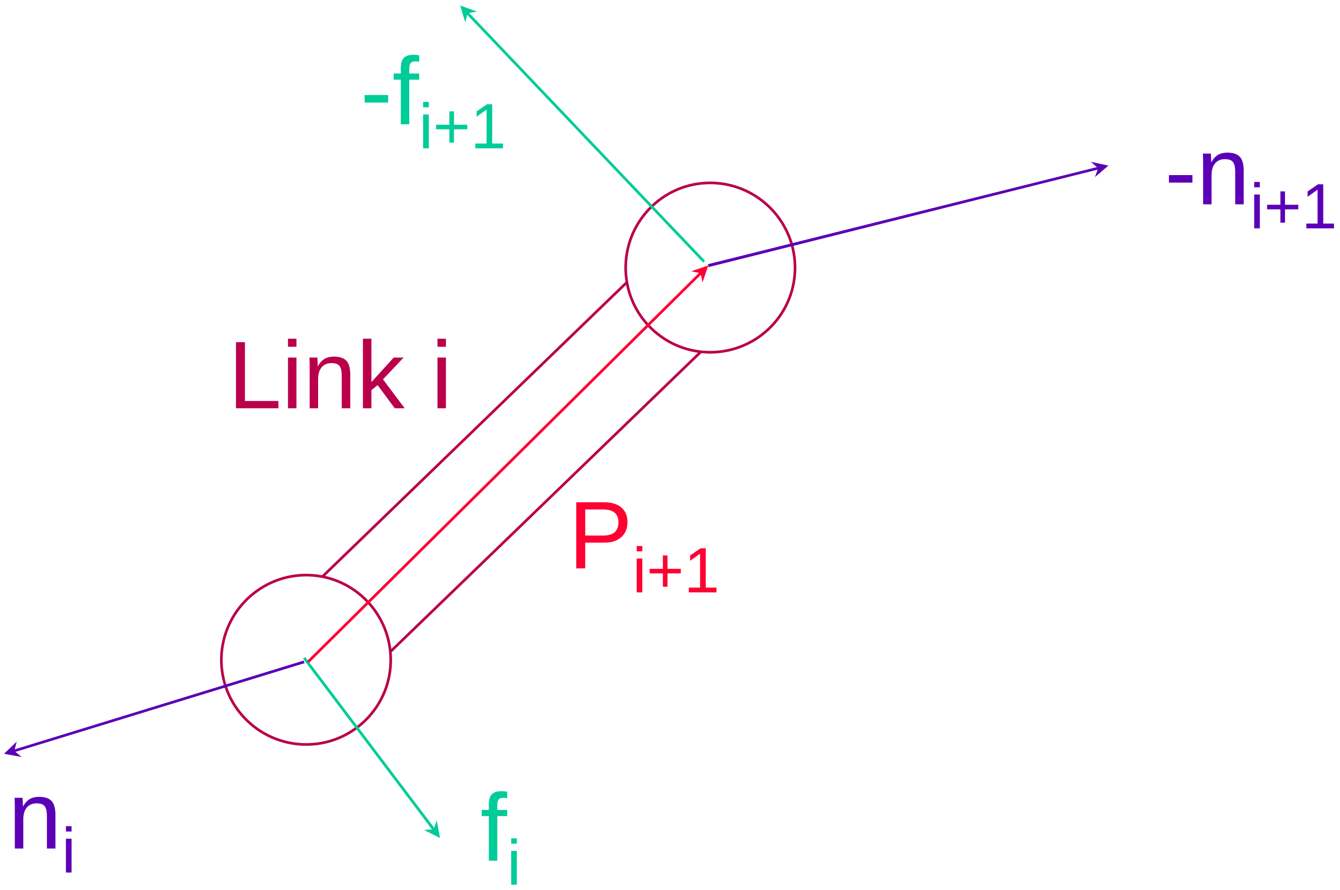
Elimination of
Internal forces

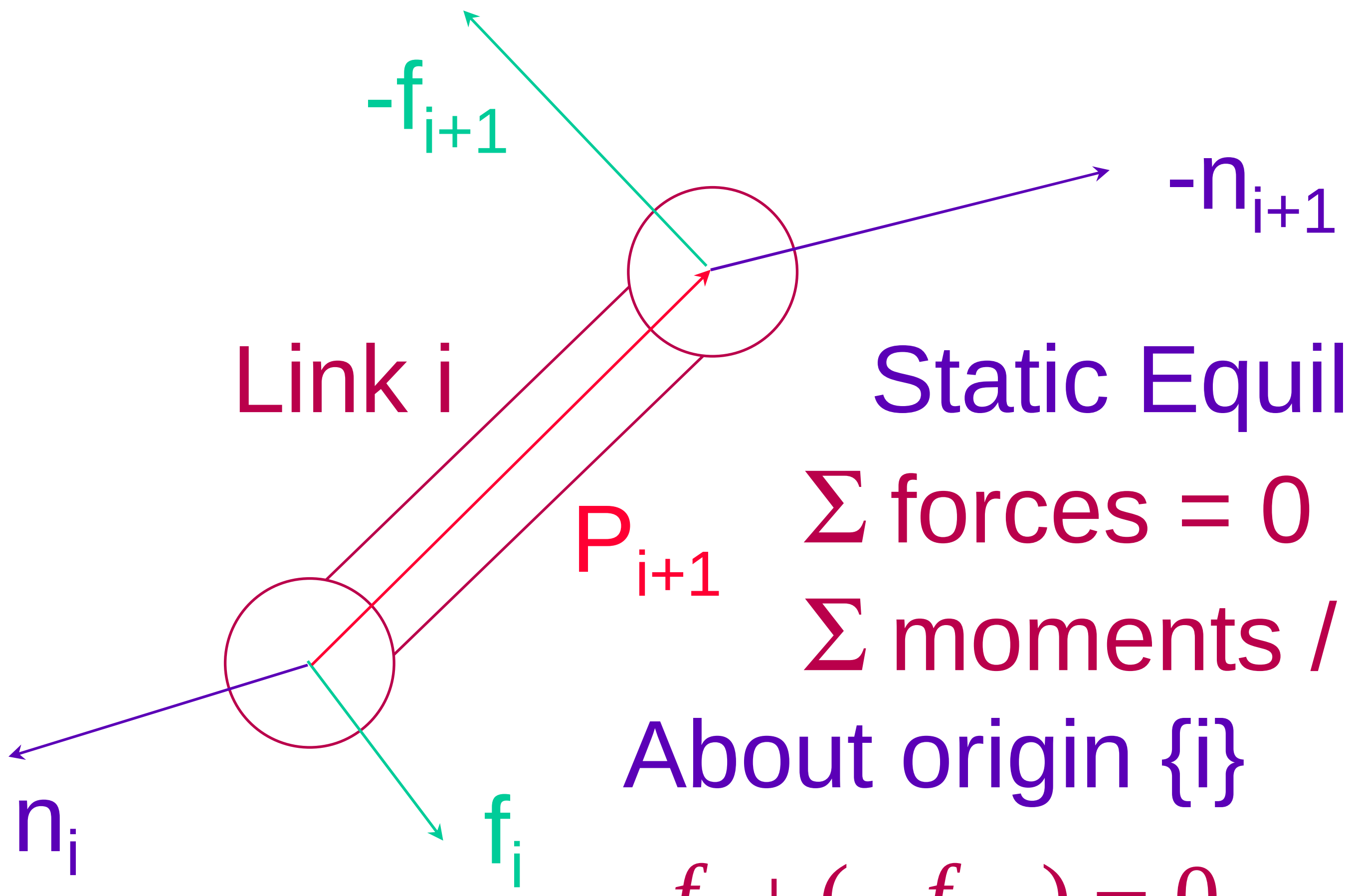
Energy Analysis

Virtual Work

Static Equilibrium


n
f
τ_3
τ_2
τ_1
$-n_1$
$-f_1$
n
f
-f
-n
link 3
n_3
f_3
$-f_3$
$-n_3$
n_2
link 2
$-f_2$
f_1
f_2
$-n_2$
n_1
link 1


$-f_{i+1}$
$-n_{i+1}$
Link i
P_{i+1}
n_i
f_i



Static Equilibrium

Σ forces = 0

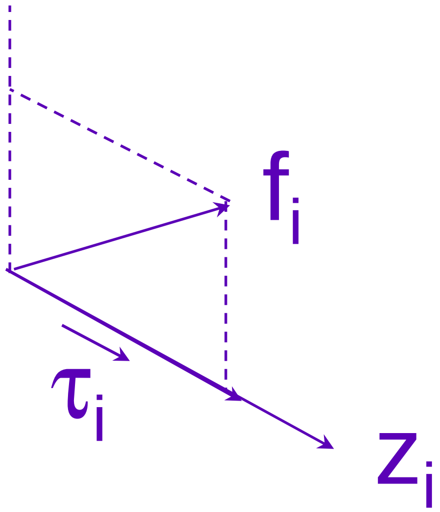
Σ moments / a point = 0

About origin {i}

$$f_i + (-f_{i+1}) = 0$$

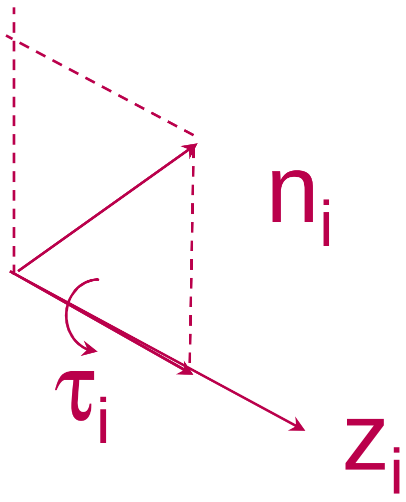
$$n_i + (-n_{i+1}) + P_{i+1} \times (-f_{i+1}) = 0$$

$$\left\| \begin{aligned} f_i &= f_{i+1} \\ n_i &= n_{i+1} + P_{i+1} \times f_{i+1} \end{aligned} \right.$$



Prismatic Joint

$$\tau_i = f_i^{\,T} Z_i$$



Revolute Joint

$$\tau_i = n_i^{\,T} Z_i$$

Algorithm

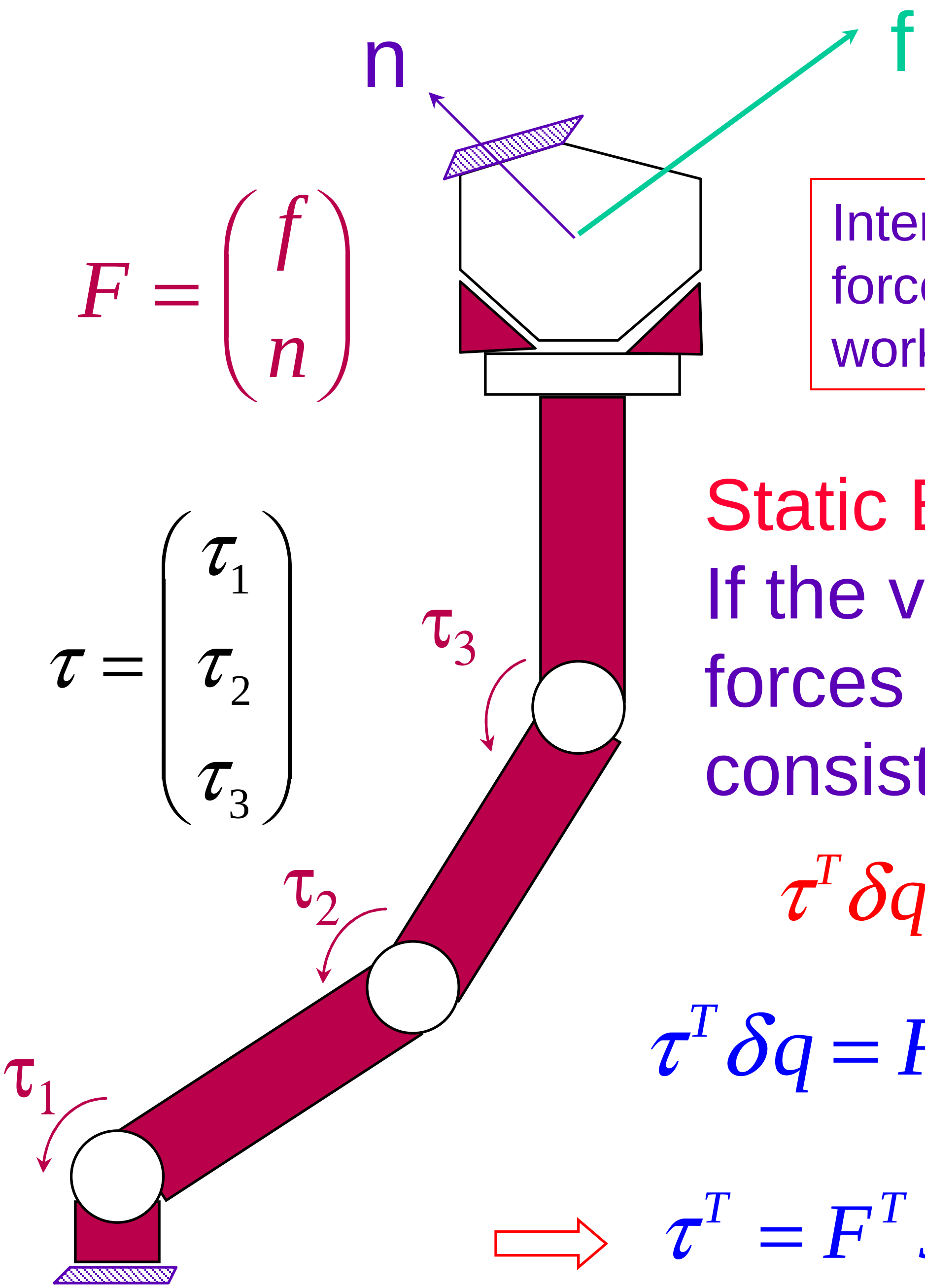
$$^{n}f_n = {}^{n}f$$

$$^{n}n_n = {}^{n}n + {}^{n}P_{n+1} \times {}^{n}f$$

$$^{i}f_i = {}^{i}_{i+1}R . {}^{i+1}f_{i+1}$$

$$^{i}n_i = {}^{i}_{i+1}R . {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i$$

Virtual Work Principal



$$F = \begin{pmatrix} f \\ n \end{pmatrix}$$

$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}$$

Internal forces are workless

$$\delta w = \sum_i \underset{\text{applied forces}}{f_i} \; \underset{\text{virtual displacements}}{\delta x_i}$$

Static Equilibrium:
If the virtual work done by applied forces is zero in displacements consistent with constraints

$$\tau^T \delta q + (-F)^T \delta x = 0$$

$\tau^T \delta q = F^T \delta x$ using $\delta x = J\,\delta q$

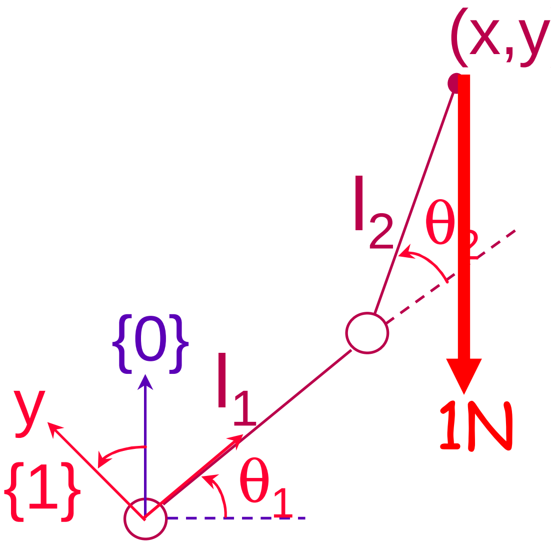
$$\Rightarrow \quad \tau^T = F^T J \quad \Rightarrow \quad \boxed{\tau = J^T F}$$

Velocity/Force Duality

$$\boxed{\dot{x} = J\,\dot{\theta}}$$

$$\boxed{\tau = J^T F}$$

Example (Static Forces)



$$J=\begin{pmatrix}-\left(l_1S1+l_2S12\right) & -l_2S12\\ l_1C1+l_2C12 & l_2C12\end{pmatrix}$$

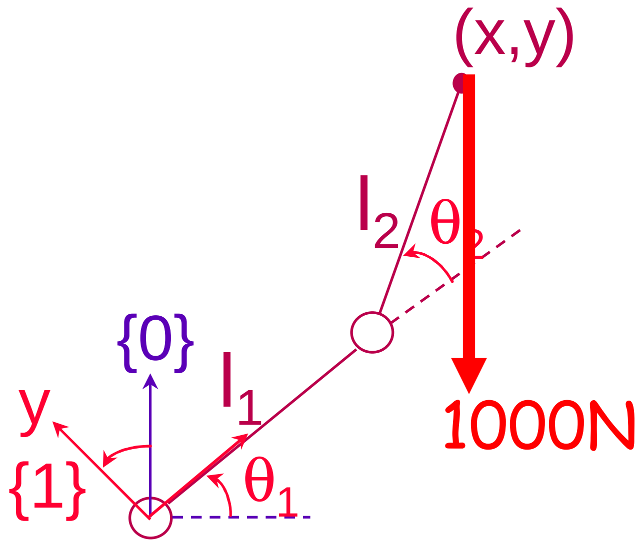
$$J^T=\begin{pmatrix}-\left(l_1S1+l_2S12\right) & l_1C1+l_2C12\\ -l_2S12 & l_2C12\end{pmatrix}$$

$$\boxed{\tau=J^TF}$$

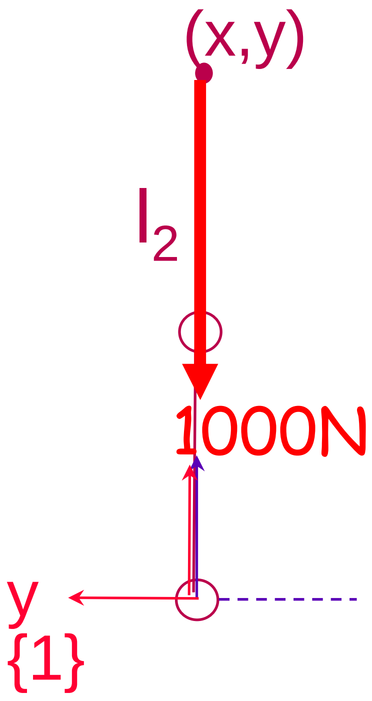
$$l_1=l_2=1;\quad \theta_1=0;\ \theta_2=60^\circ$$

$$\tau=\begin{pmatrix}-\left(l_1S1+l_2S12\right) & l_1C1+l_2C12\\ -l_2S12 & l_2C12\end{pmatrix}\begin{bmatrix}0\\-1\end{bmatrix}=-\begin{bmatrix}l_1C1+l_2C12\\ l_2C12\end{bmatrix}=-\begin{bmatrix}3/2\\1/2\end{bmatrix}$$

Example (Static Forces)



$$\boxed{\tau = J^T F}$$



$$\tau = \begin{pmatrix} -\left(l_1 S1 + l_2 S12\right) & l_1 C1 + l_2 C12 \\ -l_2 S12 & l_2 C12 \end{pmatrix} \begin{bmatrix} 0 \\ -1K \end{bmatrix} = \begin{bmatrix} l_1 C1 + l_2 C12 \\ l_2 C12 \end{bmatrix} (-1K) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$l_1 = l_2 = 1; \quad \theta_1 = 90; \theta_2 = 0^\circ$$