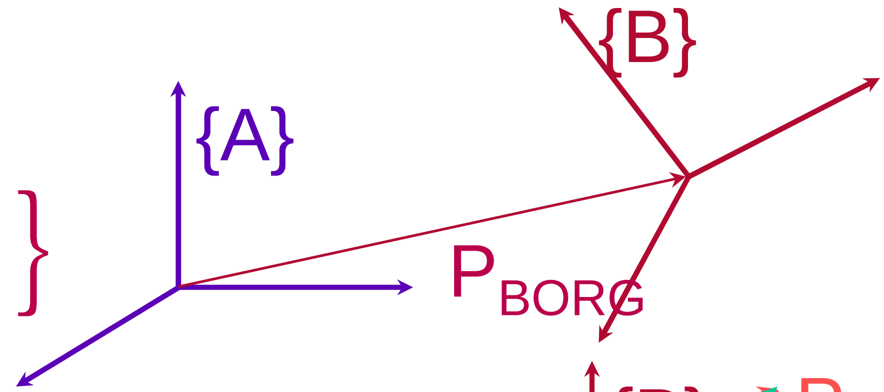
Homogeneous Transform Interpretations

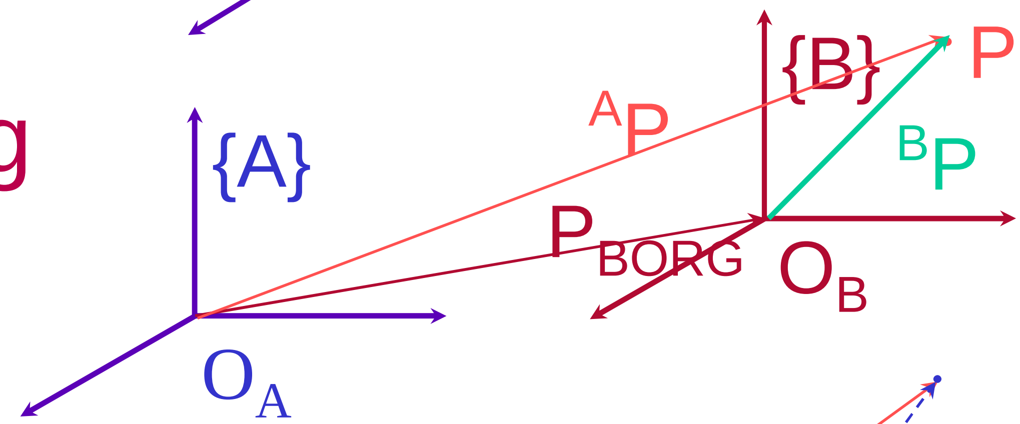
Description of a frame

$${}^A_B T: \quad \{B\} = \left\{ {}^A_B R \quad {}^A P_{Borg} \right\}$$



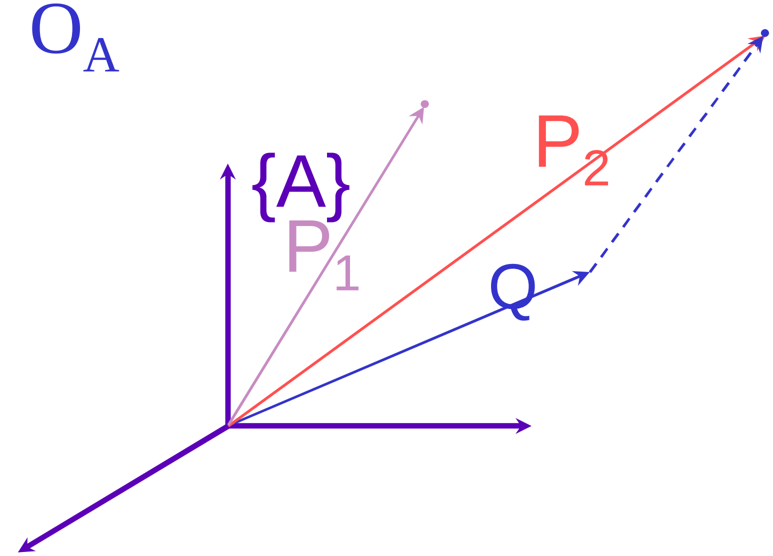
Transform mapping

$${}^A_B T : \quad {}^B P \rightarrow {}^A P$$

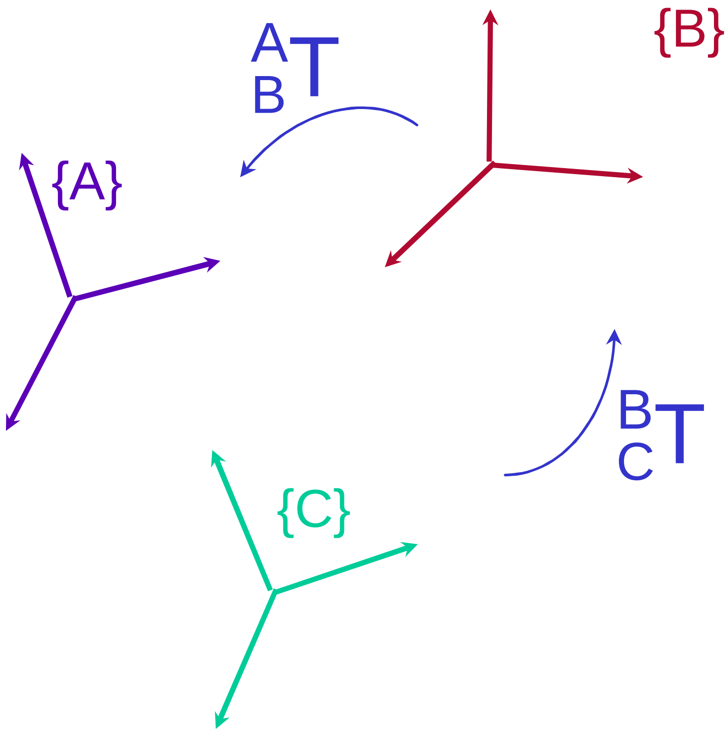


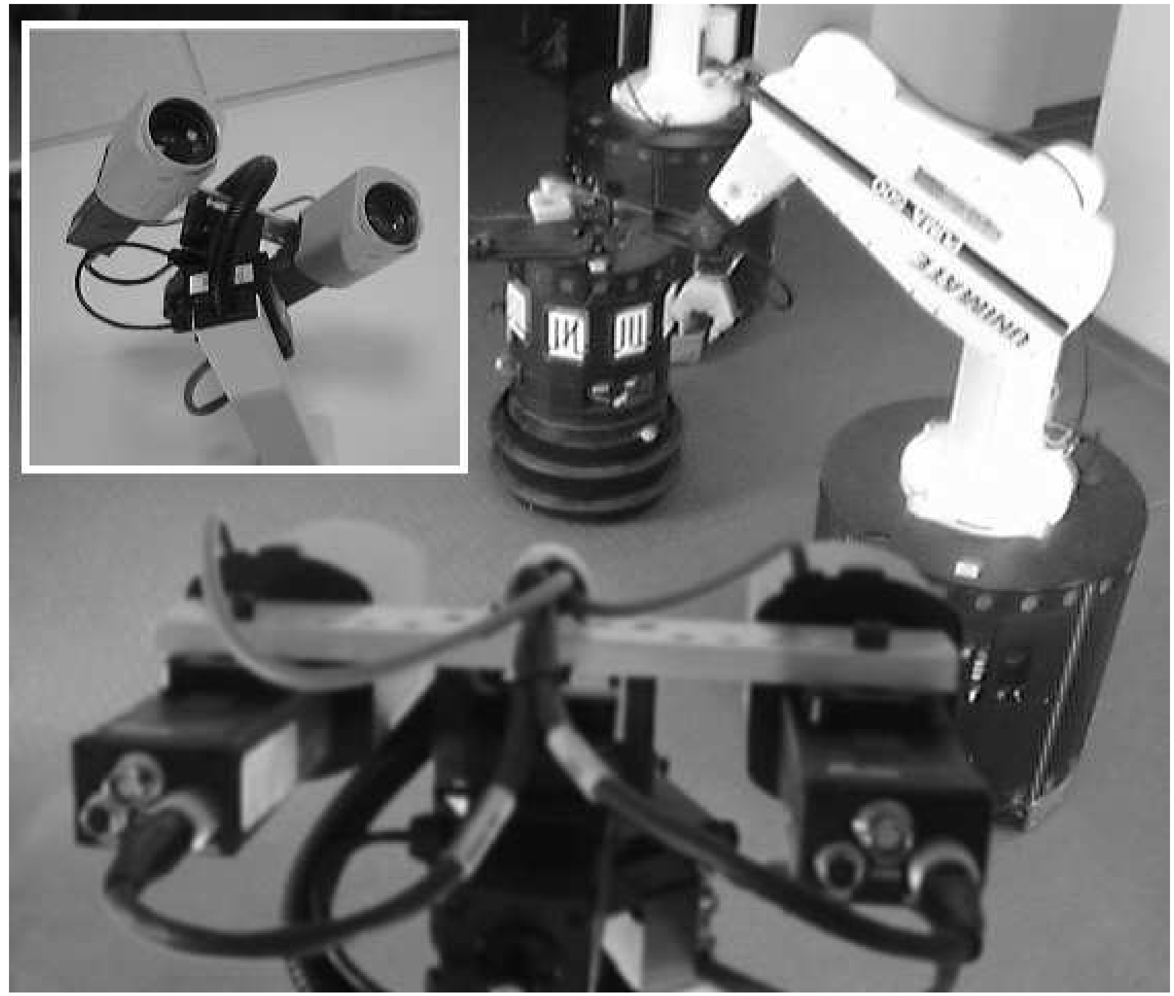
Transform operator

$$T : \quad P_1 \rightarrow P_2$$

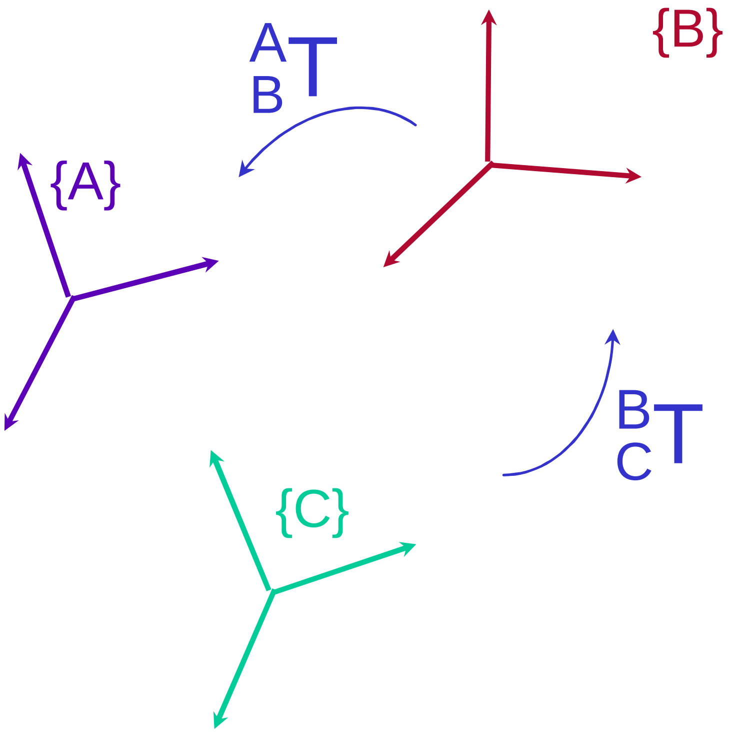


Transform Equation

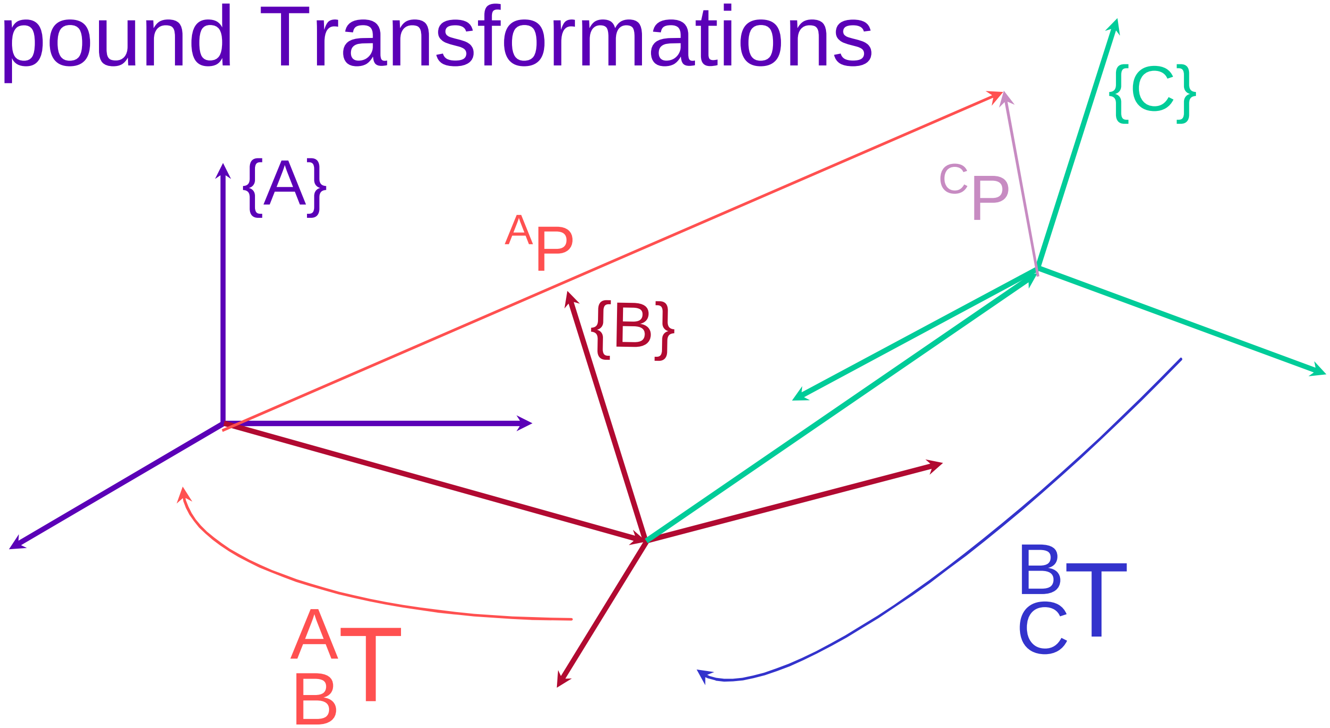




Transform Equation



Compound Transformations



$$^{B}P = {}^{B}_{C}T \, {}^{C}P$$

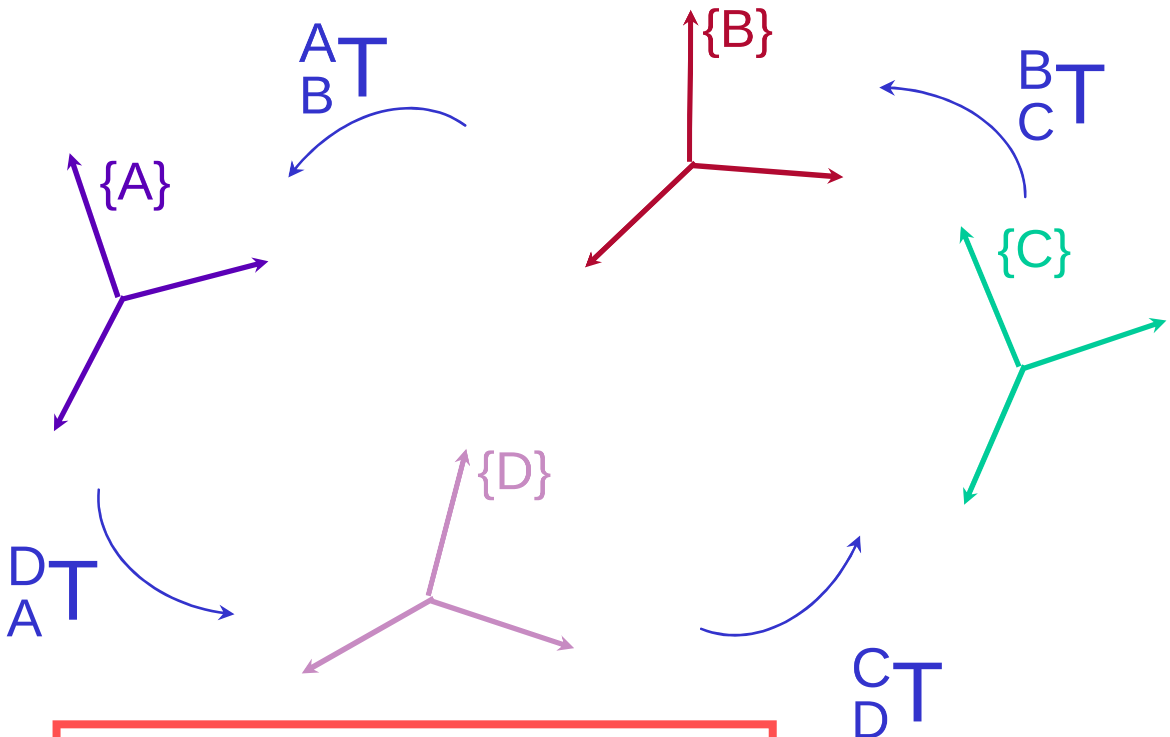
$$^{A}P = {}^{A}_{B}T \, {}^{B}P$$

$$^{A}P = {}^{A}_{B}T \, {}^{B}_{C}T \, {}^{C}P \quad \Rightarrow \quad {}^{A}_{C}T = {}^{A}_{B}T \, {}^{B}_{C}T$$

$$ {}^{A}_{C}T = {}^{A}_{B}T \, {}^{B}_{C}T $$

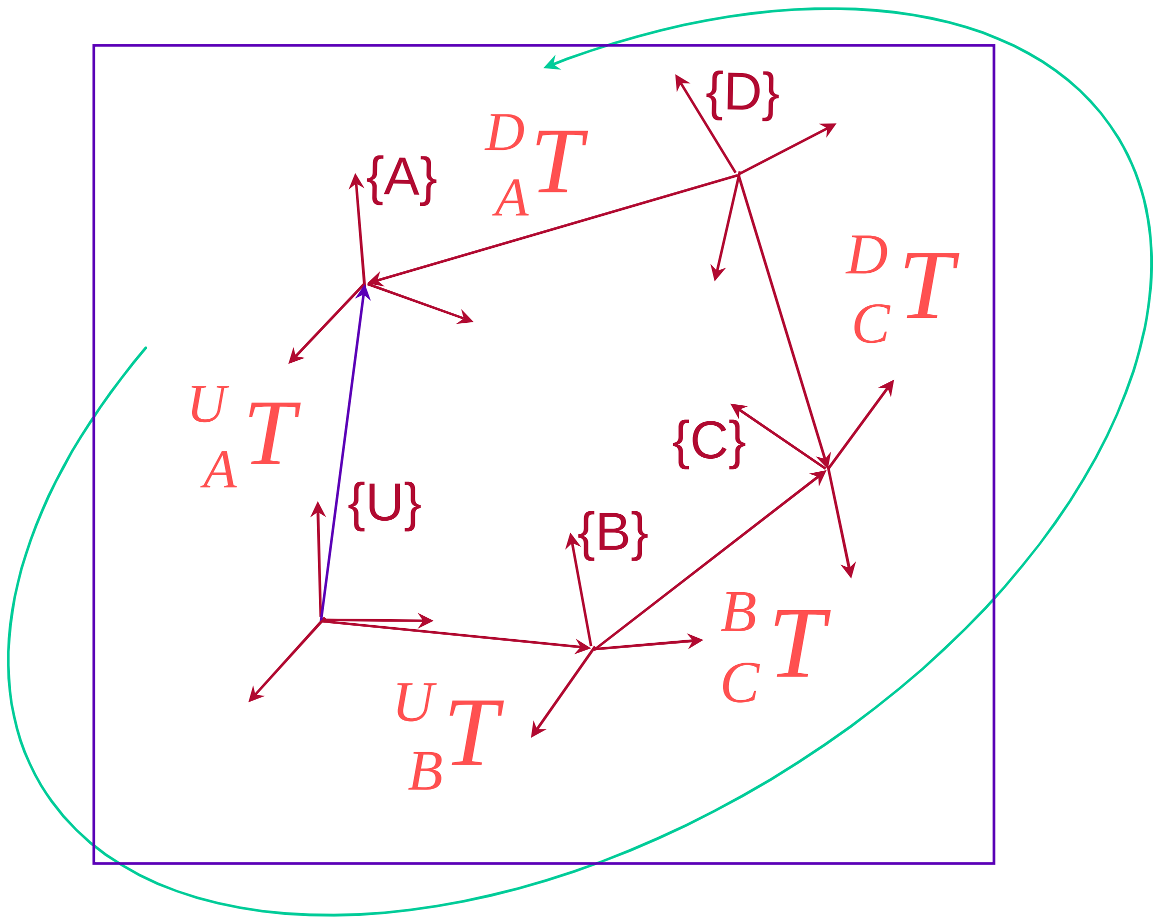
$$ {}^{A}_{C}T = \begin{bmatrix} {}^{A}_{B}R \, {}^{B}_{C}R & {}^{A}_{B}R \, {}^{B}P_{Corg} + {}^{A}P_{Borg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

Transform Equation



$$ {}^A_B T \; {}^B_C T \; {}^C_D T \; {}^D_A T \equiv I $$

$$ \Rightarrow \quad {}^B_A T = {}^B_C T \; {}^C_D T \; {}^D_A T $$



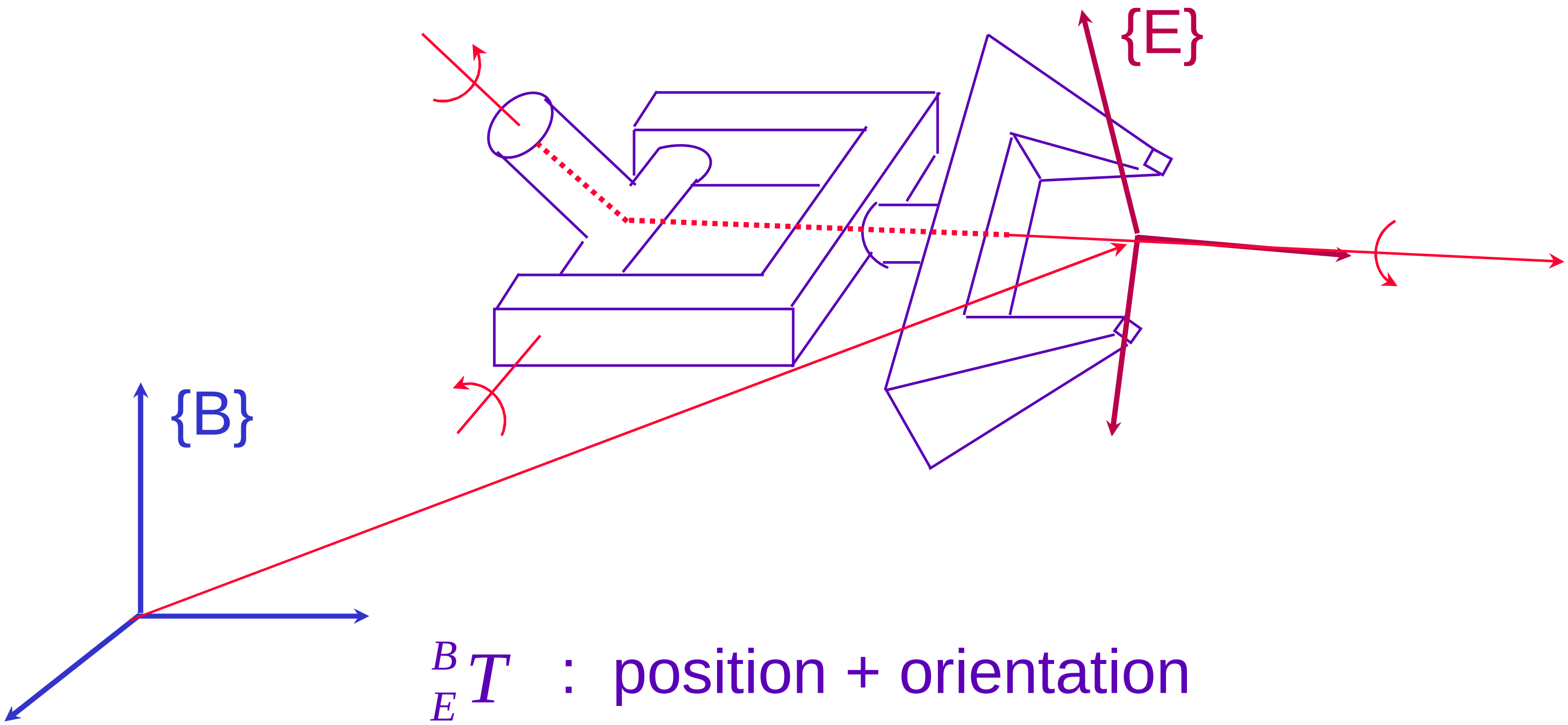
$$ {}^D_A T^{-1} \cdot {}^D_C T \cdot {}^B_C T^{-1} \cdot {}^U_B T^{-1} \cdot {}^U_A T \equiv I $$

$$ {}^U_A T = {}^U_B T \cdot {}^B_C T \cdot {}^D_C T^{-1} \cdot {}^D_A T $$

Spatial Descriptions

- Task Description
- Transformations
- Representations

End-Effector Configuration



$^{B}_{E}T$: position + orientation

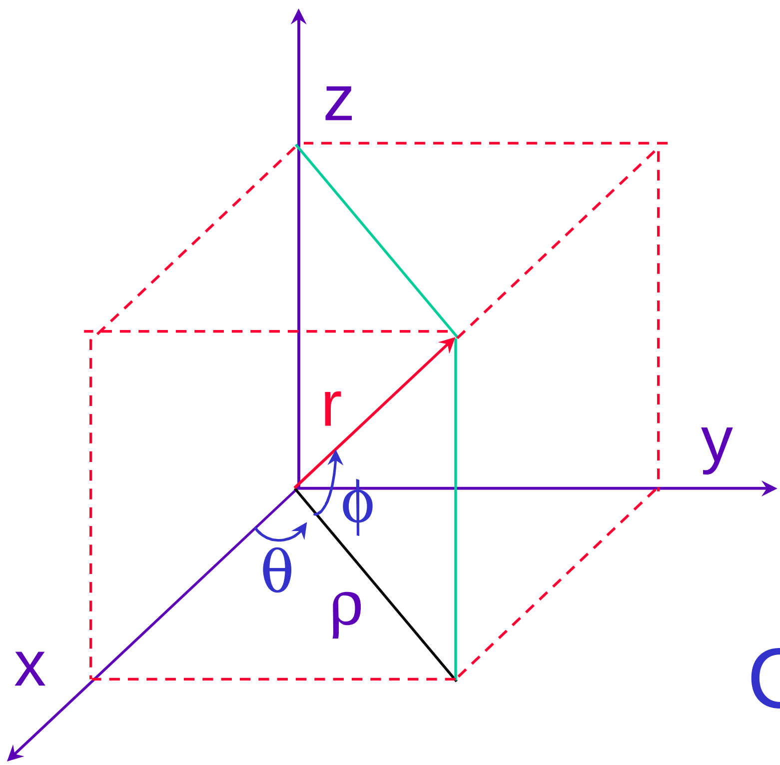
End-Effector Configuration Parameters

$$X = \begin{bmatrix} X_P \\ X_R \end{bmatrix}$$

X_P ← position

X_R ← orientation

Position Representations



Cartesian: (x, y, z)

Cylindrical: (ρ, θ, z)

Spherical: (r, θ, φ)

Rotation Representations

Rotation Matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix}$$

Direction Cosines

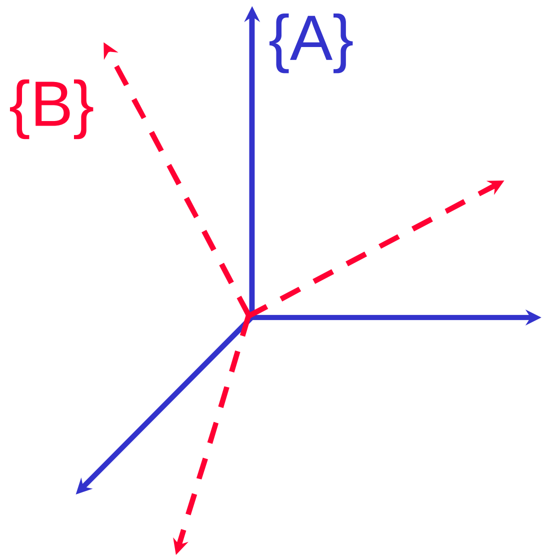
$$x_r = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}_{(9x1)}$$

Constraints

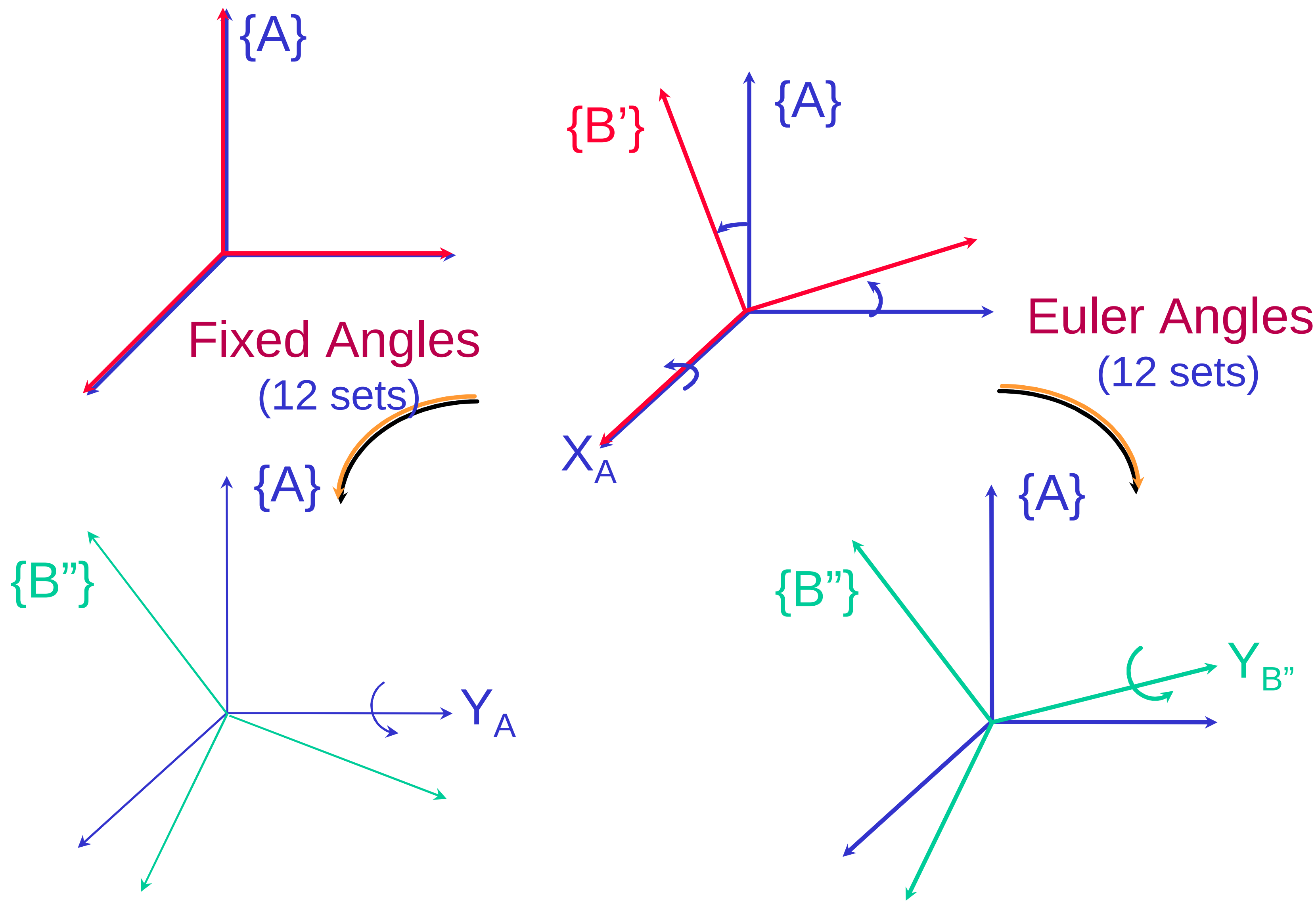
$$|\mathbf{r}_1| = |\mathbf{r}_2| = |\mathbf{r}_3| = 1$$

$$\mathbf{r}_1 . \mathbf{r}_2 = \mathbf{r}_1 . \mathbf{r}_3 = \mathbf{r}_2 . \mathbf{r}_3 = 0$$

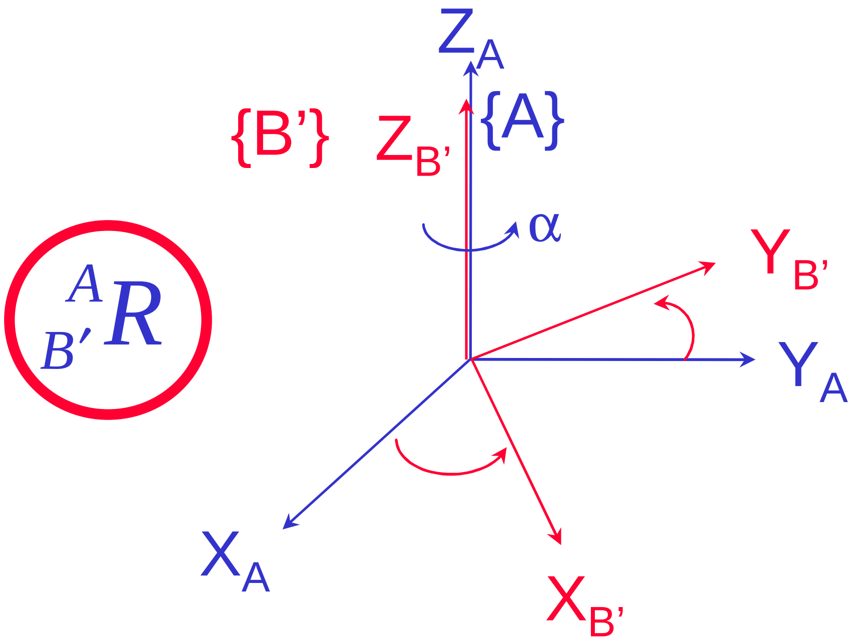
Three Angle Representations

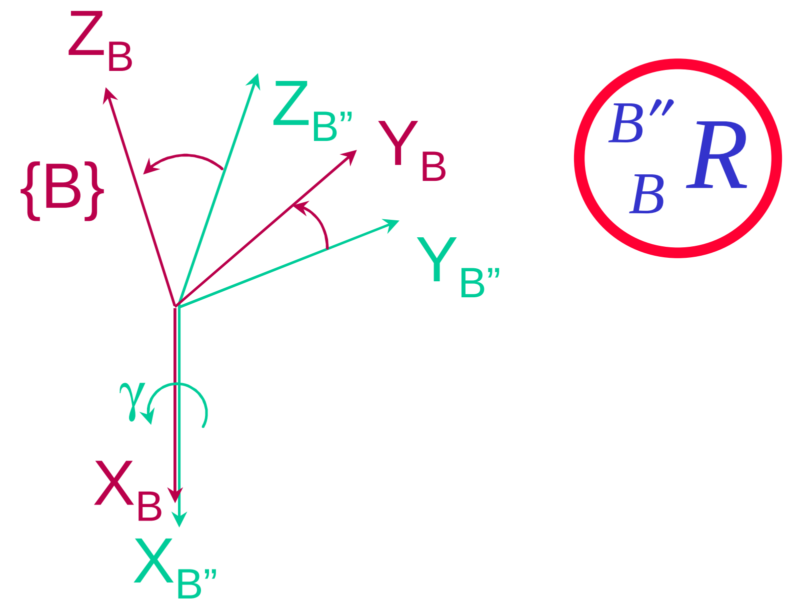


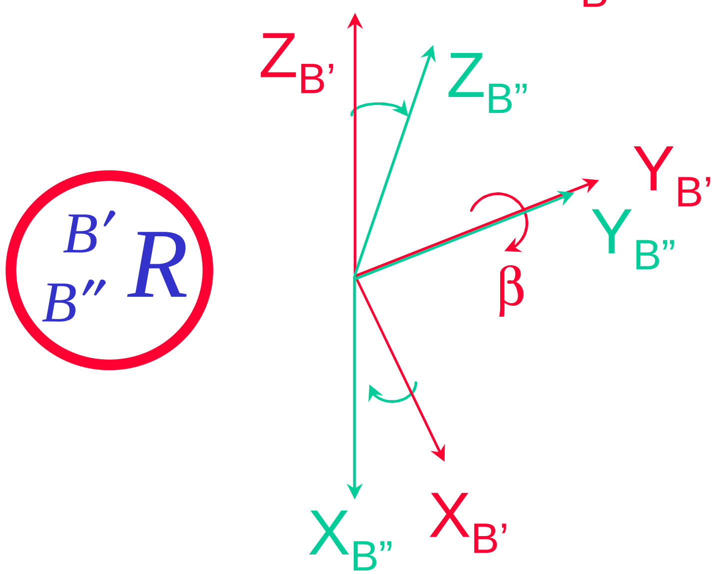
Three Angle Representations



Euler Angles (Z-Y-X)



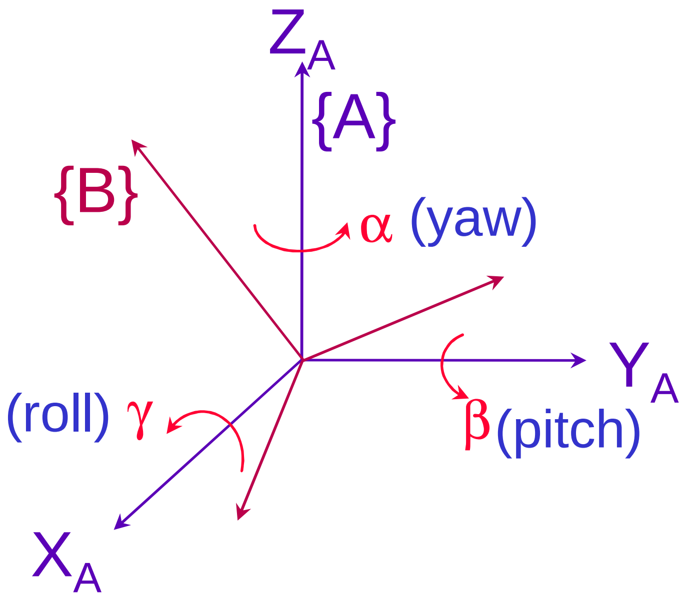




$${}^{A}_{B}R = {}^{A}_{B'}R . {}^{B'}_{B''}R . {}^{B''}_{B}R$$

$${}^{A}_{B}R = R_Z(\alpha) . R_Y(\beta) . R_X(\gamma)$$

X-Y-Z Fixed Angles



$$R_X(\gamma):\ v \rightarrow R_X(\gamma).v$$

$$R_Y(\beta):\ (R_X(\gamma).v) \rightarrow R_Y(\beta).(R_X(\gamma).v)$$

$$R_Z(\alpha):\ (R_Y(\beta).R_X(\gamma).v) \rightarrow R_Z(\alpha).(R_Y(\beta).R_X(\gamma).v)$$

$$\boxed{{}^A_BR = {}^A_BR_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)}$$

Z-Y-X Euler Angles

$$ {}^{A}_{B}R = R_{Z'}(\alpha).R_{Y'}(\beta).R_{X'}(\gamma) $$

$$ \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

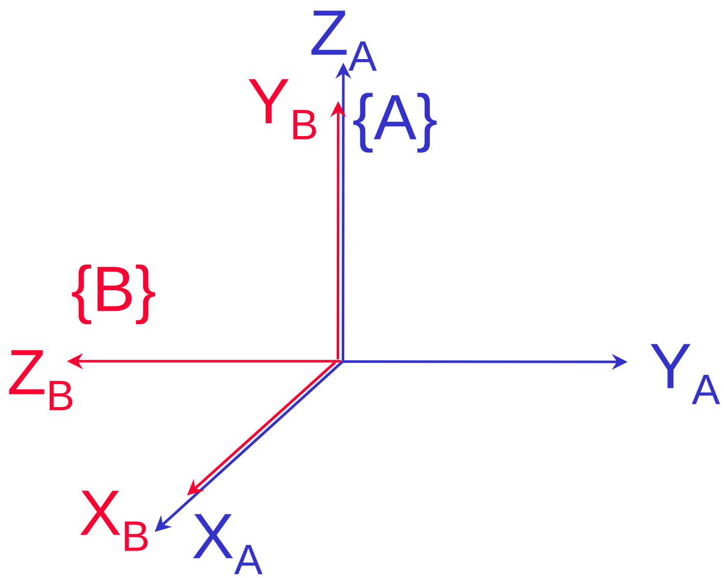
$$ {}^{A}_{B}R = {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha.c\beta & X & X \\ s\alpha.c\beta & X & X \\ -s\beta & c\beta.s\gamma & c\beta.c\gamma \end{bmatrix} $$

Z-Y-Z Euler Angles

$$ {}^{A}_{B}R = R_{Z'}(\alpha).R_{Y'}(\beta).R_{Z'}(\gamma) $$

$$ {}^{A}_{B}R = {}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} X & X & c\alpha.s\beta \\ X & X & s\alpha.s\beta \\ -s\beta.c\gamma & s\beta.s\gamma & c\beta \end{bmatrix} $$

Example



$$R_{Z'Y'X'}(\alpha,\beta,\gamma):$$

$$\alpha = 0$$

$$\beta = 0$$

$$\gamma = 90^o$$

Fixed & Euler Angles

X-Y-Z Fixed Angles

$$R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

Z-Y-X Euler Angles

$$R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

$$\boxed{R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_{XYZ}(\gamma,\beta,\alpha)}$$

Inverse Problem

Given ${}^{A}_{B}R$ find (α, β, γ)

$R_{Z'Y'X'}$

$$
{}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha.c\beta & c\alpha.s\beta.s\gamma - s\alpha.c\gamma & c\alpha.s\beta.c\gamma + s\alpha.s\gamma \\ s\alpha.c\beta & s\alpha.s\beta.s\gamma + c\alpha.c\gamma & s\alpha.s\beta.c\gamma - c\alpha.s\gamma \\ -s\beta & c\beta.s\gamma & c\beta.c\gamma \end{bmatrix}
$$

$$
\left.\begin{array}{c} \cos\beta = c\beta = \sqrt{r_{11}^2 + r_{21}^2} \\ \sin\beta = s\beta = -r_{31} \end{array}\right\} \rightarrow \beta = A\tan 2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})
$$

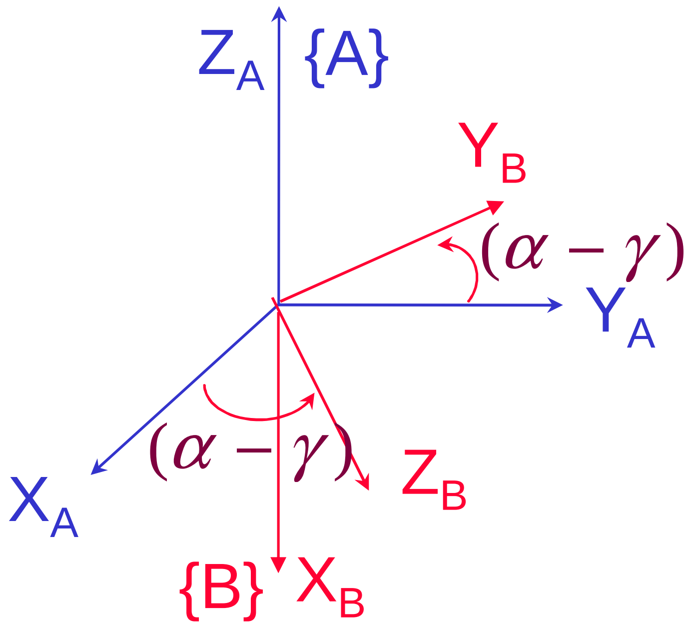
if $c\beta = 0$ $(\beta = \underline{+}90^o)$ ⇨ Singularity of the representation

⇨ Only $(\alpha + \gamma)$ or $(\alpha - \gamma)$ is defined

Singularities - Example $(R_{Z'Y'X'})$

$c\beta = 0,\ s\beta = +1$

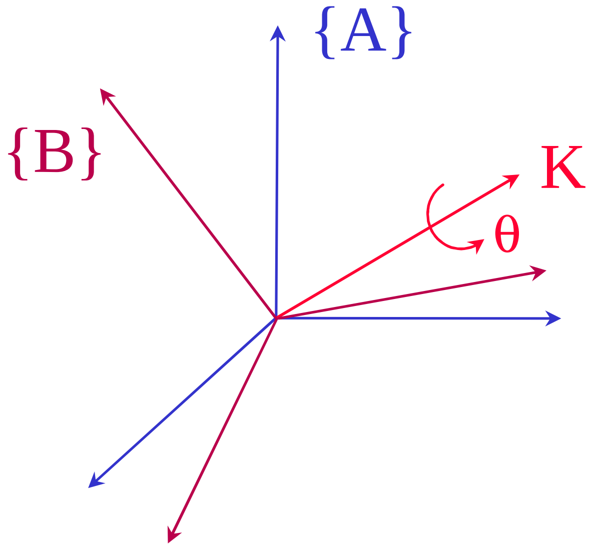
$$ {}^{A}_{B}R = \begin{pmatrix} 0 & -s(\alpha-\gamma) & c(\alpha-\gamma) \\ 0 & c(\alpha-\gamma) & s(\alpha-\gamma) \\ -1 & 0 & 0 \end{pmatrix} $$



$c\beta = 0,\ s\beta = -1$

$$ {}^{A}_{B}R = \begin{pmatrix} 0 & -s(\alpha+\gamma) & -c(\alpha+\gamma) \\ 0 & c(\alpha+\gamma) & -s(\alpha+\gamma) \\ 1 & 0 & 0 \end{pmatrix} $$

Equivalent angle-axis representation, $R_K(\theta)$



$$X_r = \theta.K = \begin{bmatrix} \theta.k_X \\ \theta.k_Y \\ \theta.k_Z \end{bmatrix}$$

$$R_K(\theta) = \begin{bmatrix} k_x.k_x.v\theta + c\theta & k_x.k_y.v\theta - k_z.s\theta & k_x.k_z.v\theta + k_y.s\theta \\ k_x.k_y.v\theta + k_z.s\theta & k_y.k_y.v\theta + c\theta & k_y.k_z.v\theta - k_x.s\theta \\ k_x.k_z.v\theta - k_y.s\theta & k_y.k_z.v\theta + k_x.s\theta & k_z.k_z.v\theta + c\theta \end{bmatrix}$$

with $v\theta = 1 - c\theta$

$$R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\theta = Ar\cos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$${}^{A}K = \frac{1}{2.\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix},$$

singularity for $\sin\theta = 0$

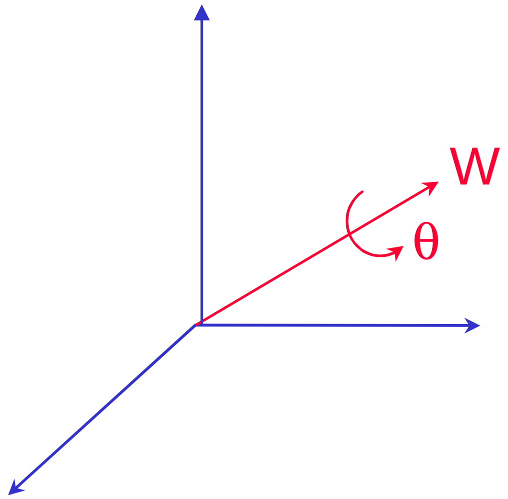
Euler Parameters

$$\varepsilon_1 = W_x . \sin\frac{\theta}{2}$$

$$\varepsilon_2 = W_y . \sin\frac{\theta}{2}$$

$$\varepsilon_3 = W_z . \sin\frac{\theta}{2}$$

$$\varepsilon_4 = \cos\frac{\theta}{2}$$



Normality Condition

$$|W| = 1, \quad \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$$

ε : point on a unit hypershpere in four-dimensional space

Inverse Problem Given ${}^A_B R$ find ε

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \equiv \begin{bmatrix} 1-2\varepsilon_2^2-2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_2^2 \end{bmatrix}$$

$$r_{11}+r_{22}+r_{33} = 3-4(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2)$$

$$(1-\varepsilon_4^2)$$

$$\varepsilon_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}$$

$$\varepsilon_1 = \frac{r_{32}-r_{23}}{4\varepsilon_4}, \quad \varepsilon_2 = \frac{r_{13}-r_{31}}{4\varepsilon_4}, \quad \varepsilon_3 = \frac{r_{21}-r_{12}}{4\varepsilon_4}$$

$$\underline{\underline{\varepsilon_4 = 0?}}$$

Lemma For all rotations one of the Euler Parameters is greater than or equal to 1/2

$$\left(\sum_{1}^{4} \varepsilon_i^2 = 1\right)$$

Algorithm Solve with respect to $\max_i \{\varepsilon_i\}$

- $\varepsilon_1 = \max_i \{\varepsilon_i\}$

$$\varepsilon_1 = \frac{1}{2}\sqrt{r_{11} - r_{22} - r_{33} + 1}$$

$$\varepsilon_2 = \frac{(r_{21} + r_{12})}{4\varepsilon_1}, \quad \varepsilon_3 = \frac{(r_{31} + r_{13})}{4\varepsilon_1}, \quad \varepsilon_4 = \frac{(r_{32} - r_{23})}{4\varepsilon_1}$$

- $\varepsilon_1 = \max_i \{\varepsilon_i\}$

$$\varepsilon_1 = \frac{1}{2}\sqrt{r_{11} - r_{22} - r_{33} + 1}$$

- $\varepsilon_2 = \max_i \{\varepsilon_i\}$

$$\varepsilon_2 = \frac{1}{2}\sqrt{-r_{11} + r_{22} - r_{33} + 1}$$

- $\varepsilon_3 = \max_i \{\varepsilon_i\}$

$$\varepsilon_3 = \frac{1}{2}\sqrt{-r_{11} - r_{22} + r_{33} + 1}$$

- $\varepsilon_4 = \max_i \{\varepsilon_i\}$

$$\varepsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

Euler Parameters / Euler Angles

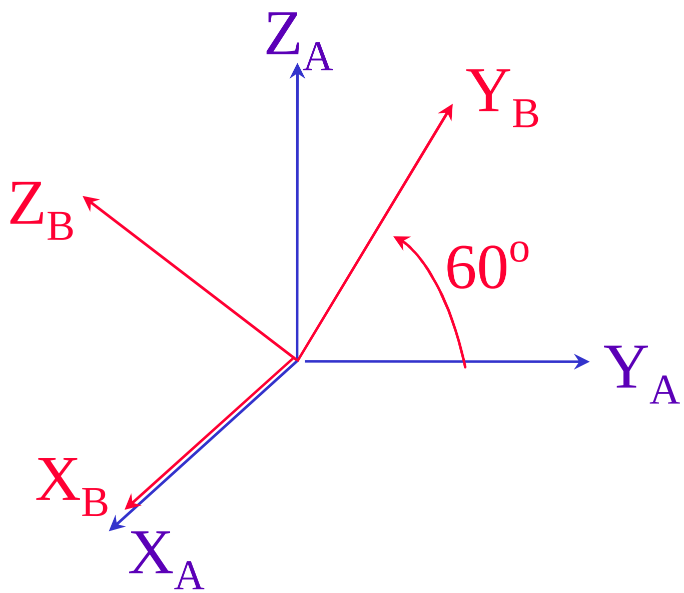
$$\varepsilon_1 = \sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2}$$

$$\varepsilon_2 = \sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2}$$

$$\varepsilon_3 = \cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}$$

$$\varepsilon_4 = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}$$

Quiz



Direction Cosines

$$x_r = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ \sqrt{3}/2 \\ 0 \\ -\sqrt{3}/2 \\ 1/2 \end{bmatrix} \begin{matrix} r_1 \\ \\ \\ r_2 \\ \\ \\ r_3 \end{matrix}$$

Euler Parameters

$$x_r = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ \sqrt{3}/2 \end{bmatrix}$$