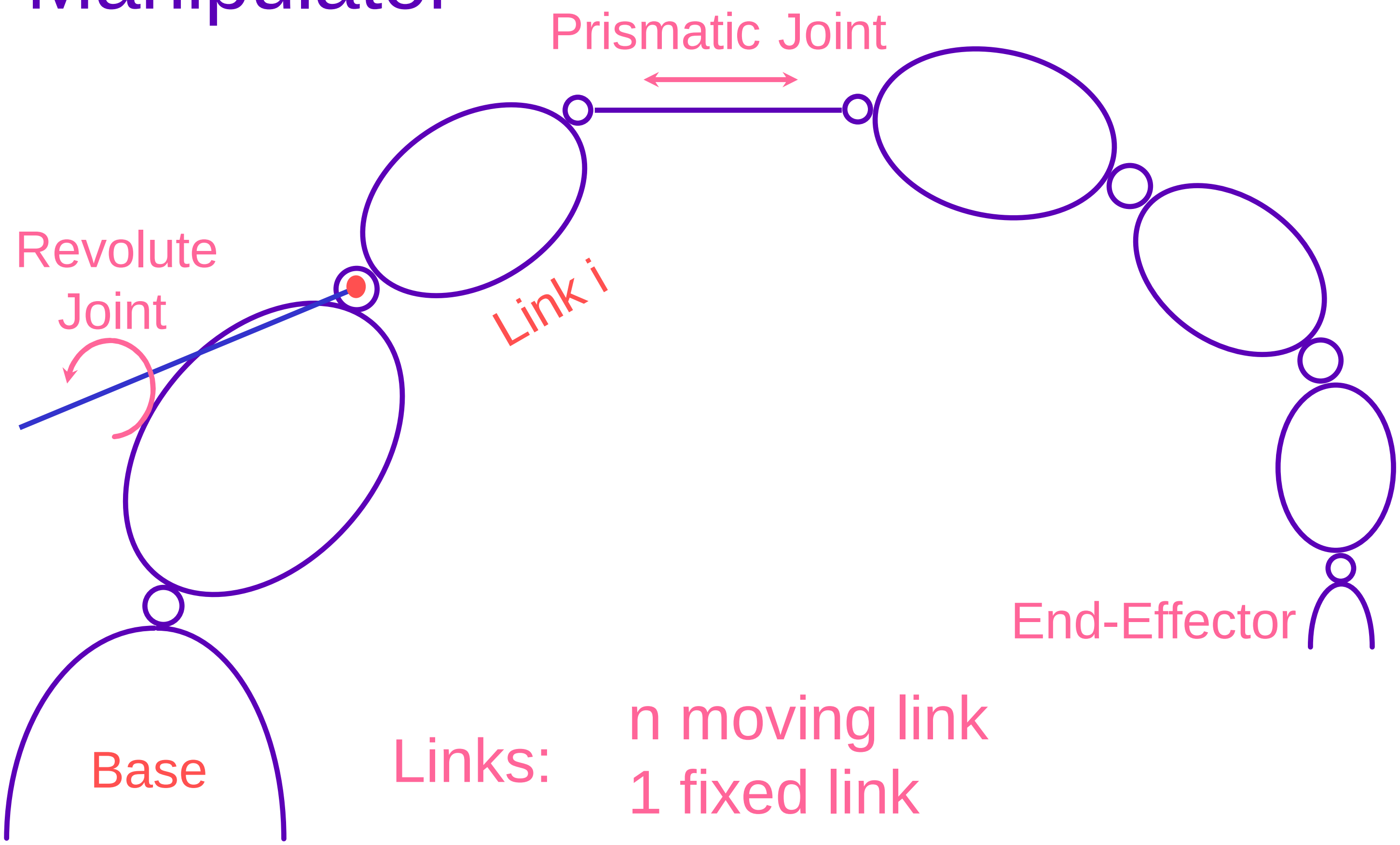
Kinematics

Spatial Descriptions

- Task Description
- Transformations
- Representations

Manipulator

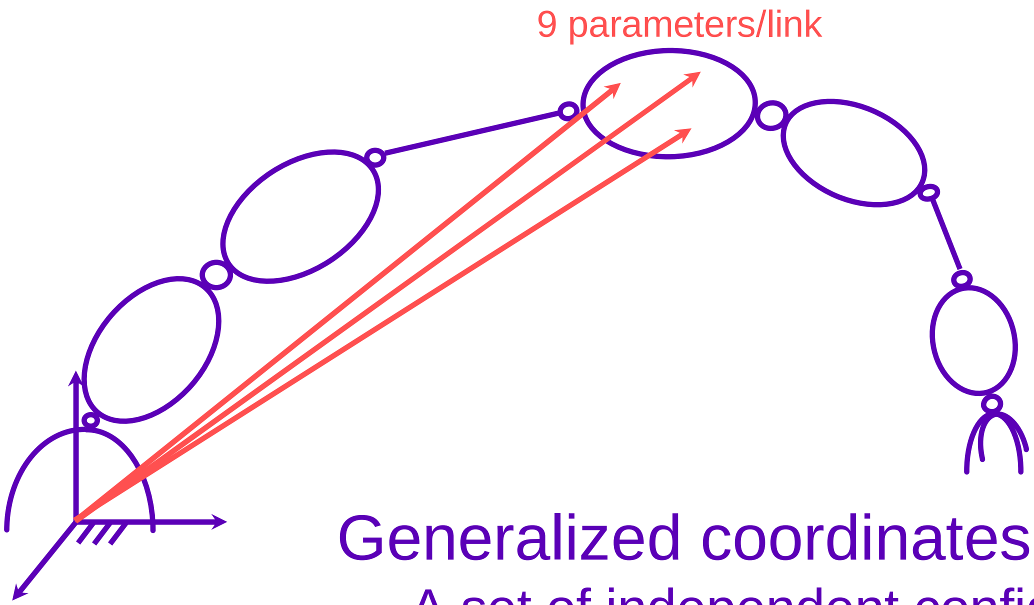


Links: n moving link
1 fixed link

Joints: Revolute (1 DOF)
Prismatic (1 DOF)

Configuration Parameters

A set of position parameters that describes
the full configuration of the system.



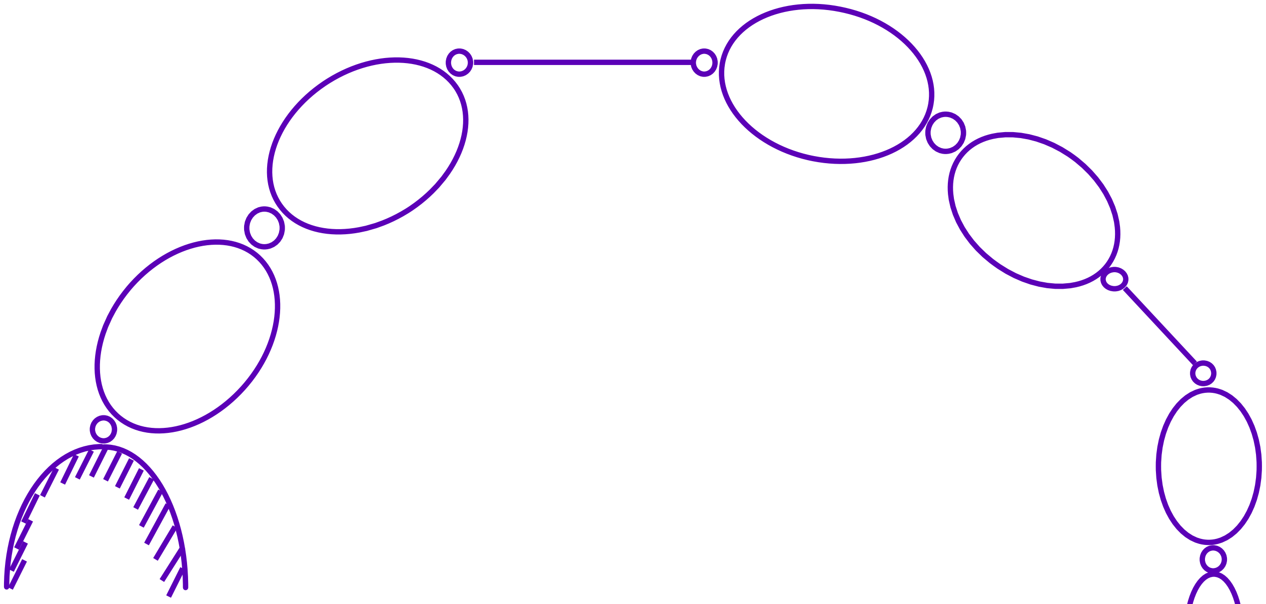
Generalized coordinates

A set of independent configuration parameters

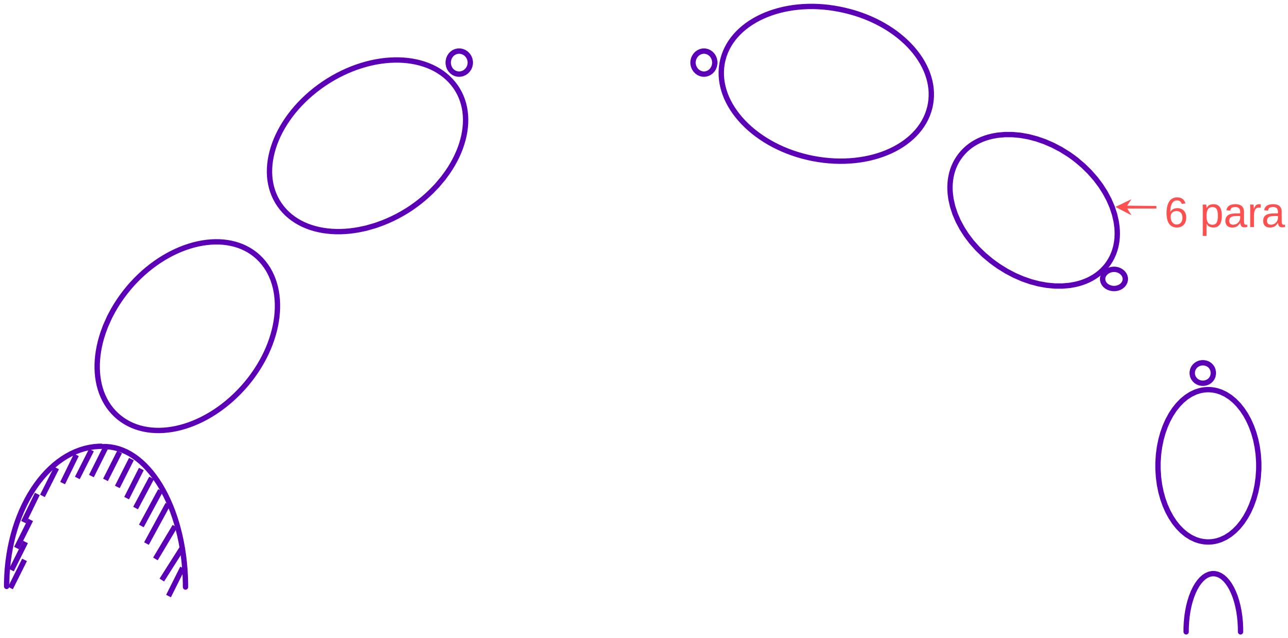
Degrees of Freedom

Number of generalized coordinates

Generalized Coordinates



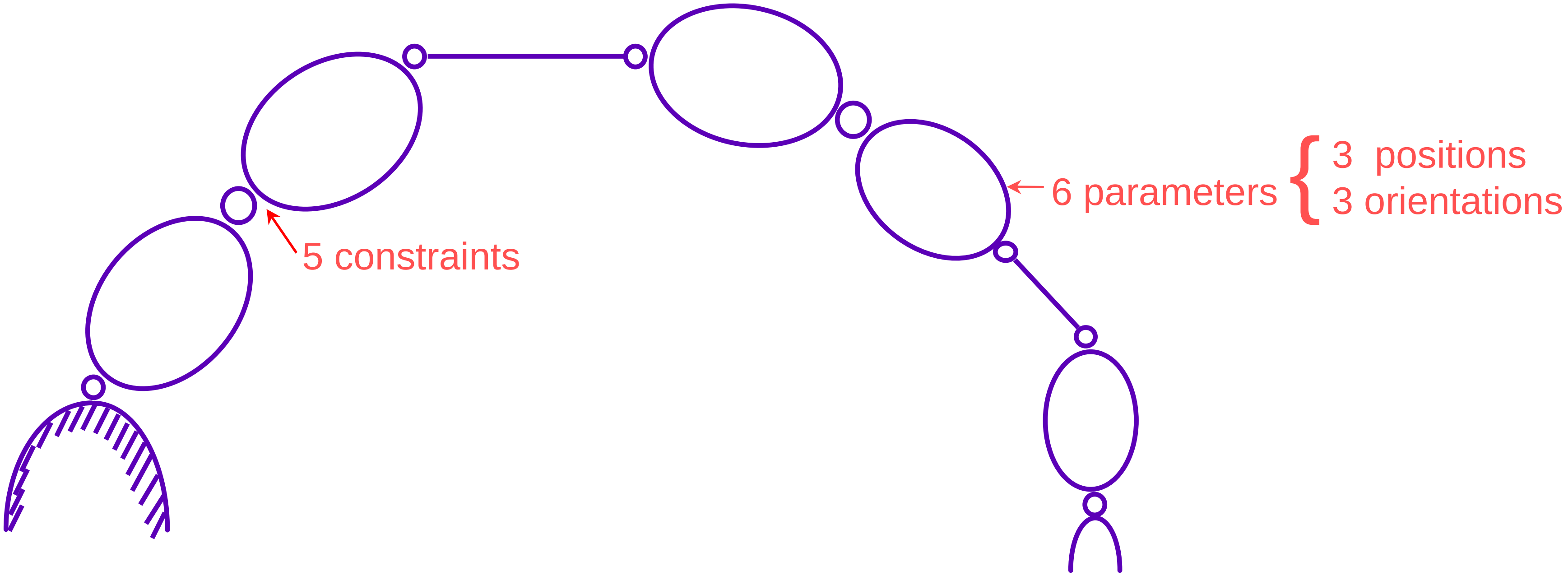
Generalized Coordinates



← 6 parameters $\left\{\begin{array}{l}\text{3 positions}\\ \text{3 orientations}\end{array}\right.$

n moving links: 6n parameters

Generalized Coordinates



5 constraints

6 parameters

3 positions

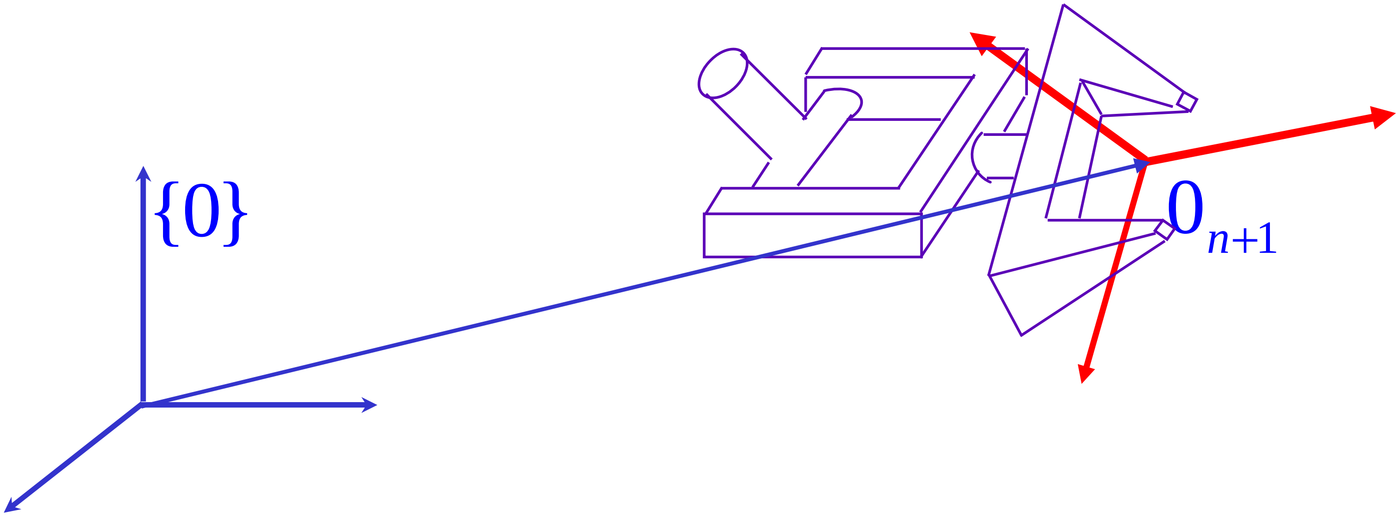
3 orientations

n moving links: 6n parameters

n 1 d.o.f. joints: 5n constraints

d.o.f. (system): 6n - 5n = n

End-Efector Configuration Parameters



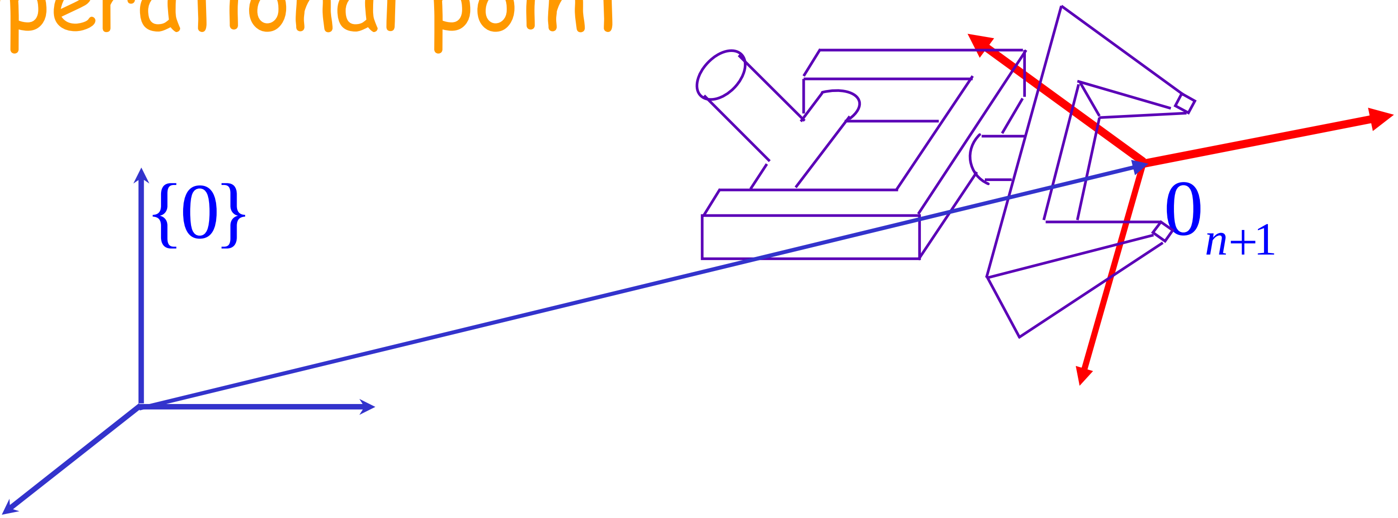
A set of m parameters:

$$(x_1, x_2, x_3, \ldots, x_m)$$

that completely specifies the end-effector position and orientation with respect to $\{0\}$

Operational Coordinates

O_{n+1} : Operational point

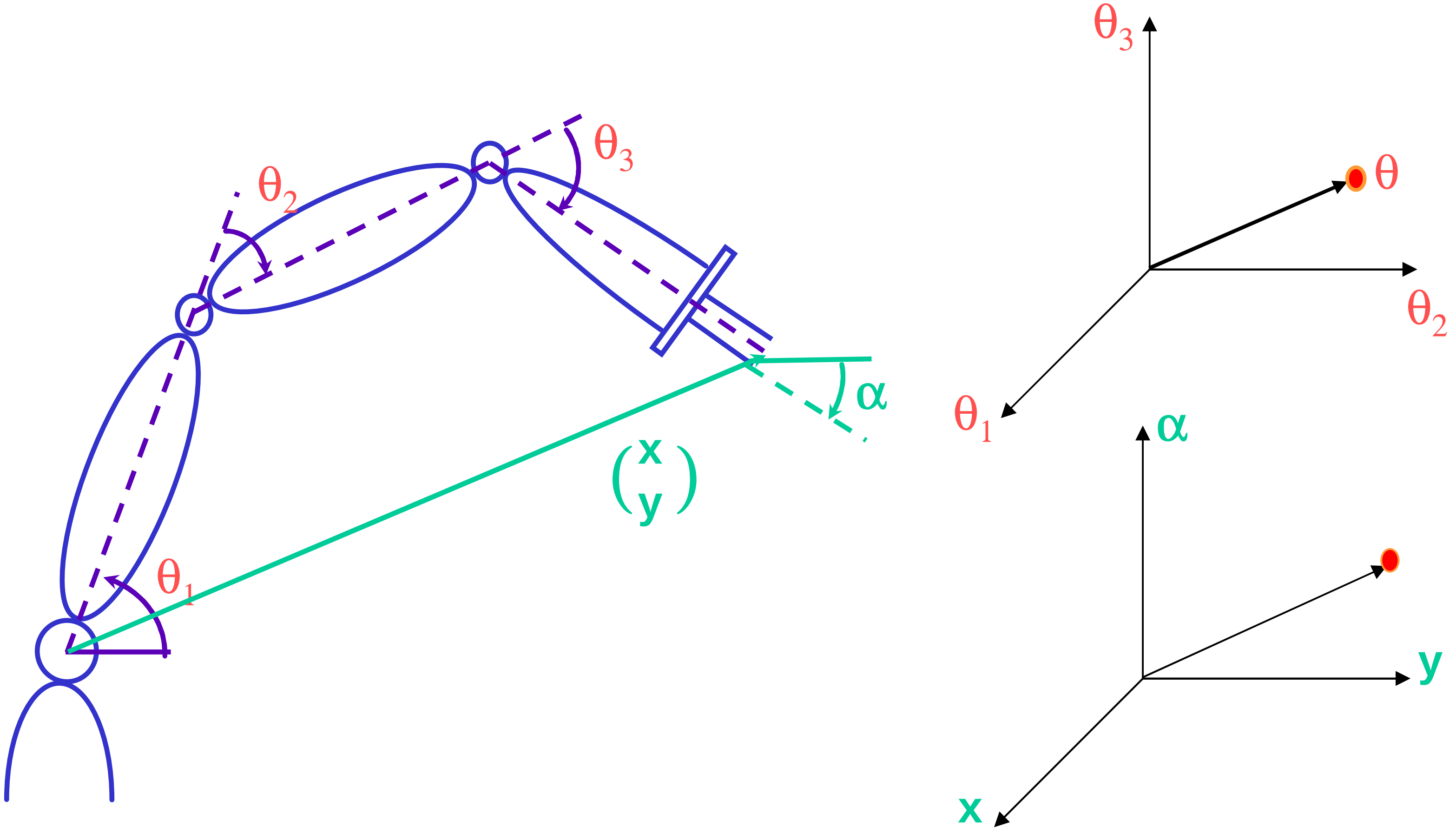


A set $x_1, x_2, \ldots, x_{m_0}$

of m_0 independent configuration parameters

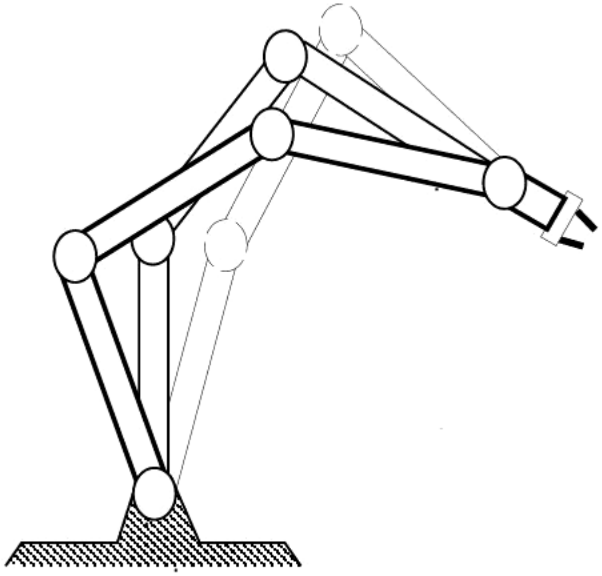
m_0 : number of degrees of freedom of the end-effector.

Joint Coordinates → Joint Space



Operational Coordinates → Operational Space

Redundancy

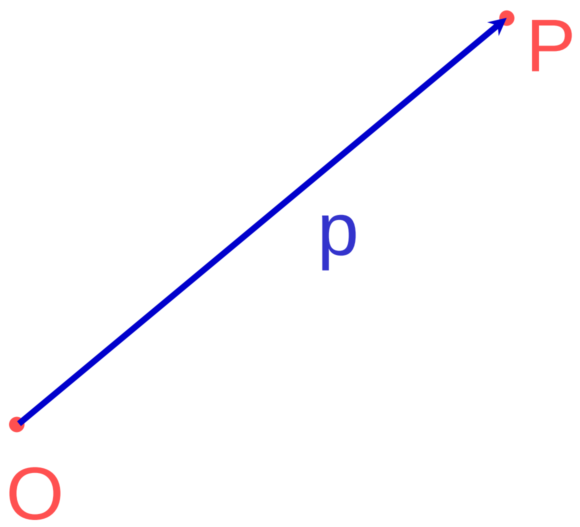


A robot is said to be redundant if

$$\boxed{n > m_0}$$

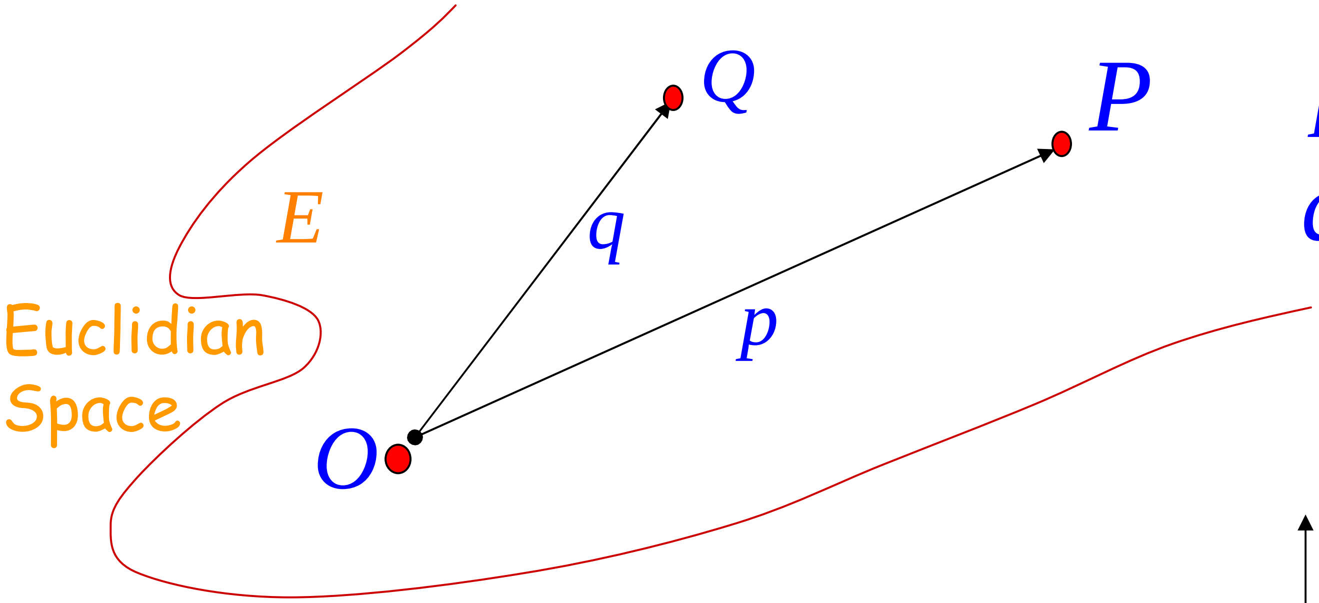
Degrees of redundancy: $n - m_0$

Position of a Point



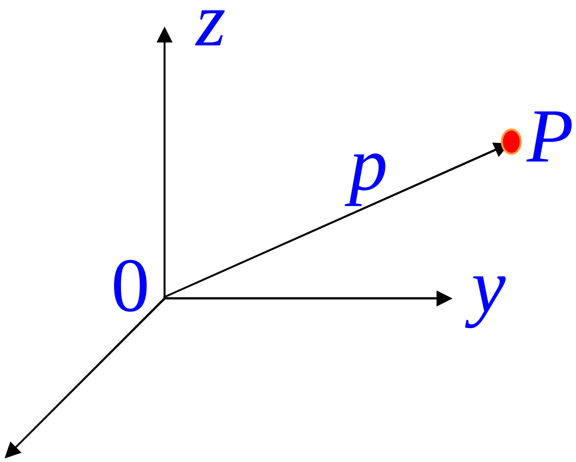
With respect to a fixed origin O, the position of a point P is described by the vector OP or simply by p.

Rigid Body Configuration

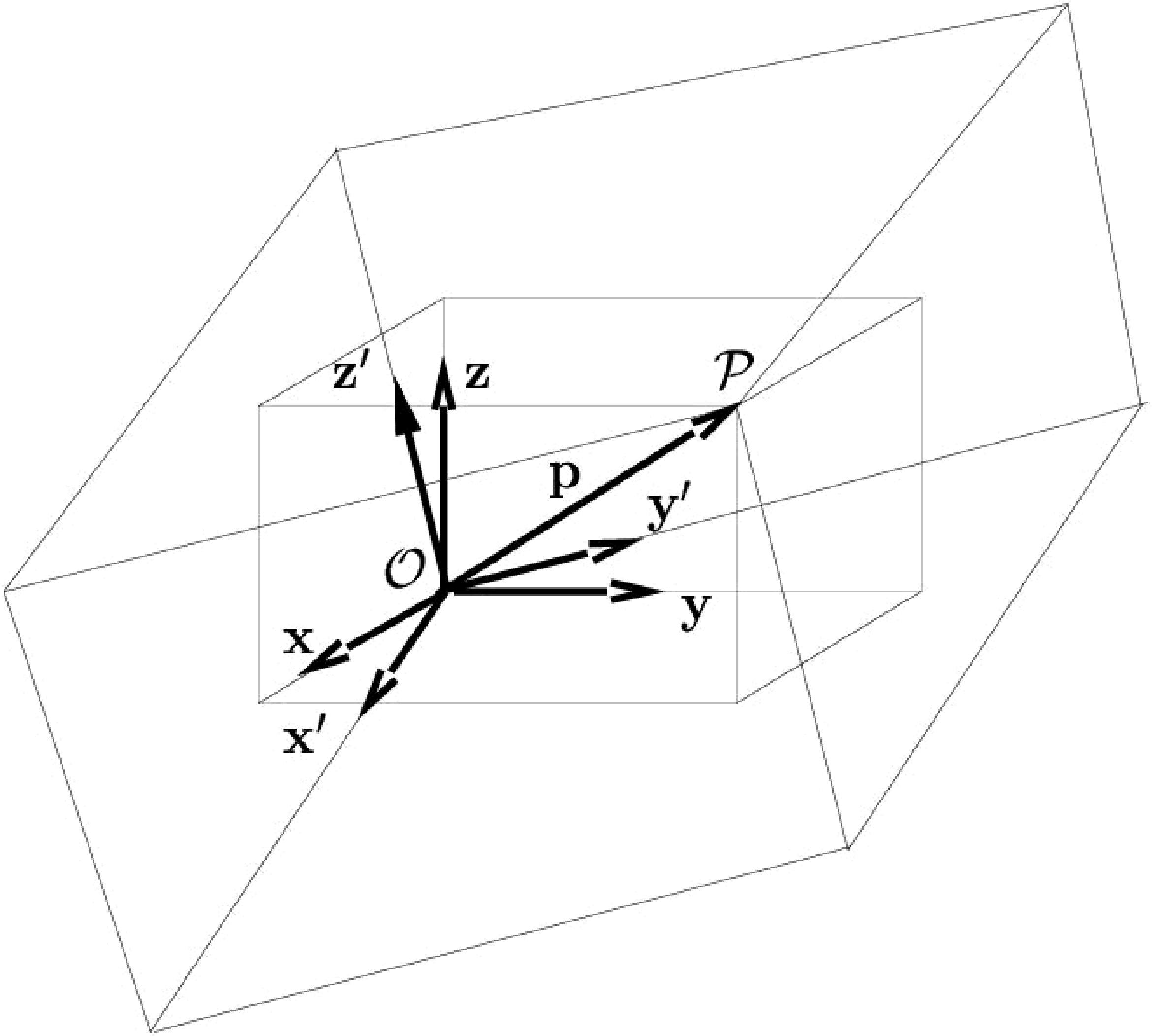


$$p = \overrightarrow{OP}$$
$$q = \overrightarrow{OQ}$$

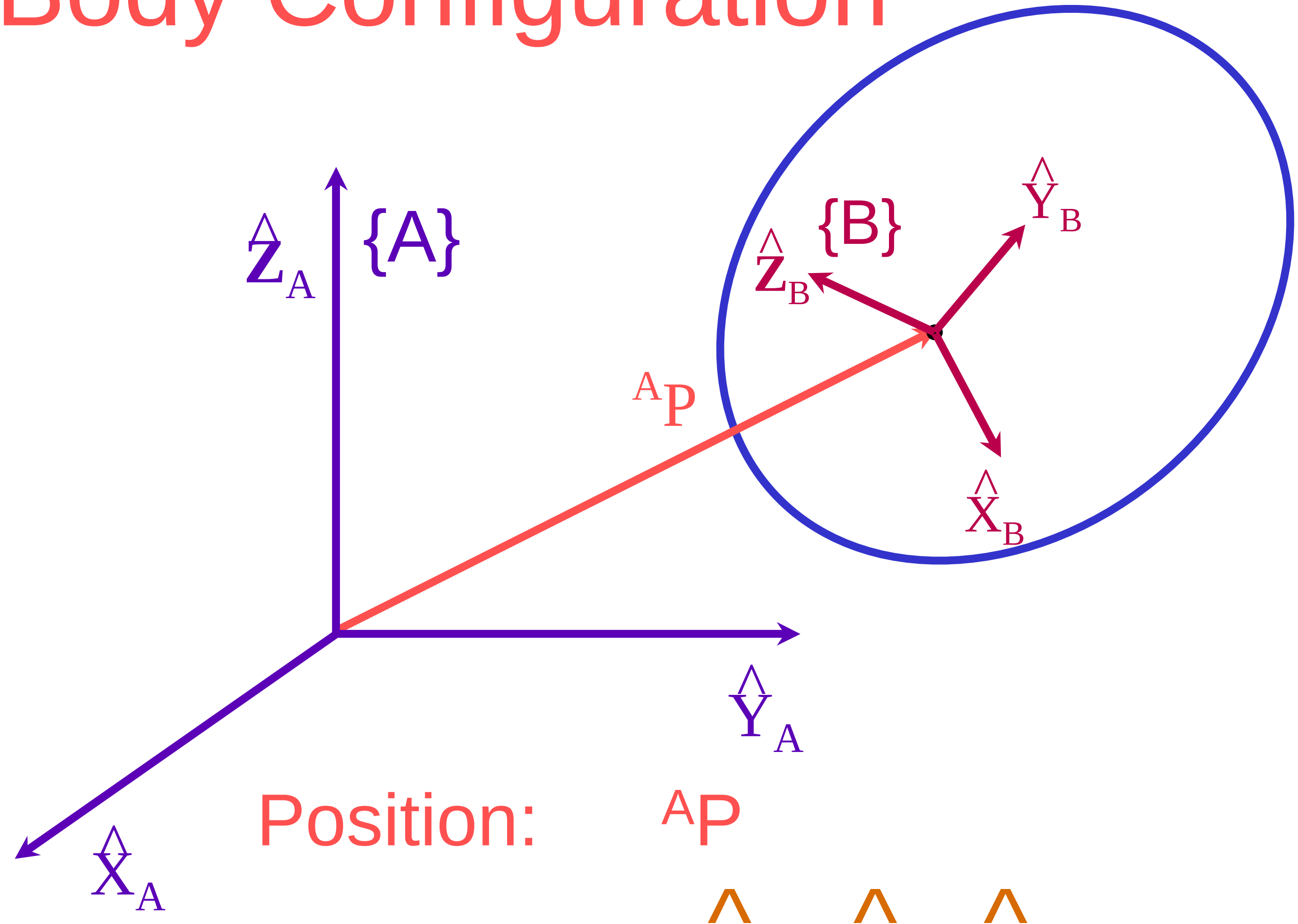
Cartesian Frame



Coordinate Frames



Rigid Body Configuration



Position: ${}^{A}P$

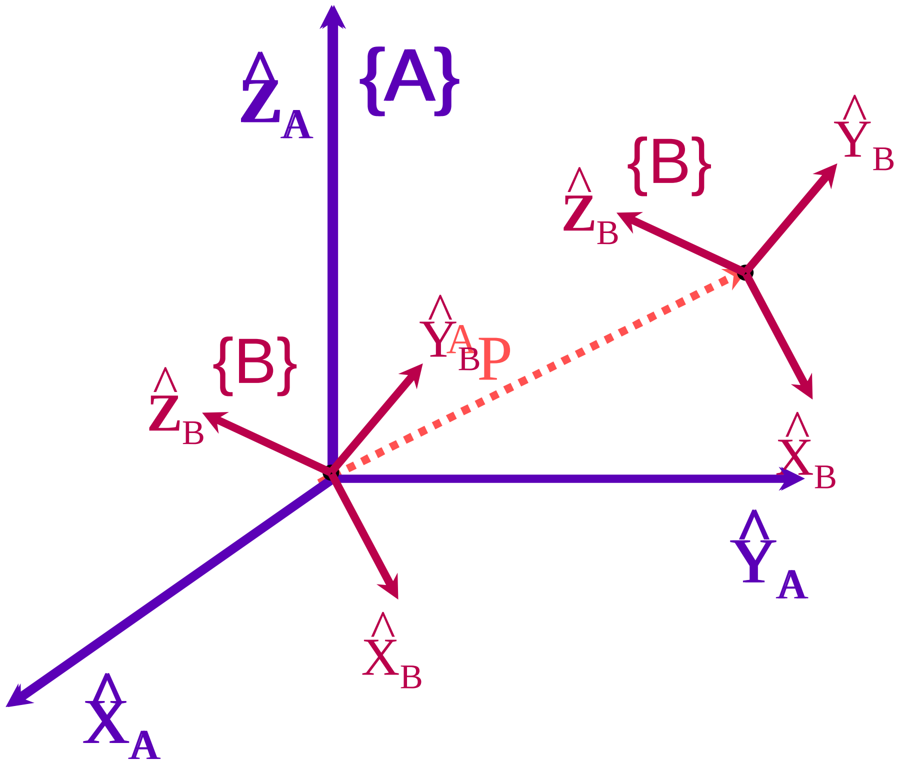
Orientation: $\{{}^{A}\hat{X}_B, {}^{A}\hat{Y}_B, {}^{A}\hat{Z}_B\}$

Describes rotations of {B} with respect to {A}

⇒ **Rotation Matrix**

Rotation Matrix

$${}^A_B R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$



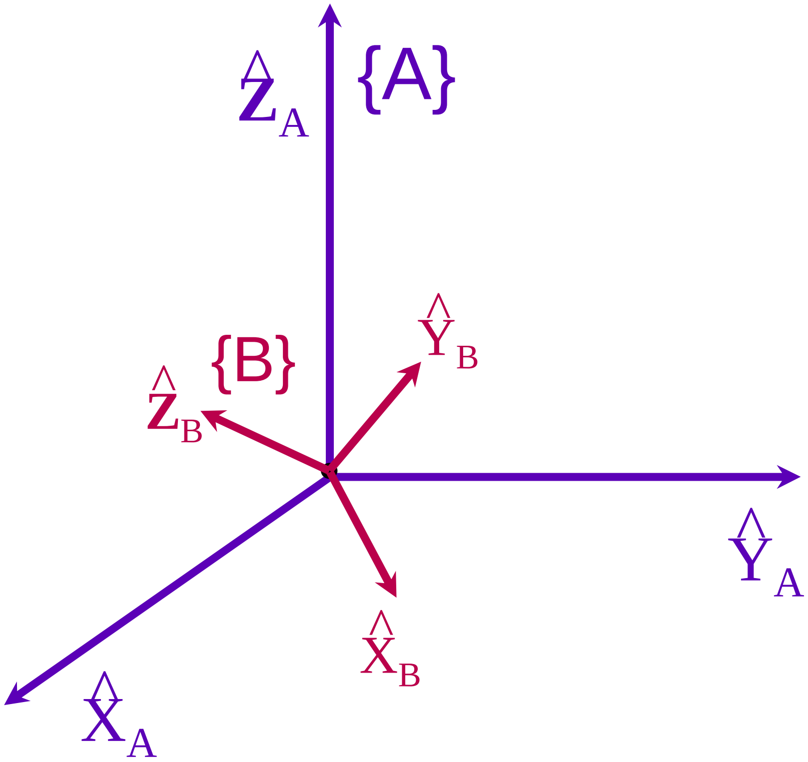
$${}^A\hat{X}_B = {}^A_B R \; {}^B\hat{X}_B$$

$${}^A\hat{X}_B = {}^A_B R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad {}^A\hat{Y}_B = {}^A_B R \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad {}^A\hat{Z}_B = {}^A_B R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$${}^A_B R = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix}$$

Rotation Matrix

$${}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix}$$



Dot Product

$${}^{A}\hat{X}_B = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A \end{bmatrix}$$

$${}^{A}_{B}R = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}$$

$${}^{B}X_A^{T}$$

Rotation Matrix

$${}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}\hat{X}_{A}^{T} \\ {}^{B}\hat{Y}_{A}^{T} \\ {}^{B}\hat{Z}_{A}^{T} \end{bmatrix} = \begin{bmatrix} {}^{B}\hat{X}_{A} & {}^{B}\hat{Y}_{A} & {}^{B}\hat{Z}_{A} \end{bmatrix}^{T} = {}^{B}_{A}R^{T}$$

$$\underline{\underline{{}^{A}_{B}R = {}^{B}_{A}R^{T}}}$$

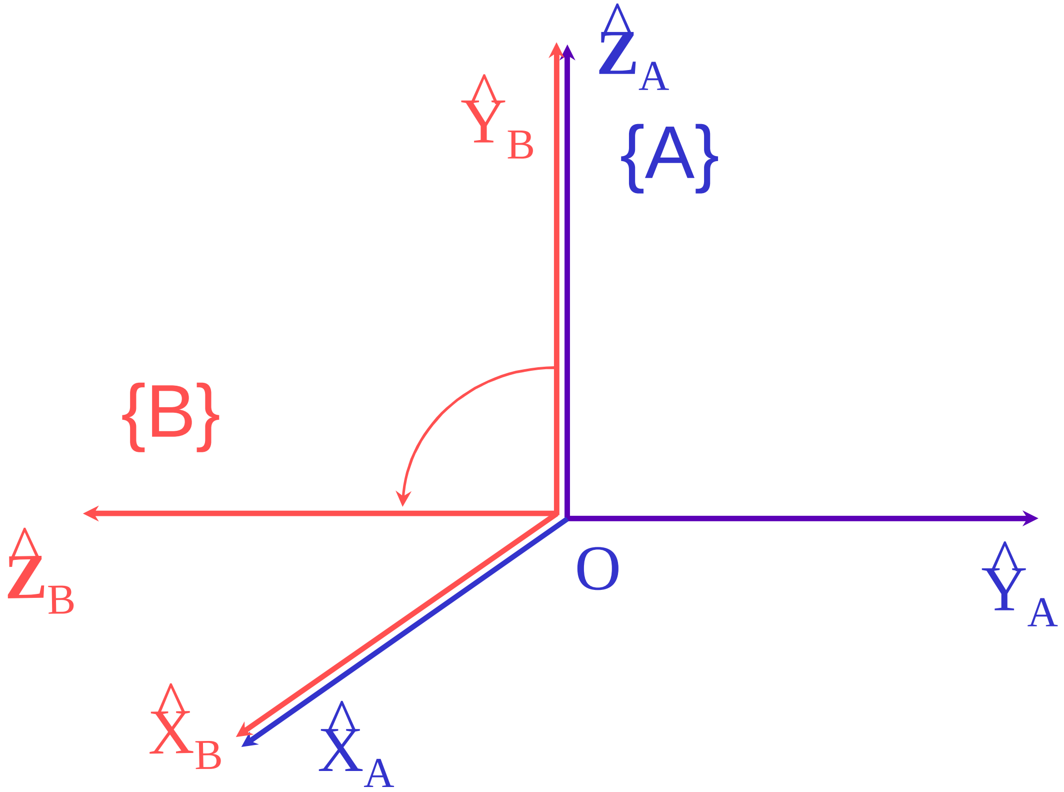
Inverse of Rotation Matrices

$${}^{A}_{B}R^{-1} = {}^{B}_{A}R = {}^{A}_{B}R^{T}$$

$$\boxed{{}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T}}$$

Orthonormal Matrix

Example

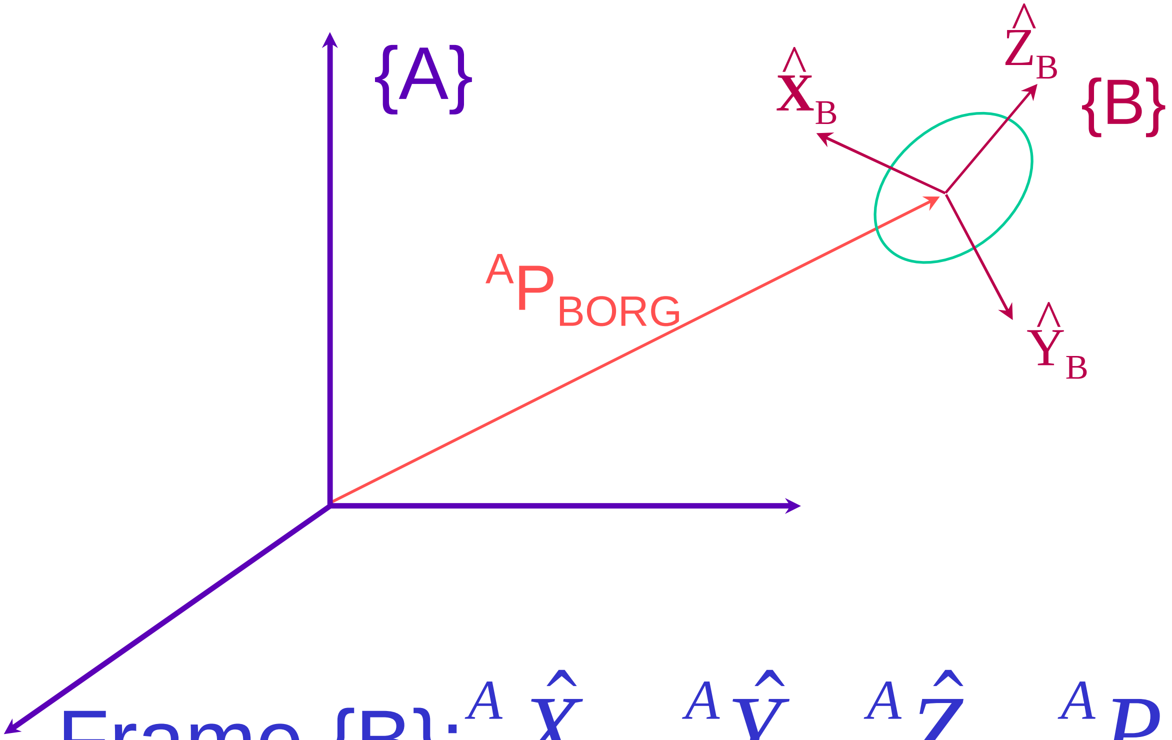


$$
{}^{A}_{B}R = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right)
\begin{array}{l} \leftarrow {}^{B}\hat{X}_{A}^{T} \\ \leftarrow {}^{B}\hat{Y}_{A}^{T} \\ \leftarrow {}^{B}\hat{Z}_{A}^{T} \end{array}
$$

$$
\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{array}
$$

Description of a Frame

with respect to reference frame



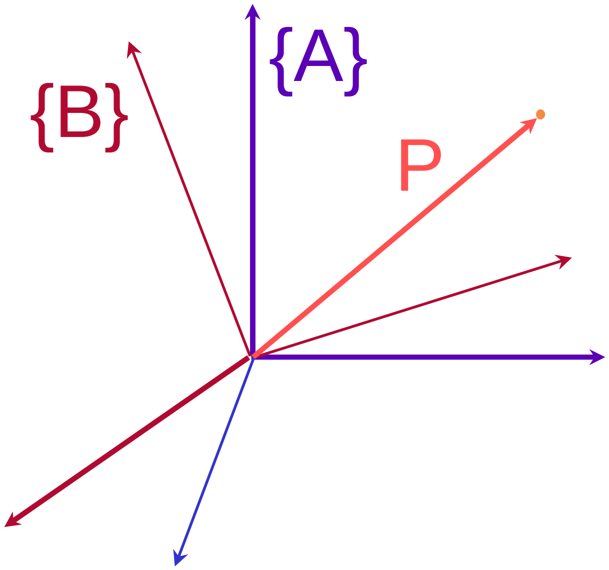
Frame {B}: $^A\hat{X}_B, {}^A\hat{Y}_B, {}^A\hat{Z}_B, {}^AP_{Borg}$

$$\{B\} = \left\{ {}^A_B R \quad {}^A P_{Borg} \right\}$$

Mapping

changing descriptions from frame to frame

Rotations



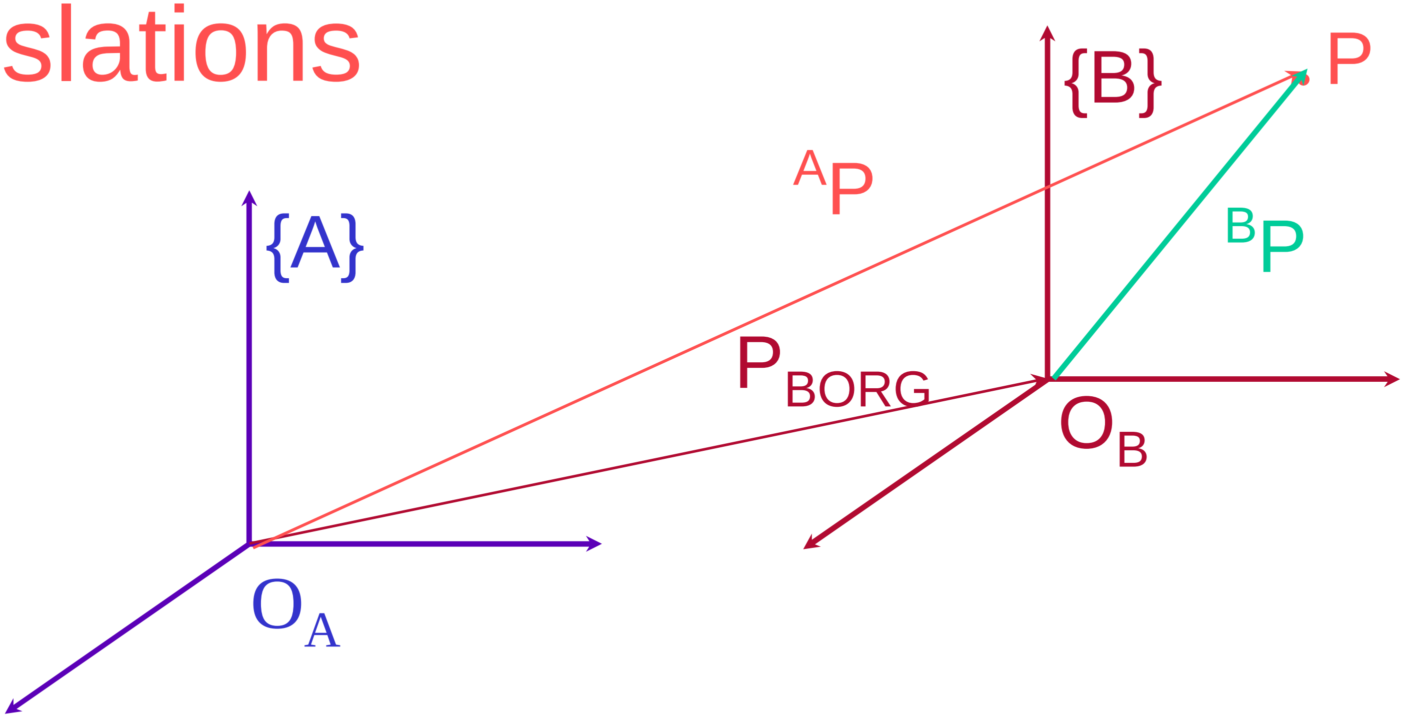
If P is given in {B}: ^{B}P

$$^{A}P = \begin{pmatrix} {}^{B}\hat{X}_{A} \cdot {}^{B}P \\ {}^{B}\hat{Y}_{A} \cdot {}^{B}P \\ {}^{B}\hat{Z}_{A} \cdot {}^{B}P \end{pmatrix} = \begin{pmatrix} {}^{B}\hat{X}_{A}^{T} \\ {}^{B}\hat{Y}_{A}^{T} \\ {}^{B}\hat{Z}_{A}^{T} \end{pmatrix} \cdot {}^{B}P$$

⇩

$$^{A}P = {}^{A}_{B}R \; {}^{B}P$$

Translations



changing the position description of a point P

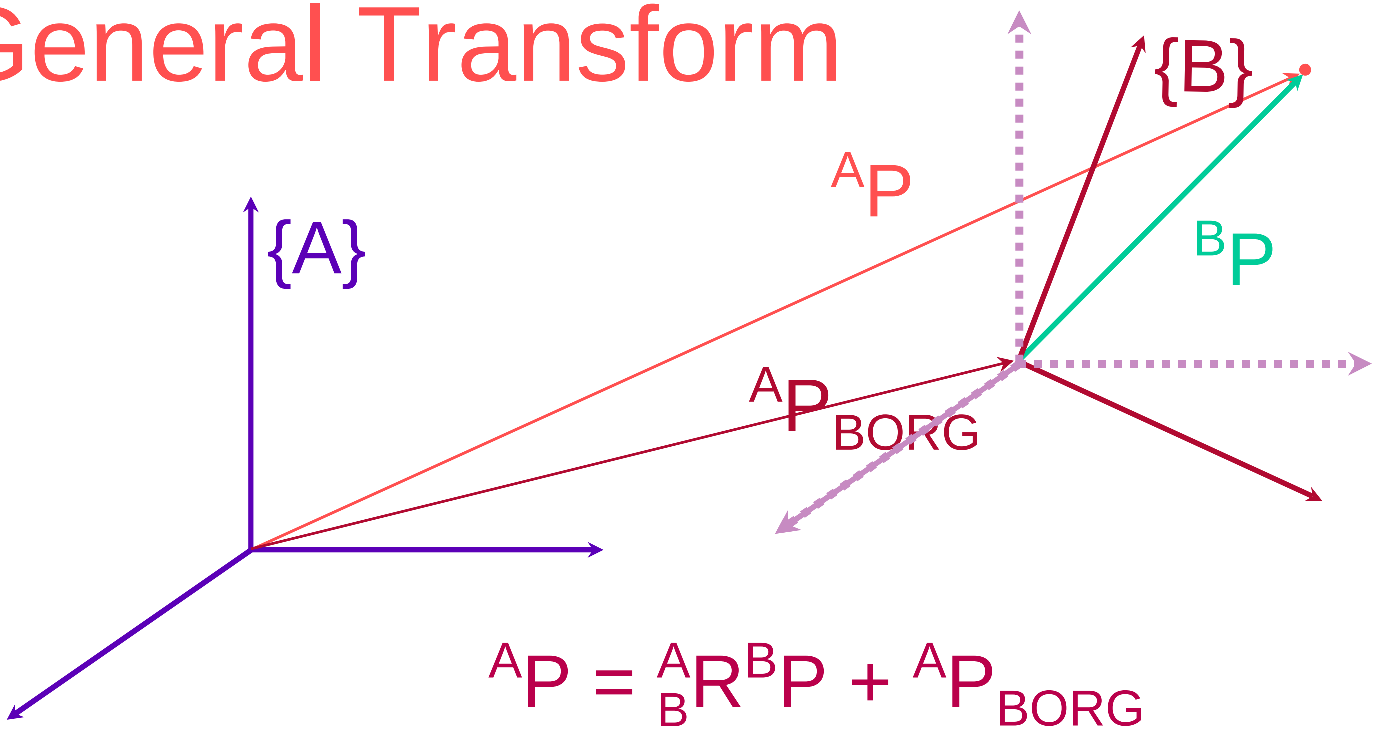
$$\overrightarrow{O_BP} \Longrightarrow \overrightarrow{O_AP}$$

(Two different vectors)

$$P_{BORG} : \quad P_{O_B} \Longrightarrow P_{O_A}$$

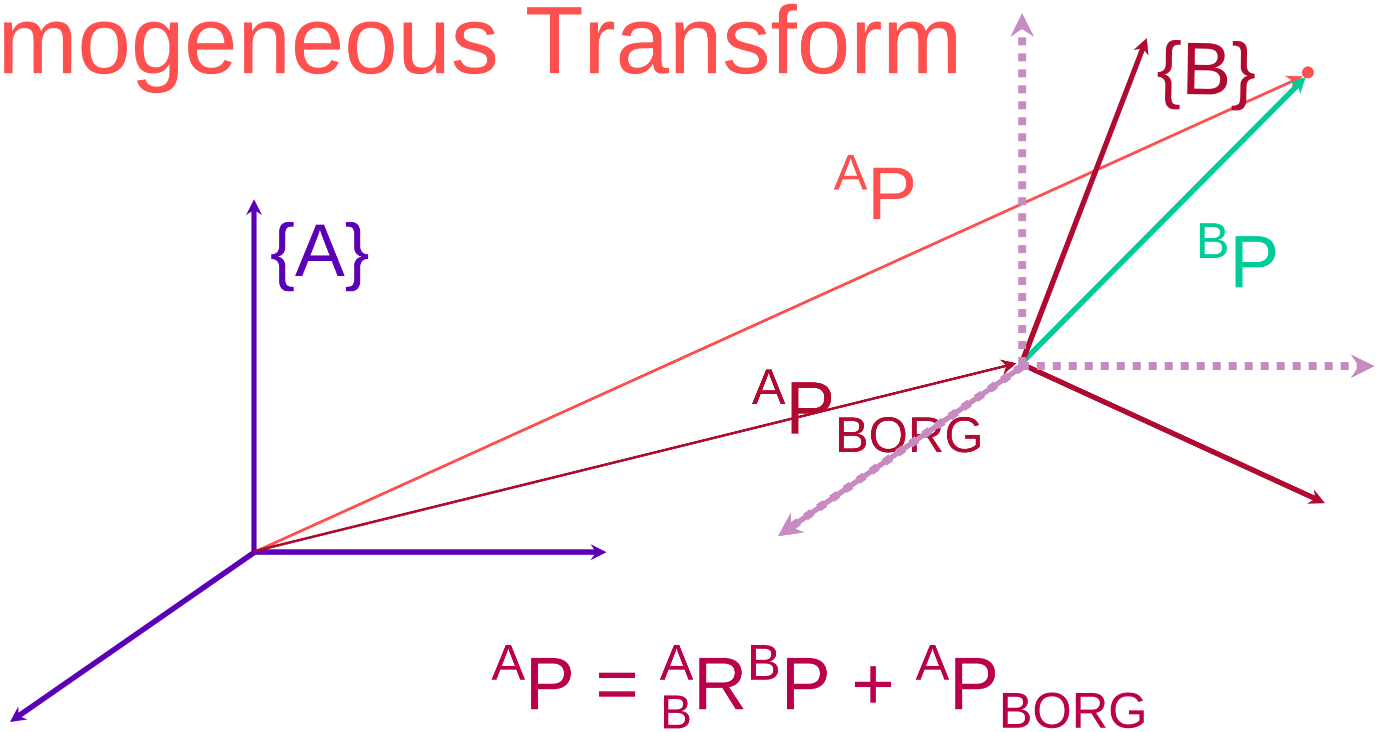
$$\boxed{{}^AP_{O_A} = \overset{\overset{B}{\shortparallel}}{{}^AP_{O_B}} + {}^AP_{BORG}}$$

General Transform



$$^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$$

Homogeneous Transform

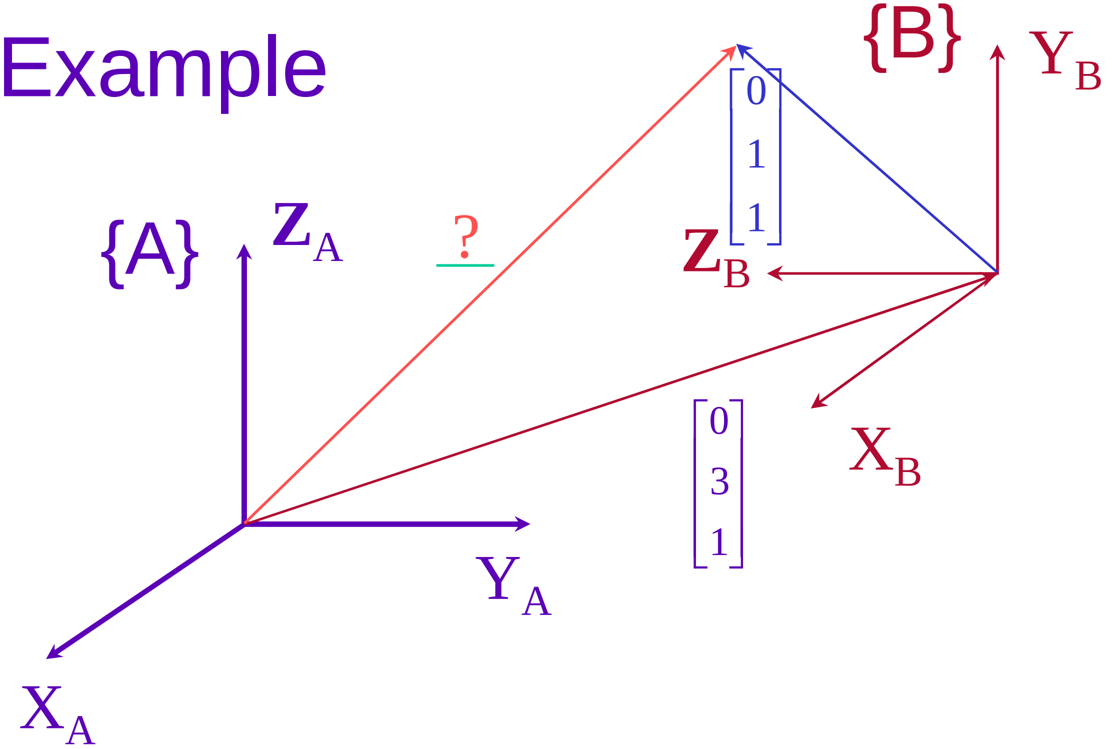


$$^AP = {}^A_BR\,{}^BP + {}^AP_{BORG}$$

$$\begin{bmatrix} {}^AP \\ \hline 1 \end{bmatrix} = \left[\begin{array}{c:c} {}^A_BR & {}^AP_{Borg} \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right] \begin{bmatrix} {}^BP \\ \hline 1 \end{bmatrix}$$

$$\underset{(4x1)}{{}^AP} = \underset{(4x4)}{{}^A_BT}\ \underset{(4x1)}{{}^BP}$$

Example



Homogeneous Transform

$$
{}^{A}_{B}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{B}P = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

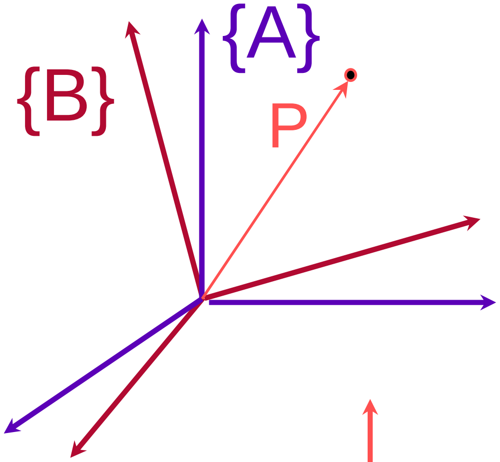
$$
{}^{A}P = {}^{A}_{B}T \,.\, {}^{B}P \quad \Rightarrow \quad {}^{A}P = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}
$$

Operators

Mapping: changing descriptions from frame to frame

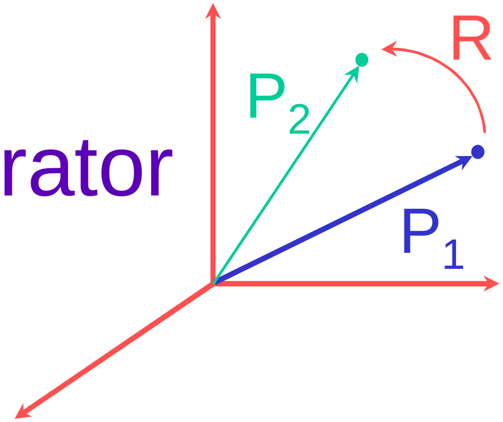
Operators: moving points (within the same frame)

Mapping



$$ {}^{A}P = {}^{A}_{B}R \; {}^{B}P $$

Rotational Operator



$$ R: \; P_1 \longrightarrow P_2 $$

$$ \boxed{P_2 = R \; P_1} $$

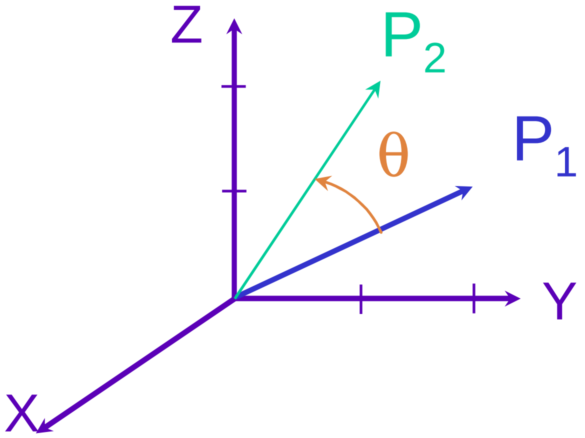
Rotational Operators

$$R_K(\theta): \quad P_1 \longrightarrow P_2$$

$$P_2 = R_K(\theta)\, P_1$$

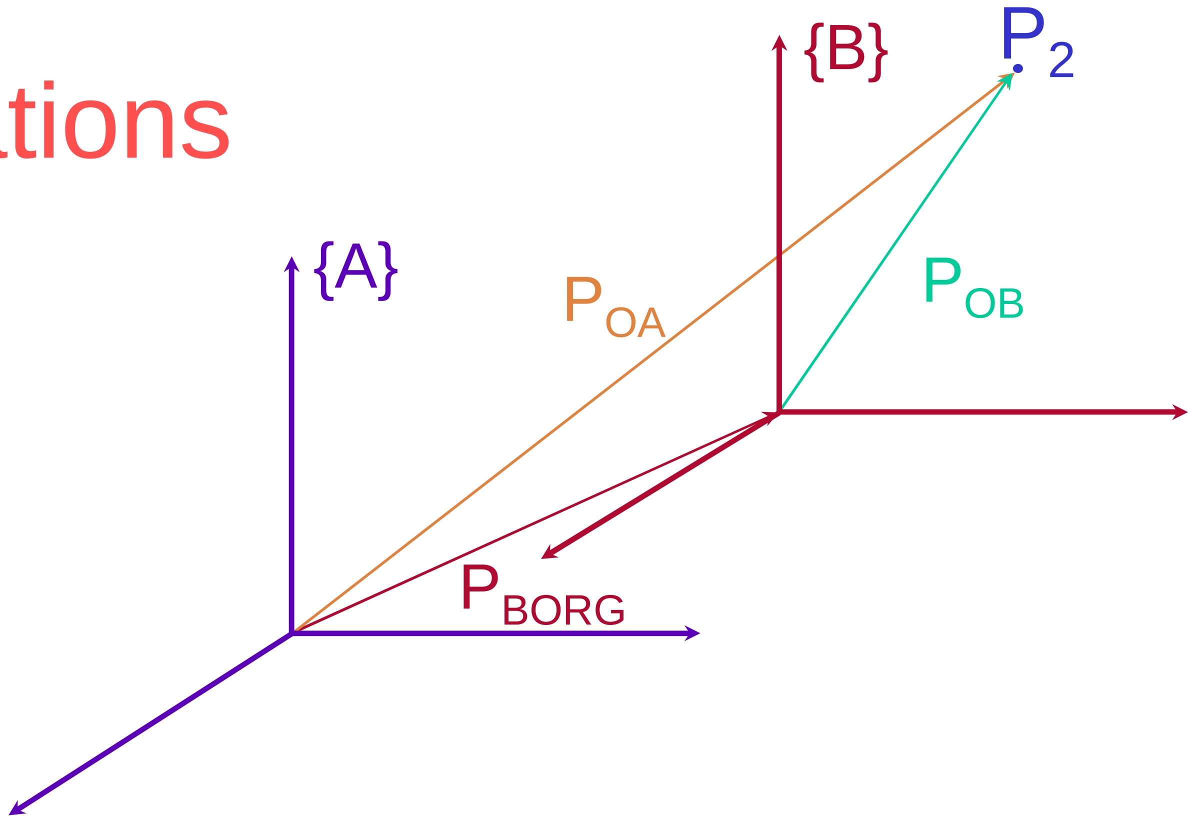
Example

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



$$P_2 = R_X(\theta) P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Translations

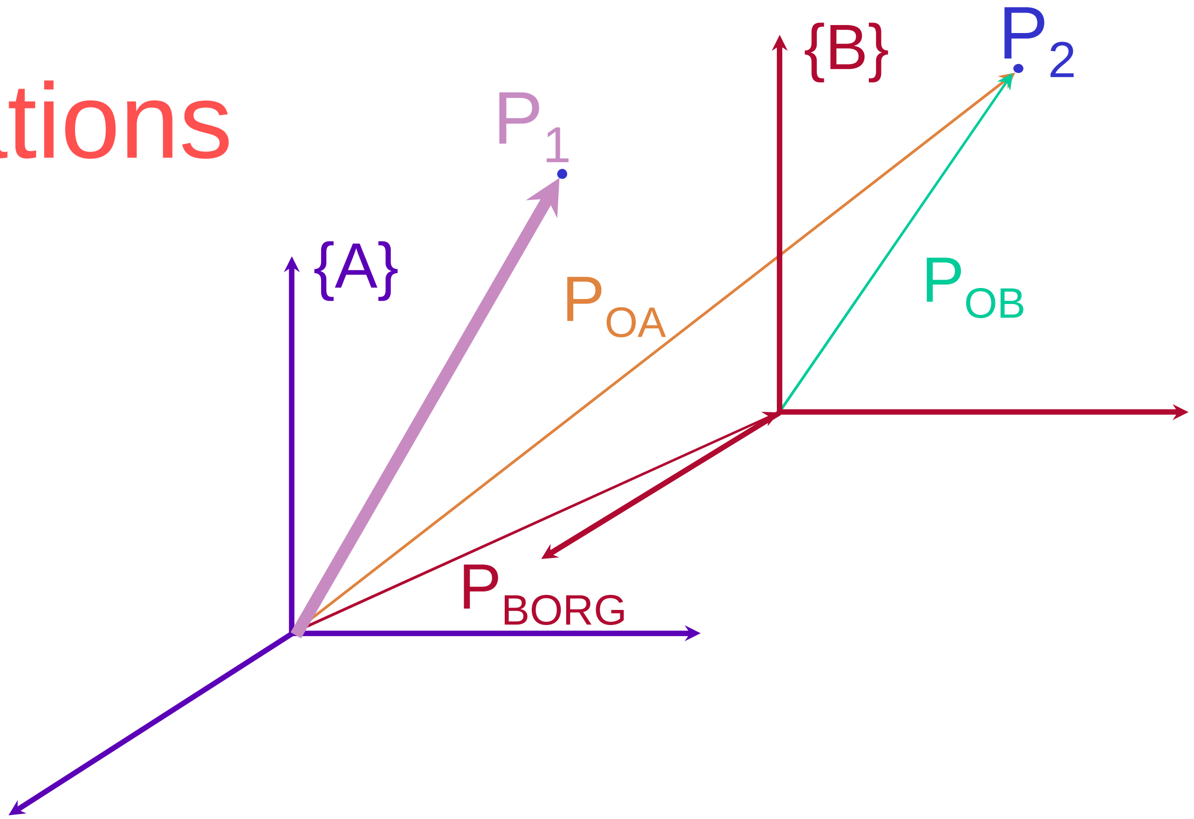


Mapping: $P_{BORG}: \; P_{OB} \longrightarrow P_{OA}$ (same point)

2 diff. vectors

$$P_{OA} = P_{OB} + P_{BORG}$$

Translations



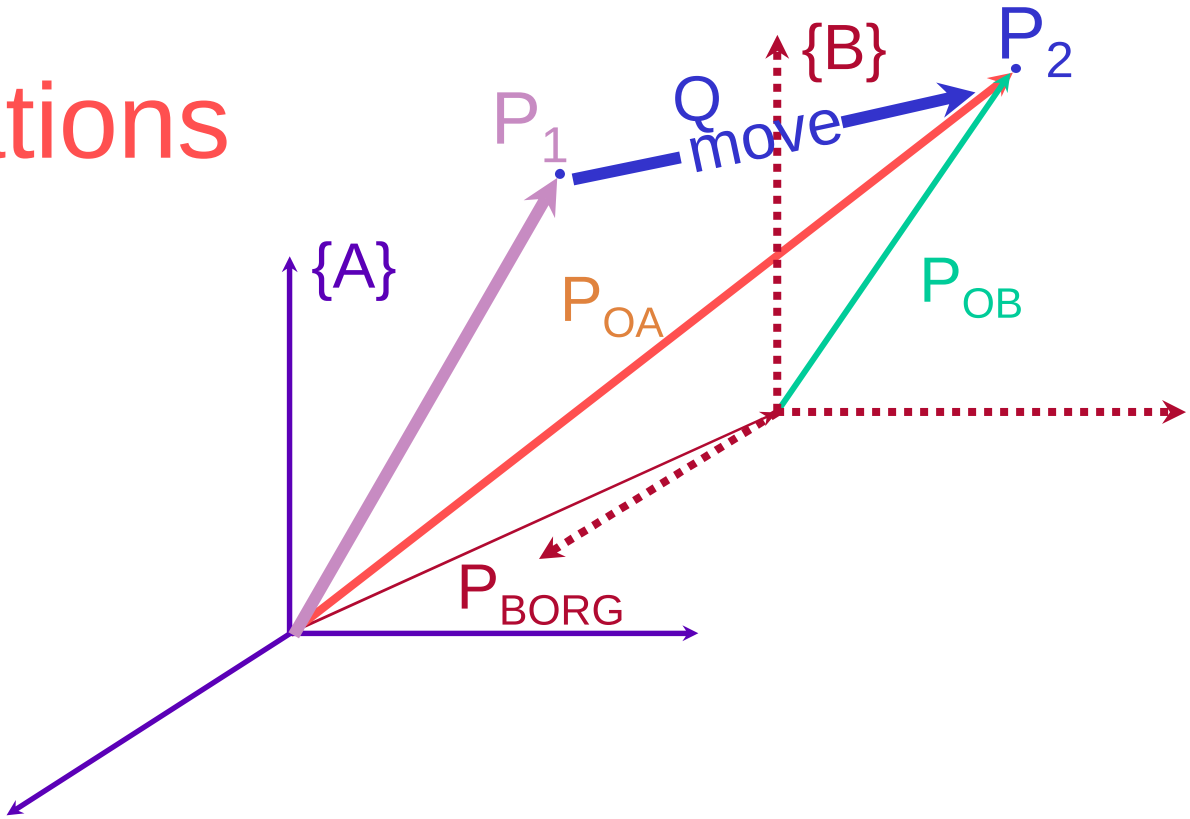
Mapping: P_{BORG} : $P_{OB} \longrightarrow P_{OA}$ (same point)

2 diff. vectors

$$P_{OA} = P_{OB} + P_{BORG}$$

Translational Operator:

Translations



Mapping: P_{BORG} : $P_{OB} \longrightarrow P_{OA}$ (same point)

2 diff. vectors

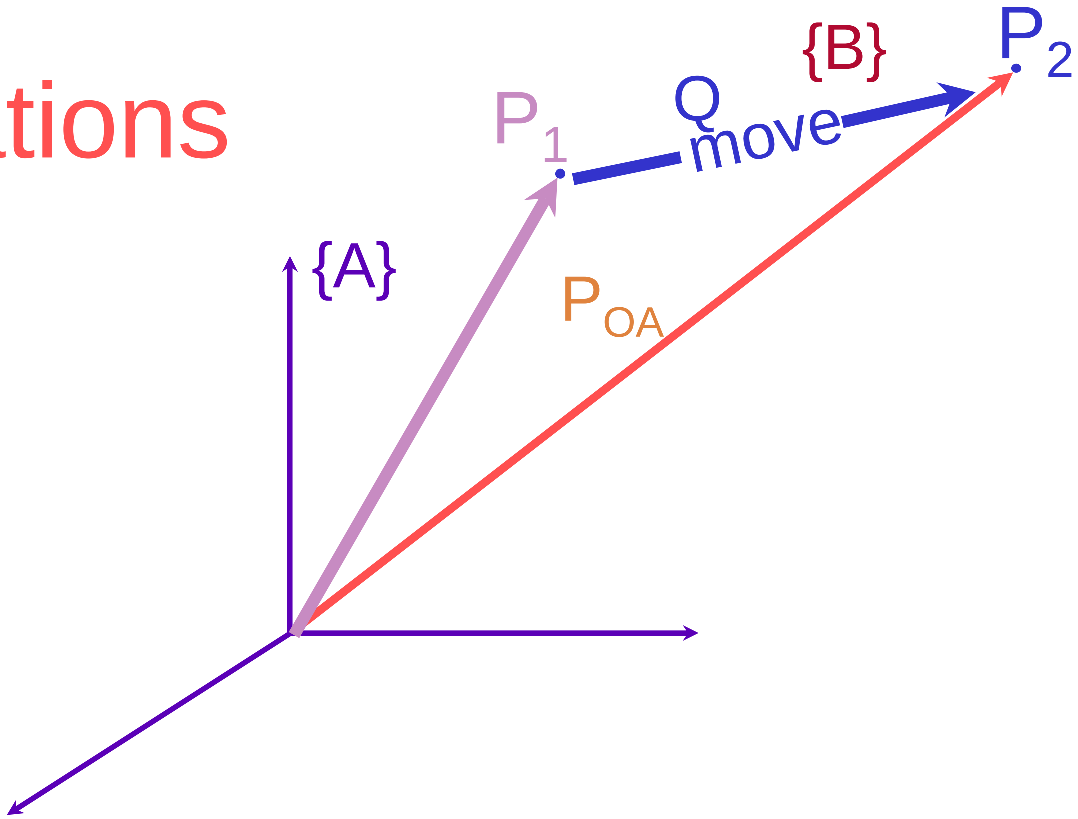
$$P_{OA} = P_{OB} + P_{BORG}$$

Translational Operator:

Q : $P_1 \longrightarrow P_2$ (2 points, 2 diff vectors)

$$P_2 = P_1 + Q$$

Translations

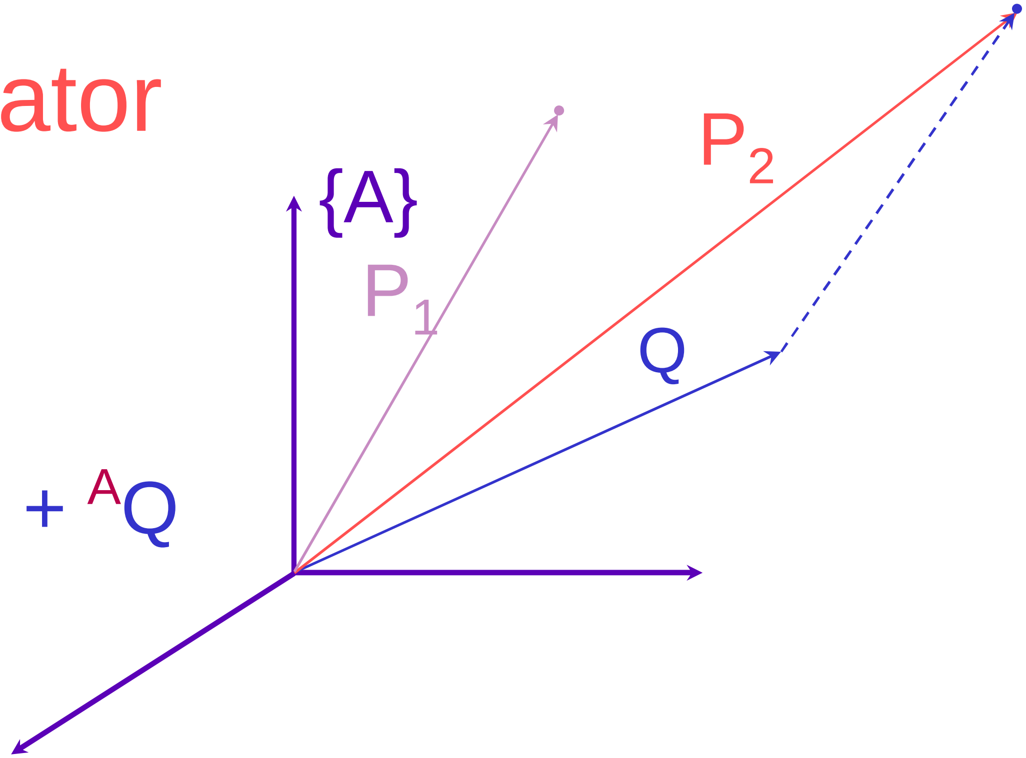


Translational Operator:

$$Q : \quad P_1 \longrightarrow P_2 \text{ (2 points, 2 diff vectors)}$$

$$P_2 = P_1 + Q$$

Translation Operator



Operator: ${}^{A}P_{2} = {}^{A}P_{1} + {}^{A}Q$

Homogeneous Transform:

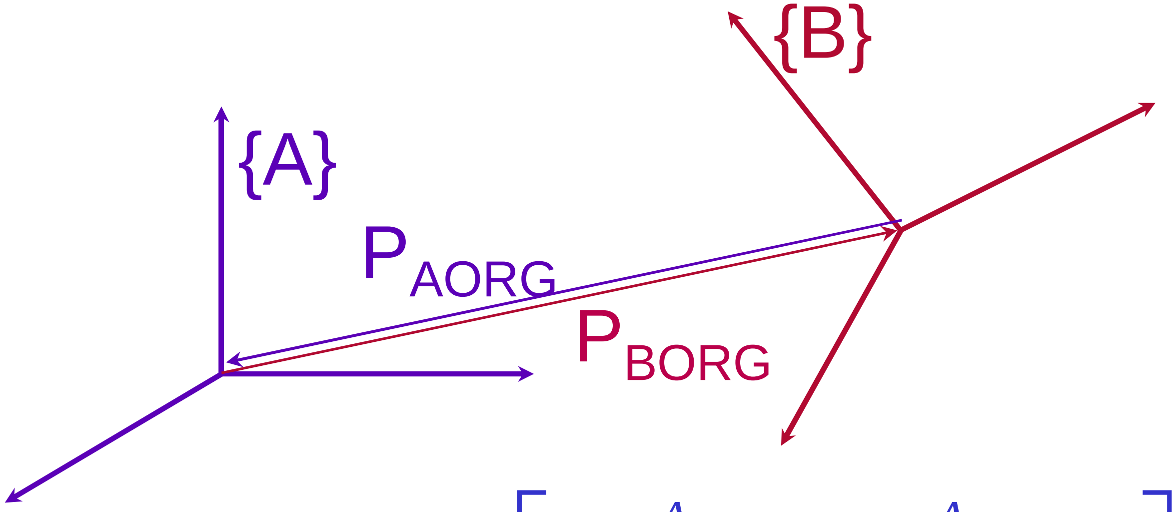
$$D_Q = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad {}^{A}P_{2} = {}^{A}D_{Q}\,{}^{A}P_{1}$$

General Operators

$$P_2 = \left(\begin{array}{ccc|c} \multicolumn{3}{c|}{R_K(\theta)} & Q \\ \hline 0 & 0 & 0 & 1 \end{array}\right) P_1$$

$$\boxed{P_2 = T\,P_1}$$

Inverse Transform



$${}^{A}_{B}T = \begin{bmatrix} & {}^{A}_{B}R & & {}^{A}P_{Borg} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R^{-1} = R^{T} \quad (T^{-1} \neq T^{T})$$

$$\boxed{{}^{B}P_{AORG}}$$

$${}^{A}_{B}T^{-1} = {}^{B}_{A}T = \begin{bmatrix} & {}^{A}_{B}R^{T} & & \boxed{-{}^{A}_{B}R^{T} \cdot {}^{A}P_{Borg}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$