Proportional-Derivative Control (PD)

$$m\ddot{x} = f = -k_p(x - x_d) - k_v\dot{x}$$

$$m\ddot{x} + k_v\dot{x} + k_p(x - x_d) = 0$$

Velocity gain (k_v) Position gain (k_p)

$$1.\ \ddot{x} + \frac{k_v}{m}\dot{x} + \frac{k_p}{m}(x - x_d) = 0$$

$$1.\ \ddot{x} + 2\xi\omega\dot{x} + \omega^2(x - x_d) = 0$$

$$\xi = \frac{k_v}{2\sqrt{k_p m}}$$ closed loop damping ratio

$$\omega = \sqrt{\frac{k_p}{m}}$$ closed loop frequency

Gains

$$k_p = m\omega^2$$

$$k_v = m(2\xi\omega)$$

Gain Selection

$$\text{set} \begin{pmatrix} \xi \\ \omega \end{pmatrix} \rightarrow \begin{array}{l} k_p = m\omega^2 \\ k_v = m(2\xi\omega) \end{array}$$

Unit mass system

$$k'_p = \omega^2$$

$$k'_v = 2\xi\omega$$

m - mass system

$$k_p = m\, k'_p$$

$$k_v = m\, k'_v$$

Control Partitioning

$$m\ddot{x} = f \quad \Rightarrow \quad m\,(1.\ddot{x}) = m\,f'$$

$$f = -k_v\dot{x} - k_p(x - x_d)$$

$$f = m[\underbrace{-k'_v\dot{x} - k'_p(x - x_d)}_{f'}] = m\,f'$$

$$m\ddot{x} = m\,f'$$

$$1.\ddot{x} = f'$$ unit mass system

$$1.\ddot{x} + \underset{\underset{2\xi\omega}{\Uparrow}}{k'_v}\dot{x} + \underset{\underset{\omega^2}{\Uparrow}}{k'_p}(x - x_d) = 0$$

Non Linearities

$$m\ddot{x} + b(x, \dot{x}) = f$$

Control Partitioning

$$f = \alpha f' + \beta$$

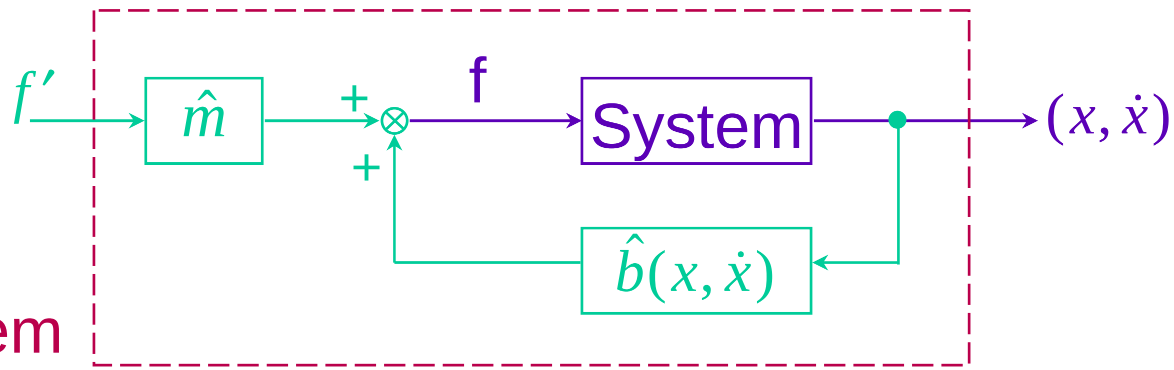
with

$$\alpha = \hat{m}$$

$$\beta = \hat{b}(x, \dot{x})$$

$$m\ddot{x} + b(x, \dot{x}) = \hat{m} f' + \hat{b}(x, \dot{x})$$

$$\Longrightarrow \quad 1.\ddot{x} = f'$$



Unit mass system

Motion Control

$$m\ddot{x} + b(x,\dot{x}) = f \Rightarrow 1.\ddot{x} = f'$$

$$f = mf' + b$$

Goal Position (x_d):

Control: $f' = -k_v'\dot{x} - k_p'(x - x_d)$

Closed-loop System: $1.\ddot{x} + k_v'\dot{x} + k_p'(x - x_d) = 0$

Trajectory Tracking

$$x_d(t);\ \dot{x}_d(t);\ \text{and}\ \ddot{x}_d(t)$$

Control: $f' = \ddot{x}_d - k_v'(\dot{x} - \dot{x}_d) - k_p'(x - x_d)$

Closed-loop System:

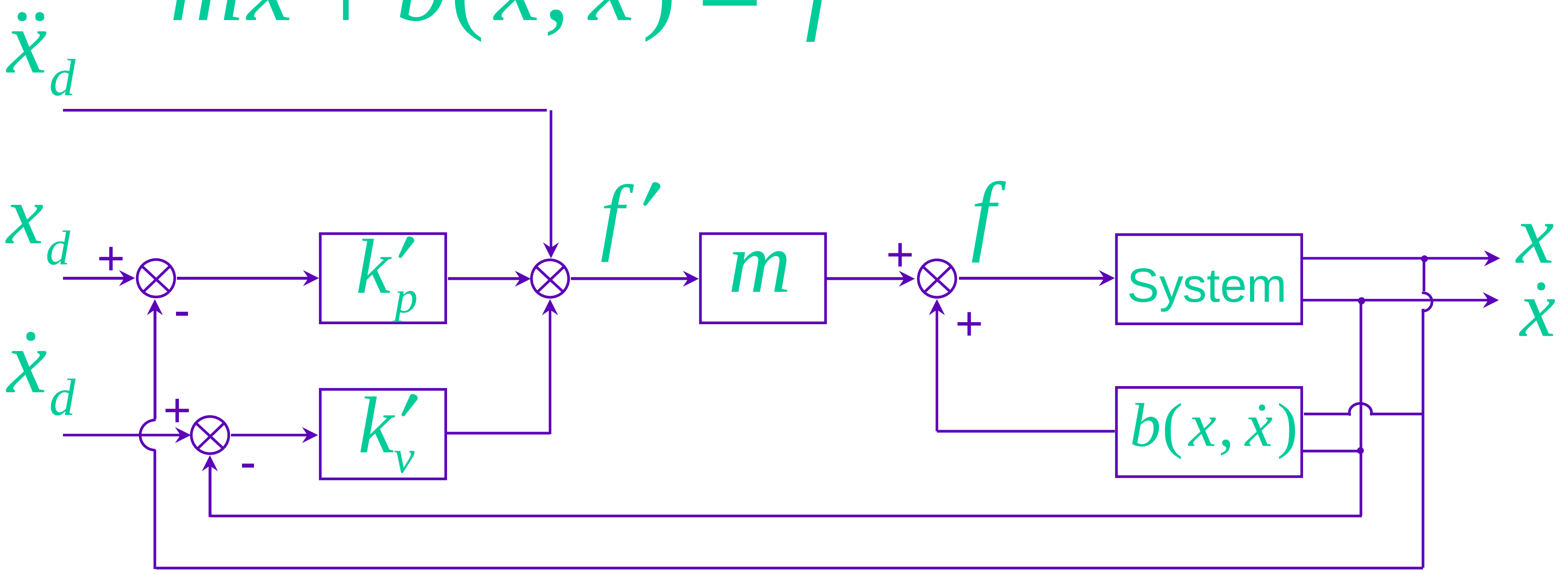
$$(\ddot{x} - \ddot{x}_d) + k_v'(\dot{x} - \dot{x}_d) + k_p'(x - x_d) = 0$$

with $e \equiv x - x_d$

$$\ddot{e} + k_v'\dot{e} + k_p'e = 0$$

Disturbance Rejection

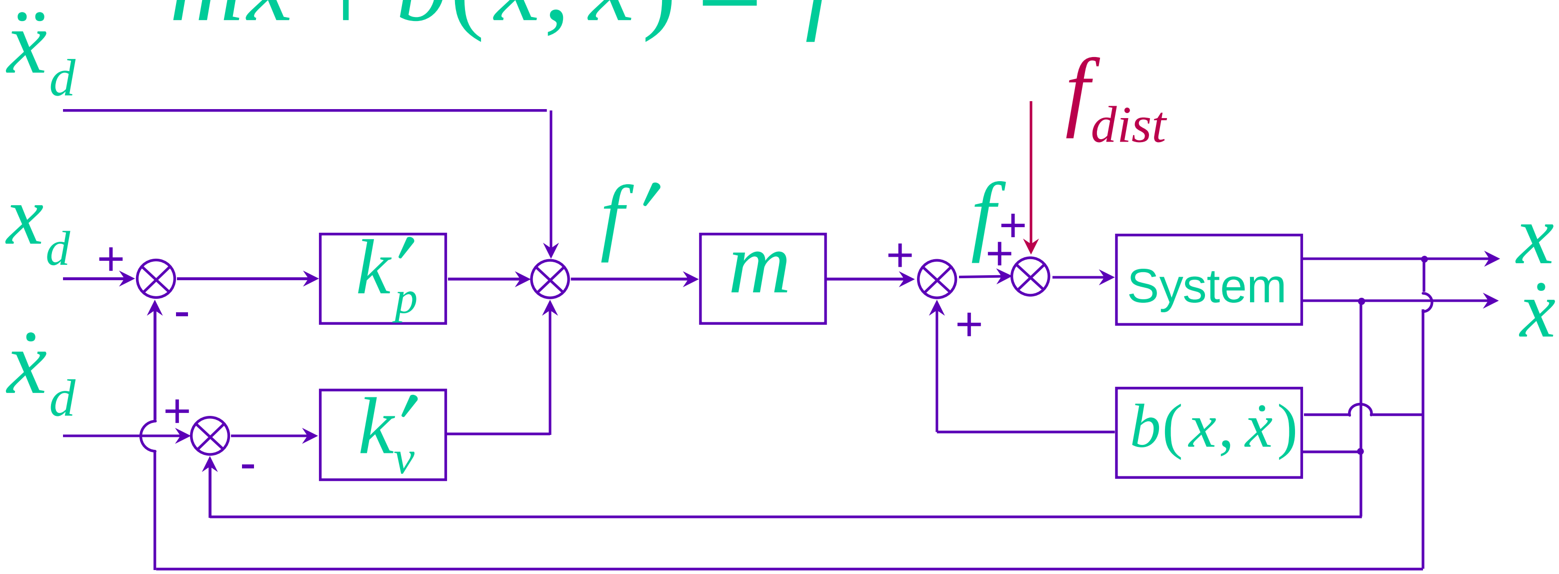
$$m\ddot{x} + b(x, \dot{x}) = f$$



$$\ddot{e} + k_v'\dot{e} + k_p'e = 0$$

Disturbance Rejection

$$m\ddot{x} + b(x,\dot{x}) = f$$



$$m\ddot{x} + b(x,\dot{x}) = f + f_{dist}$$

bounded

$$\{\forall t \, | f_{dist} | < a\}$$

Control

$$f = mf' + b(x,\dot{x})$$

Closed loop

$$\ddot{e} + k_v'\dot{e} + k_p' e = \frac{f_{dist}}{m}$$

Steady-State Error

$$\ddot{e} + k_v'\dot{e} + k_p'e = \frac{f_{dist}}{m}$$

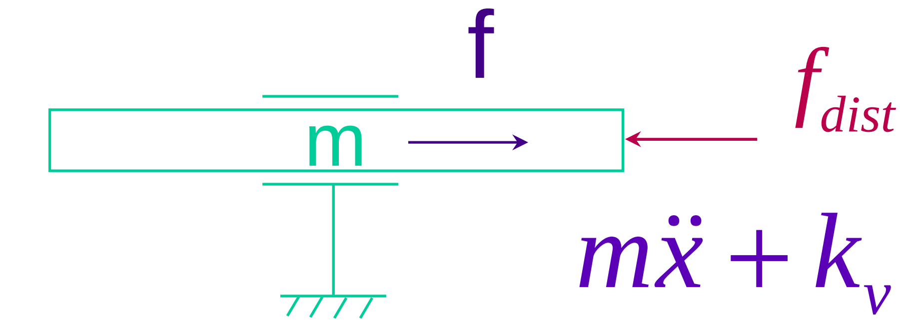
The steady-state $\left(\dot{e} = \ddot{e} = 0\right):$

$$k_p'e = \frac{f_{dist}}{m}$$

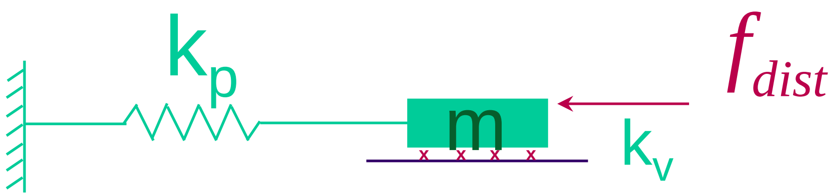
$$e = \frac{f_{dist}}{mk_p'} = \frac{f_{dist}}{k_p}$$

Closed loop position gain

Steady-State Error - Example



$$m\ddot{x} + k_v \dot{x} + k_p (x - x_d) = 0$$



$$k_p (x - x_d) = f_{dist}$$

$$x = x_d + \underbrace{\frac{f_{dist}}{k_p}}_{\Delta x}$$

$$f_{dist} = k_p \Delta x$$

$$\Delta x = \frac{f_{dist}}{k_p}$$

⟸ Closed Loop Stiffness

PID (adding Integral action)

System $$m\ddot{x} + b(x,\dot{x}) = f + f_{dist}$$

Control $$f = mf' + b(x,\dot{x})$$

$$f' = \ddot{x}_d - k'_v(\dot{x} - \dot{x}_d) - k'_p(x - x_d) - k'_i \int (x - x_d)dt$$

Closed-loop System

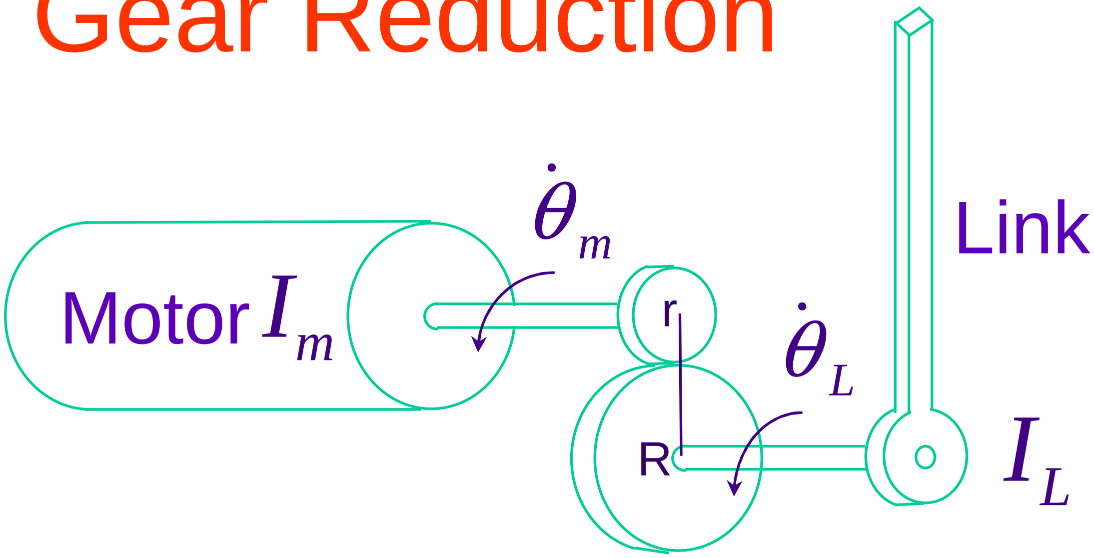
$$\ddot{e} + k'_v\dot{e} + k'_p e + k'_i \int e dt = \frac{f_{dist}}{m}$$

constant

$$\dddot{e} + k'_v\ddot{e} + k'_p\dot{e} + k'_i e = 0$$

Steady-state Error $$e = 0$$

Gear Reduction



Gear ratio $\eta = \frac{R}{r}$

$$\dot{\theta}_L = \left(\frac{1}{\eta}\right)\dot{\theta}_m$$

$$\tau_L = \eta \tau_m$$

$$\tau_m = I_m \ddot{\theta}_m + \frac{1}{\eta}(I_L \ddot{\theta}_L) + b_m \dot{\theta}_m + \frac{1}{\eta} b_L \dot{\theta}_L$$

$$\ddot{\theta}_L = \frac{1}{\eta}\ddot{\theta}_m$$

$$\tau_m = \left(I_m + \frac{I_L}{\eta^2}\right)\ddot{\theta}_m + \left(b_m + \frac{b_L}{\eta^2}\right)\dot{\theta}_m$$

$$\tau_L = \underbrace{(I_L + \eta^2 I_m)}_{\text{Effective Inertia}}\ddot{\theta}_L + \underbrace{(b_L + \eta^2 b_m)}_{\text{Effective Damping}}\dot{\theta}_L$$

Effective Inertia Effective Damping

Effective Inertia

$$I_{eff} = I_L + \eta^2 I_m$$

for a manipulator

$$I_L = I_L(q)$$

$$\eta = 1$$

Direct Drive

Gain Selection

$$k_p = (I_L + \eta^2 I_m)k'_p$$

$$k_v = (I_L + \eta^2 I_m)k'_v$$

Time Optimal Selection

$$\hat{I}_L = \frac{1}{4}(\sqrt{I_{L_{\min}}} + \sqrt{I_{L_{\max}}})^2$$