

# Lagrange Equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \tau$$

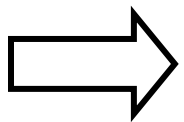
Lagrangian

Kinetic Energy

$$L = K - U$$

Potential Energy

Since  $U = U(q)$



$$\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}} \right) - \frac{\partial K}{\partial q} + \frac{\partial U}{\partial q} = \tau$$

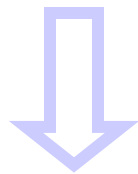
Inertial forces

Gravity vector

# Lagrange Equations

$$\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}} \right) - \frac{\partial K}{\partial q} = \tau - G; \quad G = \frac{\partial U}{\partial q}$$

Inertial forces



$$M(q)\ddot{q} + V(q, \dot{q}) = \tau - G(q)$$

# Inertial forces

$$\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}} \right) - \frac{\partial K}{\partial q} = \tau - \mathbf{G} \quad K = \frac{1}{2} \dot{q}^T M(q) \dot{q}$$

$$\frac{\partial K}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}} \left[ \frac{1}{2} \dot{q}^T M(q) \dot{q} \right] = M(q) \dot{q}$$

$$\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}} \right) = \frac{d}{dt} (M \dot{q}) = M \ddot{q} + \dot{M} \dot{q}$$

$$\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}} \right) - \frac{\partial K}{\partial q} = M \ddot{q} + \dot{M} \dot{q} - \frac{1}{2} \begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1} \dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial q_n} \dot{q} \end{bmatrix} = M \ddot{q} + V(q, \dot{q})$$

$$* \frac{\partial K}{\partial \dot{q}} = M \dot{q} \left[ K = \frac{1}{2} m \dot{x}^2; \frac{\partial}{\partial \dot{x}} \left( \frac{1}{2} m \dot{x}^2 \right) = \blacksquare \right]$$

$$K = \frac{1}{2} \dot{\mathbf{q}}^T M(\mathbf{q}) \dot{\mathbf{q}}$$

$$\mathbf{v} = M^{1/2} \dot{\mathbf{q}} \rightarrow K = \frac{1}{2} \mathbf{v}^T \mathbf{v}$$

$$\frac{\partial K}{\partial \dot{q}} = \frac{\partial K}{\partial \mathbf{v}} \frac{\partial \mathbf{v}}{\partial \dot{q}} = M^{1/2} \mathbf{v} = M \dot{q}$$

$$\frac{\partial}{\partial \mathbf{v}} \left( \frac{1}{2} \mathbf{v}^T \mathbf{v} \right) = \mathbf{v} \quad M^{1/2}$$

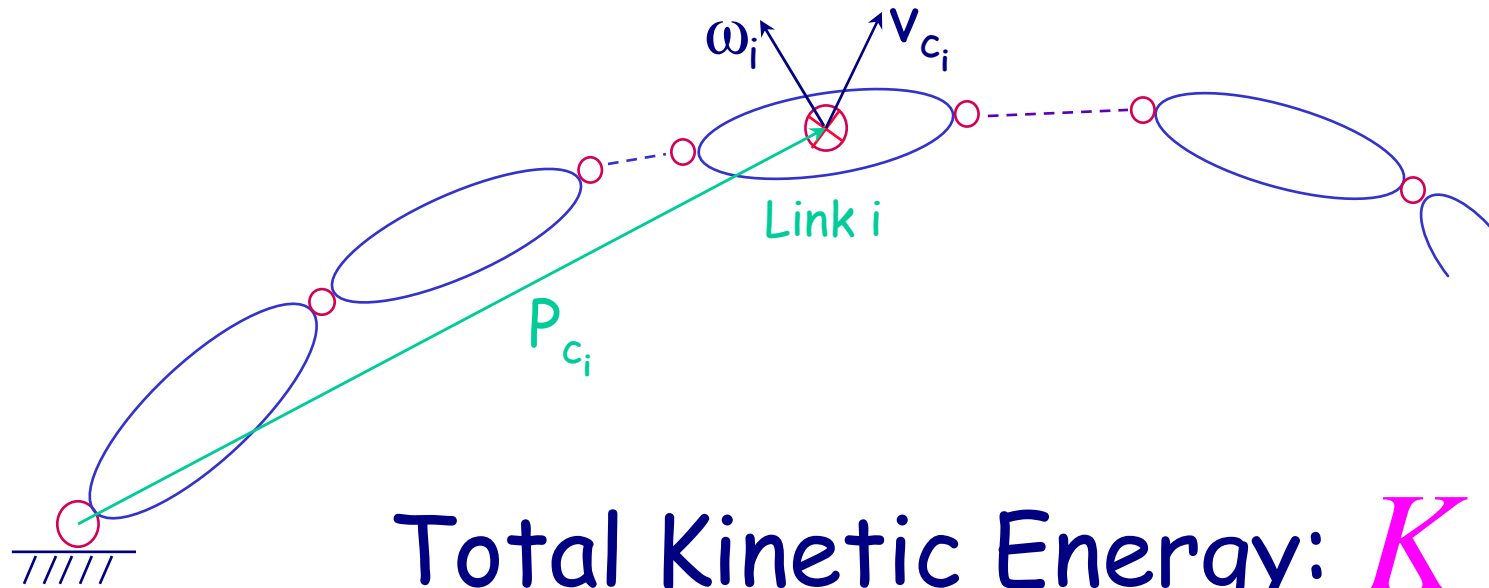
# Equations of Motion

$$\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}} \right) - \frac{\partial K}{\partial q} = M \ddot{q} + \dot{M} \dot{q} - \frac{1}{2} \begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1} \dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial q_n} \dot{q} \end{bmatrix} = M \ddot{q} + V(q, \dot{q})$$

$$M(q) \ddot{q} + V(q, \dot{q}) + G(q) = \tau$$

$$M(q): K = \frac{1}{2} \dot{q}^T M \dot{q} \quad M(q) \Rightarrow V(q, \dot{q})$$

# Equations of Motion

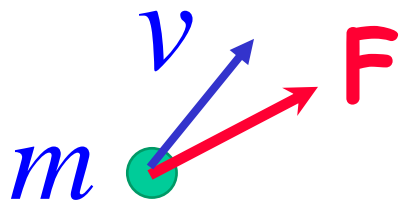


Total Kinetic Energy:  $K$

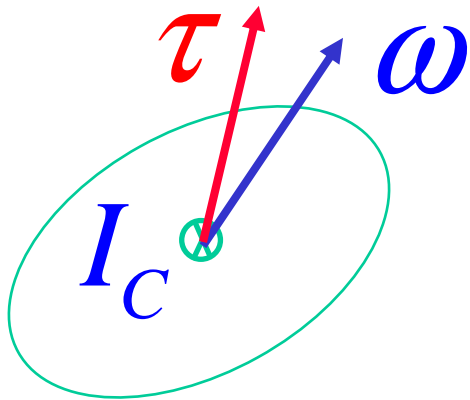
$$K = \sum K_{Linki} \equiv \frac{1}{2} \dot{q}^T M \dot{q}$$

# Kinetic Energy

Work done by external forces to bring the system from rest to its current state.



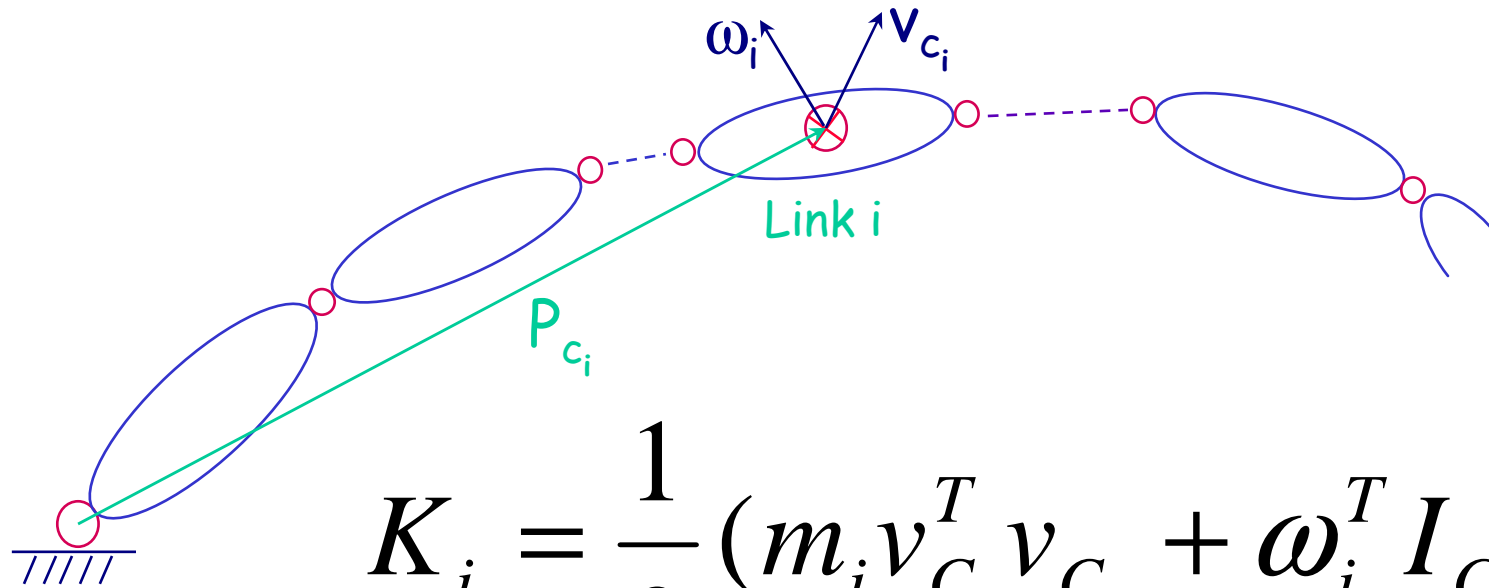
$$K = \frac{1}{2} m v^2$$



$$K = \frac{1}{2} \omega^T I_C \omega$$

# Equations of Motion

# Explicit Form

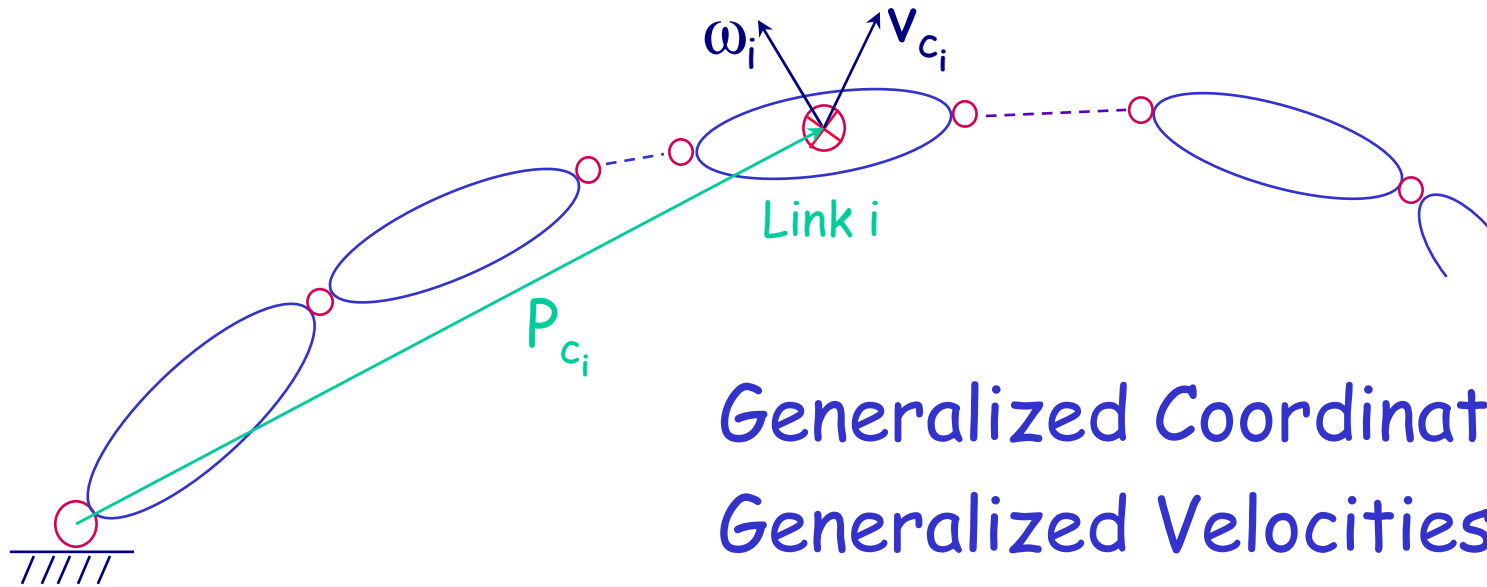


$$K_i = \frac{1}{2} (m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i} \omega_i)$$

Total Kinetic Energy  $\Rightarrow$   $K = \sum_{i=1}^n K_i$

# Equations of Motion

# Explicit Form



Generalized Coordinates  $q$

Generalized Velocities  $\dot{q}$

Kinetic Energy

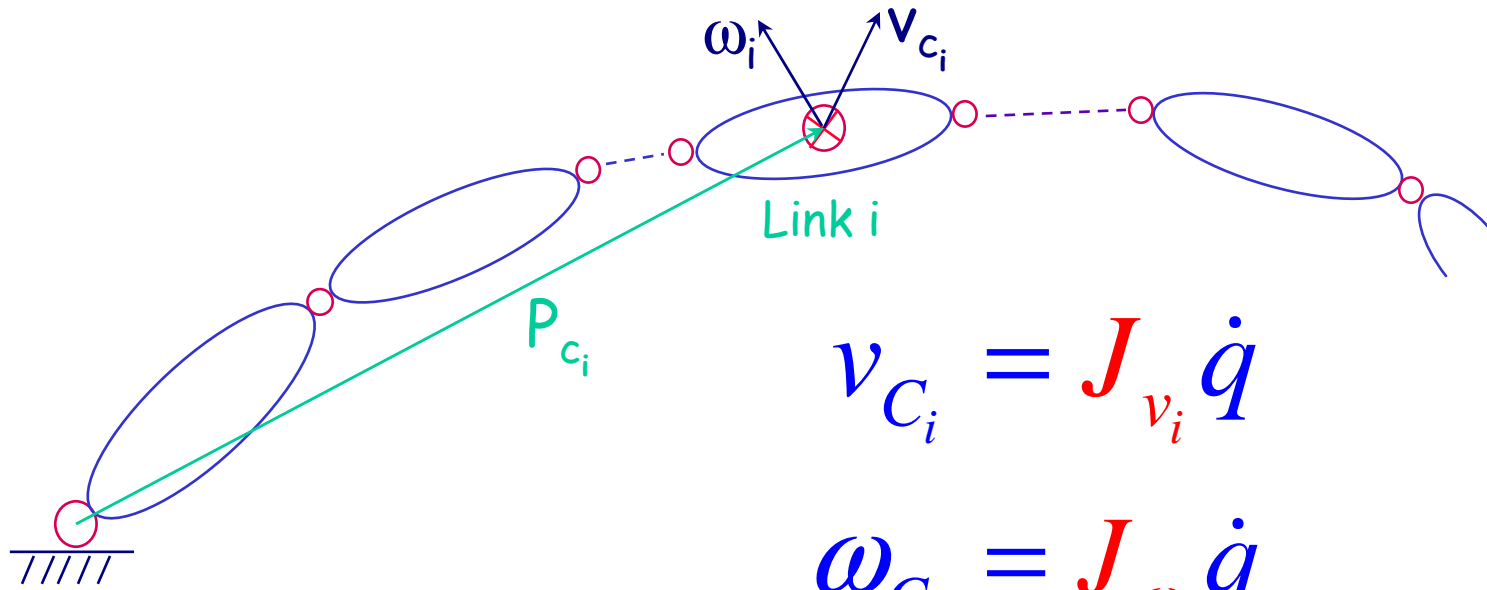
Quadratic Form of  
Generalized Velocities

$$K = \frac{1}{2} \dot{q}^T M \dot{q}$$

$$\frac{1}{2} \dot{q}^T M \dot{q} \equiv \frac{1}{2} \sum_{i=1}^n (m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i} \omega_i)$$

# Equations of Motion

# Explicit Form



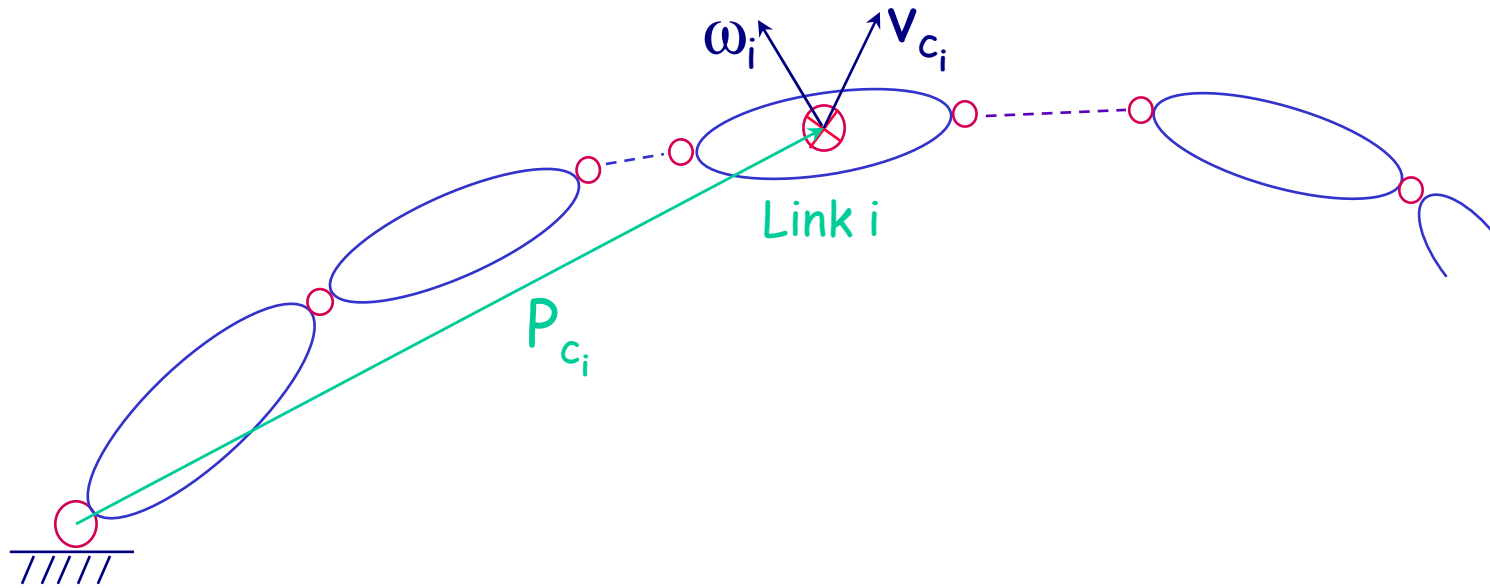
$$v_{C_i} = J_{v_i} \dot{q}$$

$$\omega_{C_i} = J_{\omega_i} \dot{q}$$

$$\begin{aligned} \frac{1}{2} \dot{q}^T M \dot{q} &= \frac{1}{2} \sum_{i=1}^n (m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i} \omega_i) \\ &= \frac{1}{2} \sum_{i=1}^n (m_i \dot{q}^T J_{v_i}^T J_{v_i} \dot{q} + \dot{q}^T J_{\omega_i}^T I_{C_i} J_{\omega_i} \dot{q}) \end{aligned}$$

# Equations of Motion

# Explicit Form

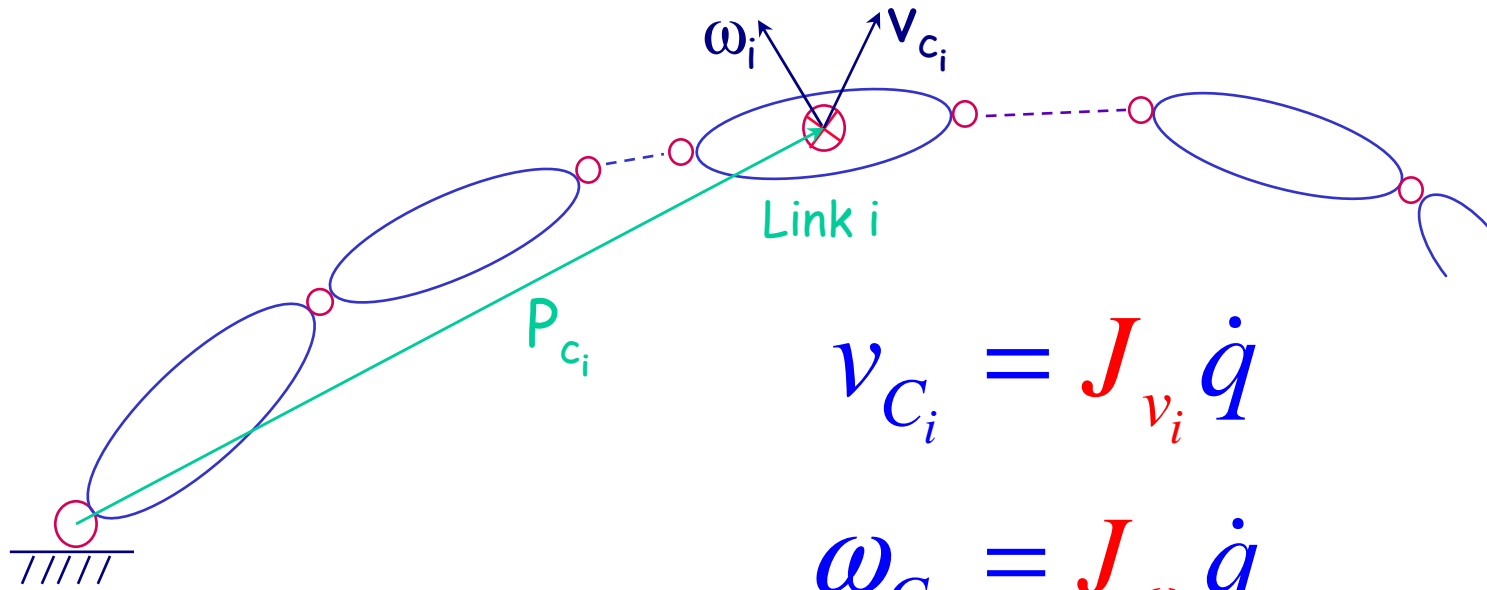


$$\frac{1}{2} \dot{q}^T M \dot{q} = \frac{1}{2} \dot{q}^T \left[ \sum_{i=1}^n (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{C_i} J_{\omega_i}) \right] \dot{q}$$

$$M = \sum_{i=1}^n (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{C_i} J_{\omega_i})$$

# Equations of Motion

# Explicit Form



$$v_{C_i} = J_{v_i} \dot{q}$$

$$\omega_{C_i} = J_{\omega_i} \dot{q}$$

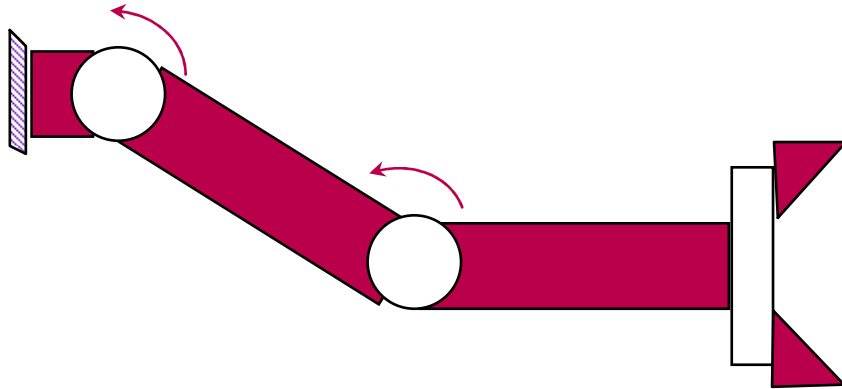
$$J_{v_i} = \begin{bmatrix} \frac{\partial p_{C_i}}{\partial q_1} & \frac{\partial p_{C_i}}{\partial q_2} & \dots & \frac{\partial p_{C_i}}{\partial q_i} & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$J_{\omega_i} = \begin{bmatrix} \bar{\epsilon}_1 z_1 & \bar{\epsilon}_2 z_2 & \dots & \bar{\epsilon}_i z_i & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$M(q) = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$

$(n \times n)$

# Vector $V(\mathbf{q}, \dot{\mathbf{q}})$ Centrifugal & Coriolis Forces



$$\begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Vector  $V(\mathbf{q}, \dot{\mathbf{q}}) \frac{\partial M}{\partial q_1}$

$$V = \dot{M}\dot{\mathbf{q}} - \frac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}^T M_{q_1} \dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T M_{q_2} \dot{\mathbf{q}} \end{bmatrix} = \begin{pmatrix} \dot{m}_{11} & \dot{m}_{12} \\ \dot{m}_{12} & \dot{m}_{22} \end{pmatrix} \dot{\mathbf{q}} - \frac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}^T \begin{pmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \begin{pmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{pmatrix} \dot{\mathbf{q}} \end{bmatrix}$$

$$\dot{m}_{11} = m_{111}\dot{q}_1 + m_{112}\dot{q}_2$$

$$V(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} \frac{1}{2}(m_{111} + m_{111} - m_{111}) & \frac{1}{2}(m_{122} + m_{122} - m_{221}) \\ \frac{1}{2}(m_{211} + m_{211} - m_{112}) & \frac{1}{2}(m_{222} + m_{222} - m_{222}) \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} m_{112} + m_{121} - m_{121} \\ m_{212} + m_{221} - m_{122} \end{bmatrix} [\dot{q}_1 \dot{q}_2]$$

# Christoffel Symbols

$$b_{ijk} = \frac{1}{2} \left( m_{ijk} + m_{ikj} - m_{jki} \right)$$

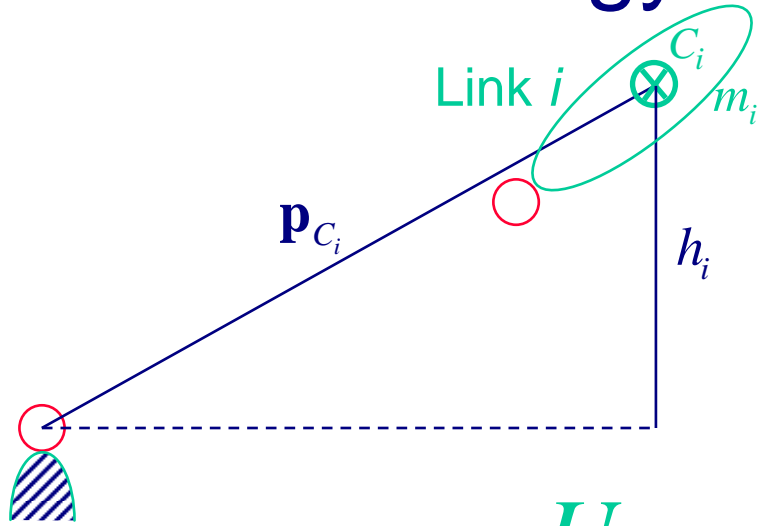
$\frac{\partial m_{ij}}{\partial q_k}$

$$V = \underbrace{\begin{bmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{bmatrix}}_{C(\mathbf{q})} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix}}_{B(\mathbf{q})} [\dot{q}_1 \dot{q}_2]$$

$$C(\mathbf{q})[\dot{\mathbf{q}}^2] = \begin{matrix} (n \times n) & (n \times 1) \end{matrix} \begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

$$B(\mathbf{q}) [\dot{\mathbf{q}}\dot{\mathbf{q}}] = \begin{matrix} (n \times \frac{(n-1)n}{2}) & (\frac{(n-1)n}{2} \times 1) \end{matrix} \begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \vdots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \dot{q}_2 \\ \dot{q}_1 \dot{q}_3 \\ \vdots \\ \dot{q}_{(n-1)} \dot{q}_n \end{bmatrix}$$

# Potential Energy



$$U_i = m_i g_0 h_i + U_0$$

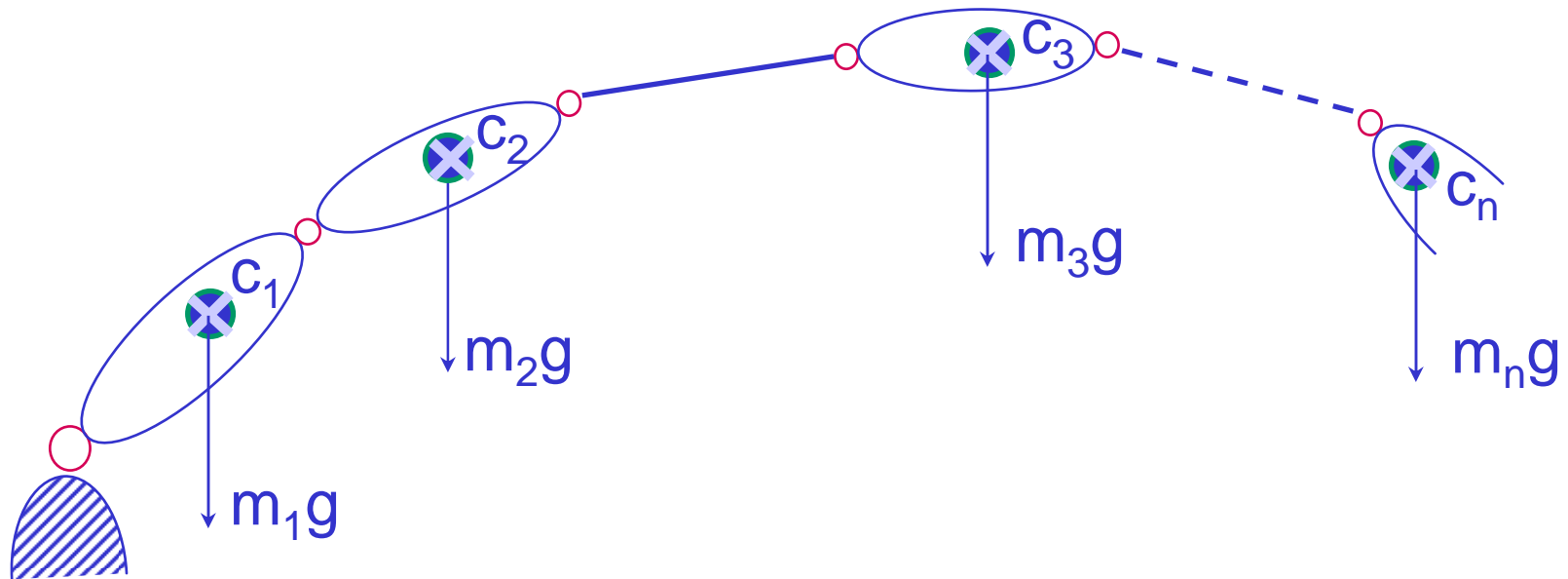
$$U_i = m_i (-g^T \mathbf{p}_{C_i}); \quad U = \sum_i U_i$$

Gravity Vector

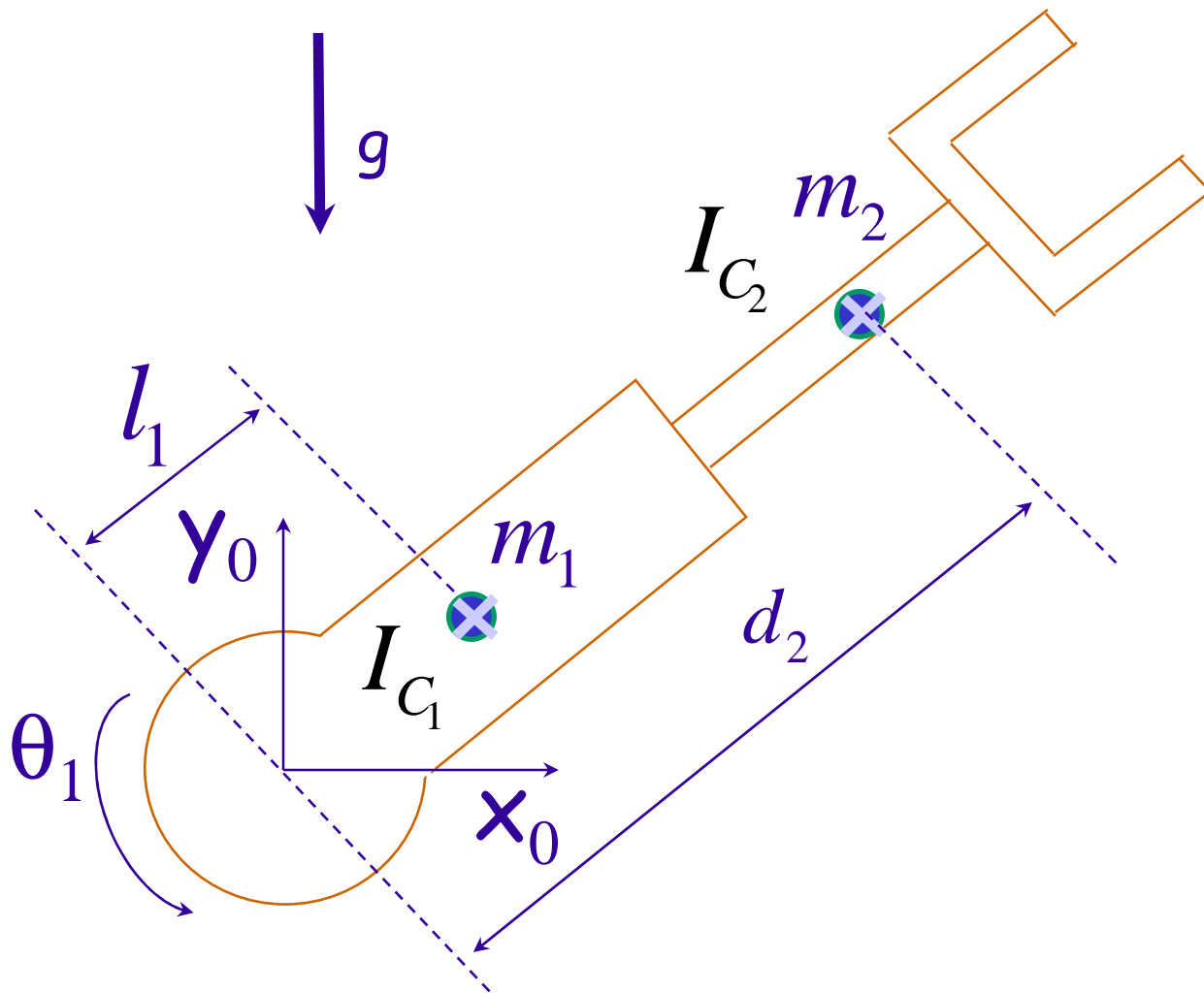
$$G_j = \frac{\partial U}{\partial q_j} = - \sum_{i=1}^n (m_i g^T \frac{\partial \mathbf{p}_{C_i}}{\partial q_j})$$

$$\mathbf{G} = - \begin{pmatrix} J_{v_1}^T & J_{v_2}^T & \dots & J_{v_n}^T \end{pmatrix} \begin{pmatrix} m_1 g \\ m_2 g \\ \vdots \\ m_n g \end{pmatrix}$$

# Gravity Vector



$$G = -(J_{v_1}^T (m_1 g) + J_{v_2}^T (m_2 g) + \dots + J_{v_n}^T (m_n g))$$



# Matrix M

$$M = m_1 J_{v_1}^T J_{v_1} + J_{\omega_1}^T I_{C_1} J_{\omega_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T I_{C_2} J_{\omega_2}$$

$J_{v_1}$  and  $J_{v_2}$ : direct differentiation of the vectors:

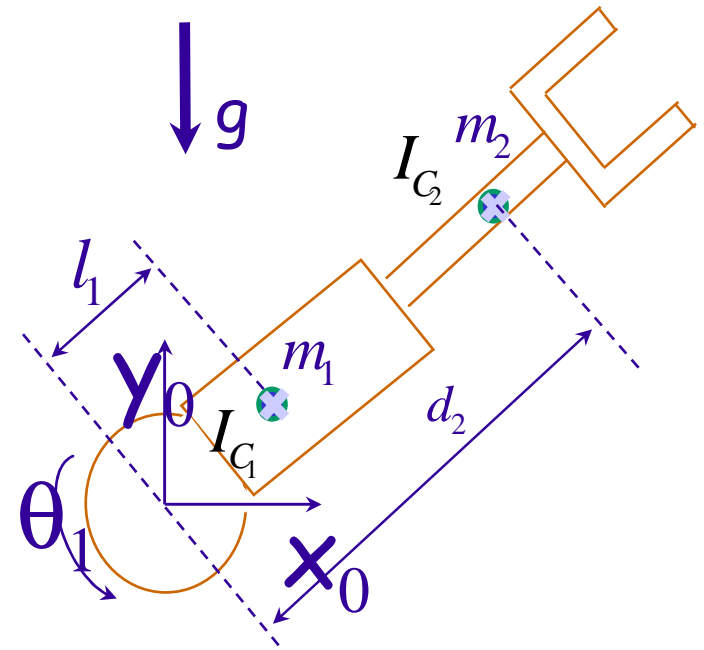
$${}^0 \mathbf{p}_{C_1} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}; \text{ and } {}^0 \mathbf{p}_{C_2} = \begin{bmatrix} d_2 c_1 \\ d_2 s_1 \\ 0 \end{bmatrix}$$

In frame {0}, these matrices are:

$${}^0 J_{v_1} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } {}^0 J_{v_2} = \begin{bmatrix} -d_2 s_1 c_1 \\ d_2 c_1 s_1 \\ 0 & 0 \end{bmatrix}$$

This yields

$$m_1 ({}^0 J_{v_1}^T {}^0 J_{v_1}) = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } m_2 ({}^0 J_{v_2}^T {}^0 J_{v_2}) = \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$



The matrices  $J_{\omega_1}$  and  $J_{\omega_2}$  are given by

$$J_{\omega_1} = [\bar{\epsilon}_1 \mathbf{z}_1 \quad 0] = \text{ and } J_{\omega_2} = [\bar{\epsilon}_1 \mathbf{z}_1 \quad \bar{\epsilon}_2 \mathbf{z}_2]$$

Joint 1 is revolute and joint 2 is prismatic:

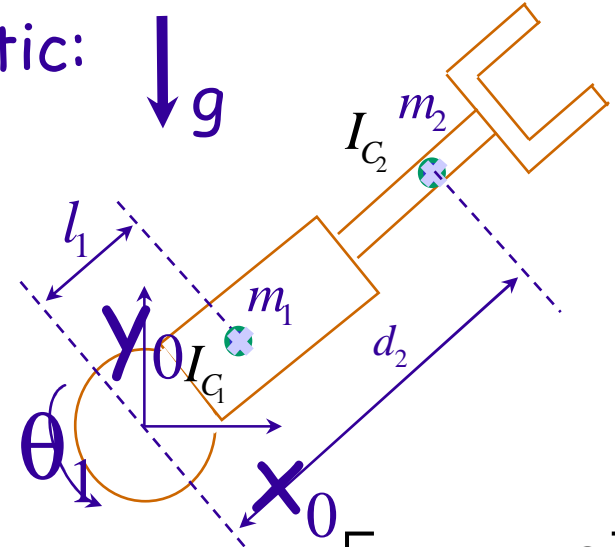
$${}^1J_{\omega_1} = {}^1J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

And

$$({}^1J_{\omega_1}^T I_{C_1} {}^1J_{\omega_1}) = \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } ({}^1J_{\omega_2}^T I_{C_2} {}^1J_{\omega_2}) = \begin{bmatrix} I_{zz2} & 0 \\ 0 & 0 \end{bmatrix}$$

Finally,

$$M = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}$$



## Centrifugal and Coriolis Vector $V$

$$b_{i,jk} = \frac{1}{2} (m_{ijk} + m_{ikj} - m_{jki})$$

where  $m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}$  ; with  $b_{iii} = 0$  and  $b_{iji} = 0$  for  $i > j$

For this manipulator, only  $m_{11}$  is configuration dependent - function of  $d_2$ . This implies that only  $m_{112}$  is non-zero,

$$m_{112} = 2m_2d_2.$$

Matrix  $B$

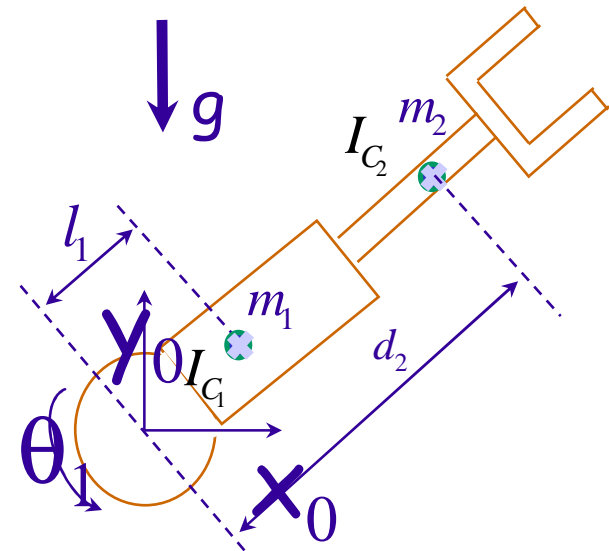
$$B = \begin{bmatrix} 2b_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} 2m_2d_2 \\ 0 \end{bmatrix}.$$

Matrix  $C$

$$C = \begin{bmatrix} 0 & b_{122} \\ b_{211} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -m_2d_2 & 0 \end{bmatrix}.$$

Matrix  $V$

$$V = \begin{bmatrix} 2m_2d_2 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{d}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -m_2d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}.$$



## Vector $V$

$$V = \begin{bmatrix} 2m_2d_2 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{d}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -m_2d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}.$$

## The Gravity Vector $G$

$$\mathbf{G} = -\left[ J_{v_1}^T m_1 \mathbf{g} + J_{v_2}^T m_2 \mathbf{g} \right].$$

In frame  $\{0\}$ ,  $\mathbf{g} = (0 \quad -g \quad 0)^T$  and the gravity vector is

$${}^0\mathbf{G} = -\begin{bmatrix} -l_1s_1 & l_1c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_1g \\ 0 \end{bmatrix} - \begin{bmatrix} -d_2s_1 & d_2c_1 & 0 \\ c_1 & s_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_2g \\ 0 \end{bmatrix}$$

and

$${}^0\mathbf{G} = \begin{bmatrix} (m_1l_1 + m_2d_2)gc_1 \\ m_2gs_1 \end{bmatrix}$$

## Equations of Motion

$$\begin{aligned} & \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix} \\ & + \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{d}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix} \\ & + \begin{bmatrix} (m_1 l_1 + m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}. \end{aligned}$$

