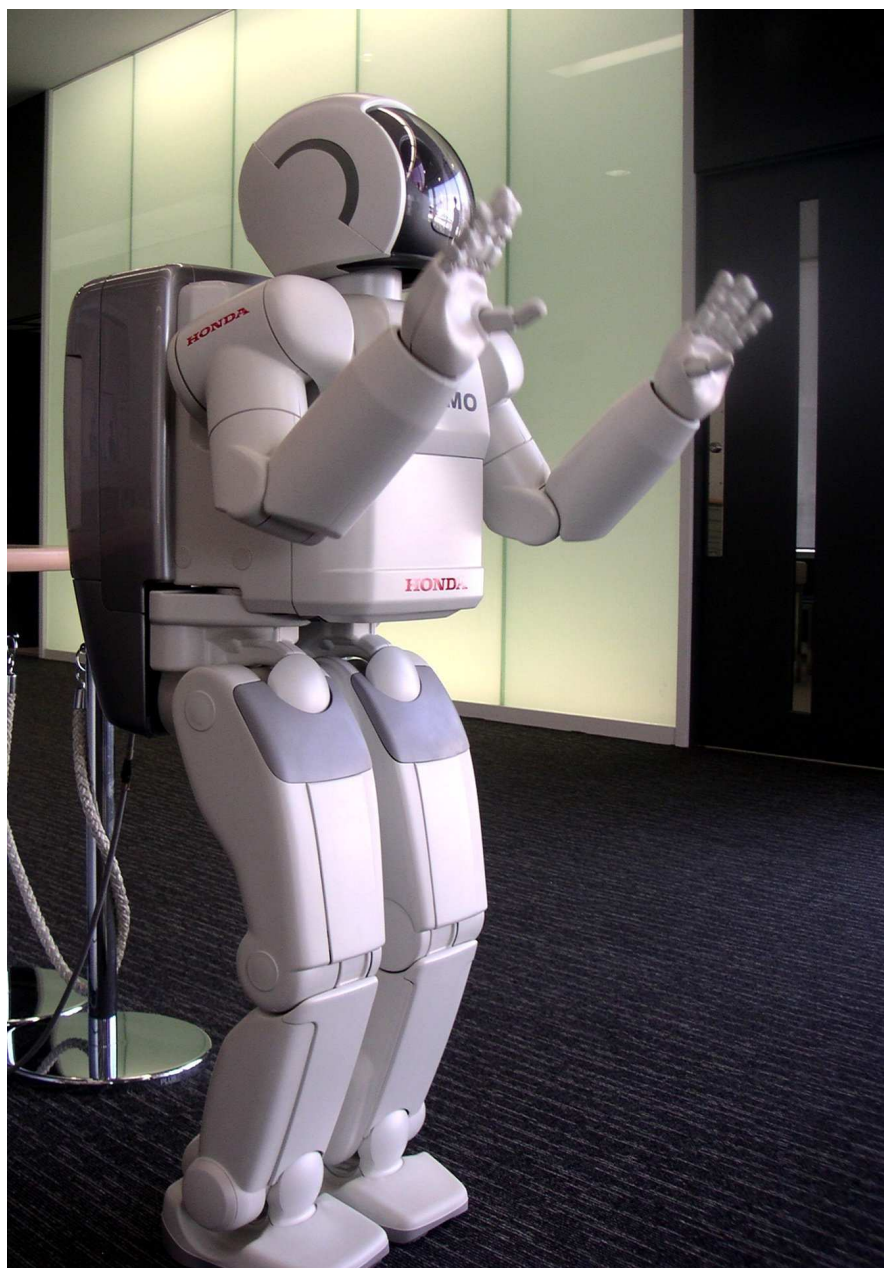


Dynamics



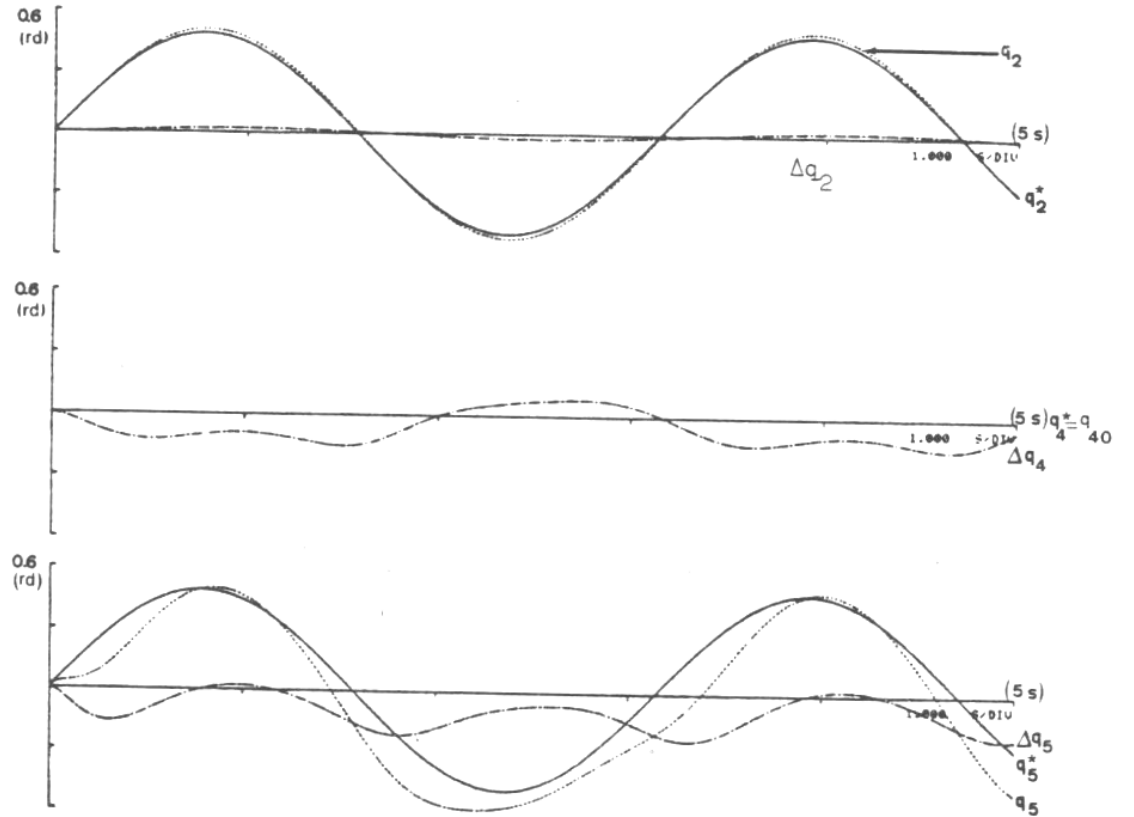
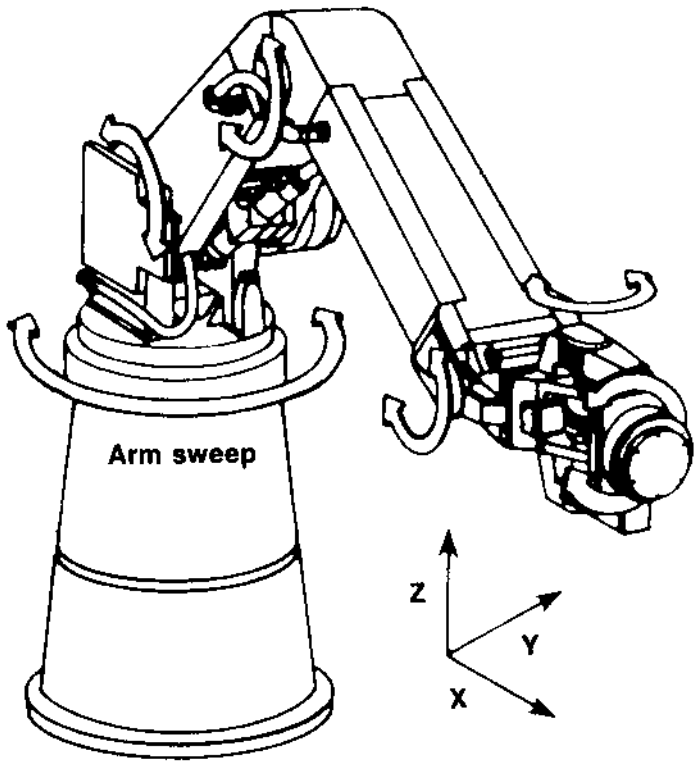
- Rigid Body Dynamics
- Newton-Euler Formulation
- Articulated Multi-Body Dynamics
- Recursive Algorithm
- Lagrange Formulation
- Explicit Form

COEFFICIENTS DYNAMIQUES DU MA 23



MA23

$$\begin{aligned}
 A_{11} &= I_{z1} + M_2 SS_2 d_2^2 + SS_2 I_{y2} + CC_2 I_{z2} + M_3 (SS_2 \ell_2^2 + 2S_2 S(2+3) \ell_2 d_3 + SS(2+3) d_3^2) \\
 &+ SS(2+3) I_{y3} + CC(2+3) I_{z3} + M_4 (SS_2 \ell_2^2 + 2S_2 S(2+3) \ell_2 h_1 + 2S_2 C(2+3) C_4 \ell_2 d_{y4} \\
 &+ SS(2+3) h_1^2 + 2SC(2+3) C_4 d_{y4} h_1 + CC(2+3) d_{y4}^2 + SS(2+3) SS_4 d_{y4}^2) + SS(2+3) SS_4 I_{x4} \\
 &+ SS(2+3) CC_4 I_{y4} - 2SC(2+3) C_4 I_{yz4} + CC(2+3) I_{z4} + M_5 (SS_2 \ell_2^2 + 2S_2 S(2+3) \ell_2 h_2 \\
 &+ 2S_2 C(2+3) C_4 \ell_2 \ell_{y4} + 2S_2 S(2+3) C_5 \ell_2 d_5 + 2S_2 C(2+3) C_4 S_5 \ell_2 d_5 + SS(2+3) h_2^2 + \\
 &2SC(2+3) C_4 \ell_{y4} h_2 + 2SS(2+3) C_5 h_2 d_5 + 2SC(2+3) C_4 S_5 h_2 d_5 + CC(2+3) \ell_{y4}^2 + \\
 &2SC(2+3) C_4 C_5 \ell_{y4} d_5 + 2CC(2+3) S_5 \ell_{y4} d_5 + SS(2+3) CC_4 CC_5 d_5^2 + 2SC(2+3) C_4 S_5 d_5^2 + \\
 &CC(2+3) SS_5 d_5^2 + SS(2+3) SS_4 \ell_{y4}^2 + 2SS(2+3) SS_4 S_5 \ell_{y4} d_5 + SS(2+3) SS_4 d_5^2) + SS(2+3) SS_4 I_{x5} \\
 &+ SS(2+3) CC_4 CC_5 I_{y5} + 2SC(2+3) C_4 S_5 I_{y5} + CC(2+3) SS_5 I_{y5} + SS(2+3) CC_4 SS_5 I_{z5} - 2SC(2+3) C_4 S_5 I_{z5} \\
 &+ CC(2+3) CC_5 I_{z5} + M_6 (SS_2 \ell_2^2 + 2S_2 S(2+3) \ell_2 d_6 + SS(2+3) d_6^2) + SS(2+3) I_{y6} + CC(2+3) I_{z6} \\
 A_{22} &= A_{33} + M_2 d_2^2 + I_{x2} + M_3 (2C_3 \ell_2 d_3 + \ell_2^2) + M_4 (2C_3 \ell_2 h_1 + \ell_2^2 - 2S_3 C_4 \ell_2 d_{y4}) \\
 &- M_5 (\ell_2^2 + 2C_3 \ell_2 h_2 + 2C_3 C_5 \ell_2 d_5 - 2S_3 C_4 \ell_2 \ell_{y4} - 2S_3 C_4 S_5 \ell_2 d_5) + M_6 (2C_3 \ell_2 d_6 + \ell_2^2) \\
 A_{33} &= M_3 d_3^2 + I_{x3} + M_4 (h_1^2 + CC_4 d_{y4}^2) + CC_4 I_{x4} + SS_4 I_{y4} + M_5 (h_2^2 + 2C_5 h_2 d_5 + SS_4 CC_5 d_5^2 \\
 &+ CC_4 \ell_{y4}^2 + 2CC_4 S_5 \ell_{y4} d_5 + CC_4 d_5^2) + CC_4 I_{x5} + SS_4 CC_5 I_{y5} + SS_4 SS_5 I_{z5} + M_6 d_6^2 + I_{x6} \\
 A_{44} &= M_4 d_{y4}^2 + I_{z4} + M_5 (\ell_{y4}^2 + 2S_5 \ell_{y4} d_5 + SS_5 d_5^2) + SS_5 I_{y5} + CC_5 I_{z5} \\
 A_{55} &= M_5 d_5^2 + I_{x5} \\
 A_{66} &= I_{z5} \\
 A_{12} &= A_{13} - M_4 C_2 S_4 \ell_2 d_{y4} + M_5 (-C_2 S_4 \ell_2 \ell_{y4} - C_2 S_4 S_5 \ell_2 d_5) \\
 A_{13} &= M_4 (-C(2+3) S_4 d_{y4} h_1 + S(2+3) SC_4 d_{y4}^2) - S(2+3) SC_4 I_{x4} - S(2+3) SC_4 I_{y4} + C(2+3) S_4 I_{yz4} \\
 &+ M_5 (-C(2+3) S_4 \ell_{y4} h_2 - C(2+3) S_4 C_5 \ell_{y4} d_5 - S(2+3) SC_4 CC_5 d_5^2 - C(2+3) S_4 S_5 h_2 d_5 \\
 &- C(2+3) S_4 S_5 d_5^2 + S(2+3) SC_4 \ell_{y4}^2 + 2S(2+3) SC_4 S_5 \ell_{y4} d_5 + S(2+3) SC_4 d_5^2) + S(2+3) SC_4 I_{x5} \\
 &- S(2+3) SC_4 CC_5 I_{y5} - C(2+3) S_4 S_5 I_{z5} - S(2+3) SC_4 SS_5 I_{z5} + C(2+3) S_4 S_5 I_{z5}
 \end{aligned}$$



Joint Space Dynamics

$$M(q)\ddot{q} + V(q, \dot{q}) + G(q) = \Gamma$$

q : Generalized Joint Coordinates

$M(q)$: Mass Matrix - Kinetic Energy Matrix

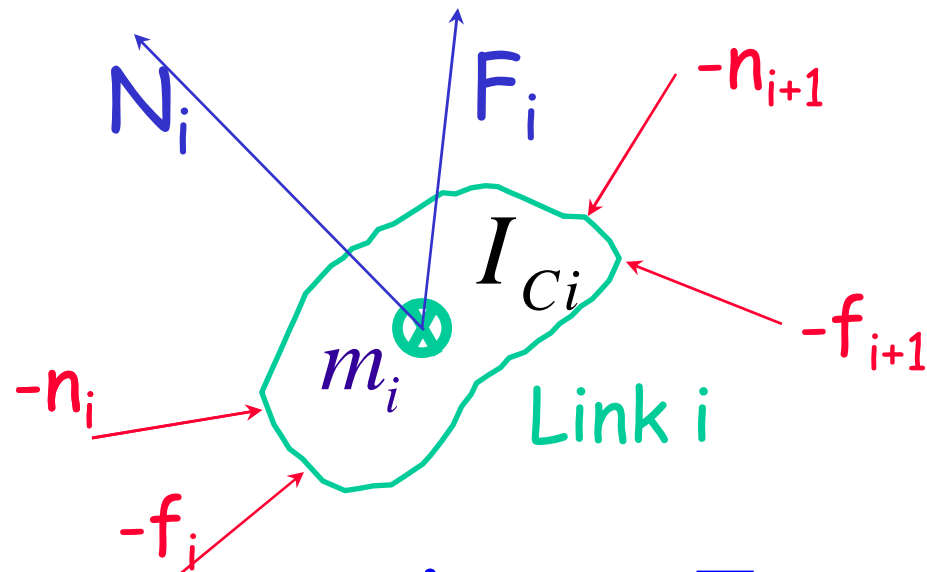
$V(q, \dot{q})$: Centrifugal and Coriolis forces

$G(q)$: Gravity forces

Γ : Generalized forces

Formulations

Newton-Euler



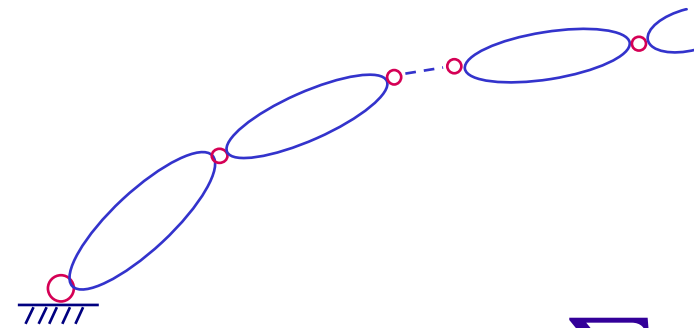
Newton: $m \dot{\mathbf{v}}_C = \mathbf{F}$

Euler: $\mathbf{N}_i = I_{C_i} \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times I_{C_i} \boldsymbol{\omega}_i$

Eliminate Internal Forces

$$\boldsymbol{\tau}_i = \begin{cases} \mathbf{n}_i^T \cdot \mathbf{Z}_i & \text{revolute} \\ \mathbf{f}_i^T \cdot \mathbf{Z}_i & \text{prismatic} \end{cases}$$

Lagrange



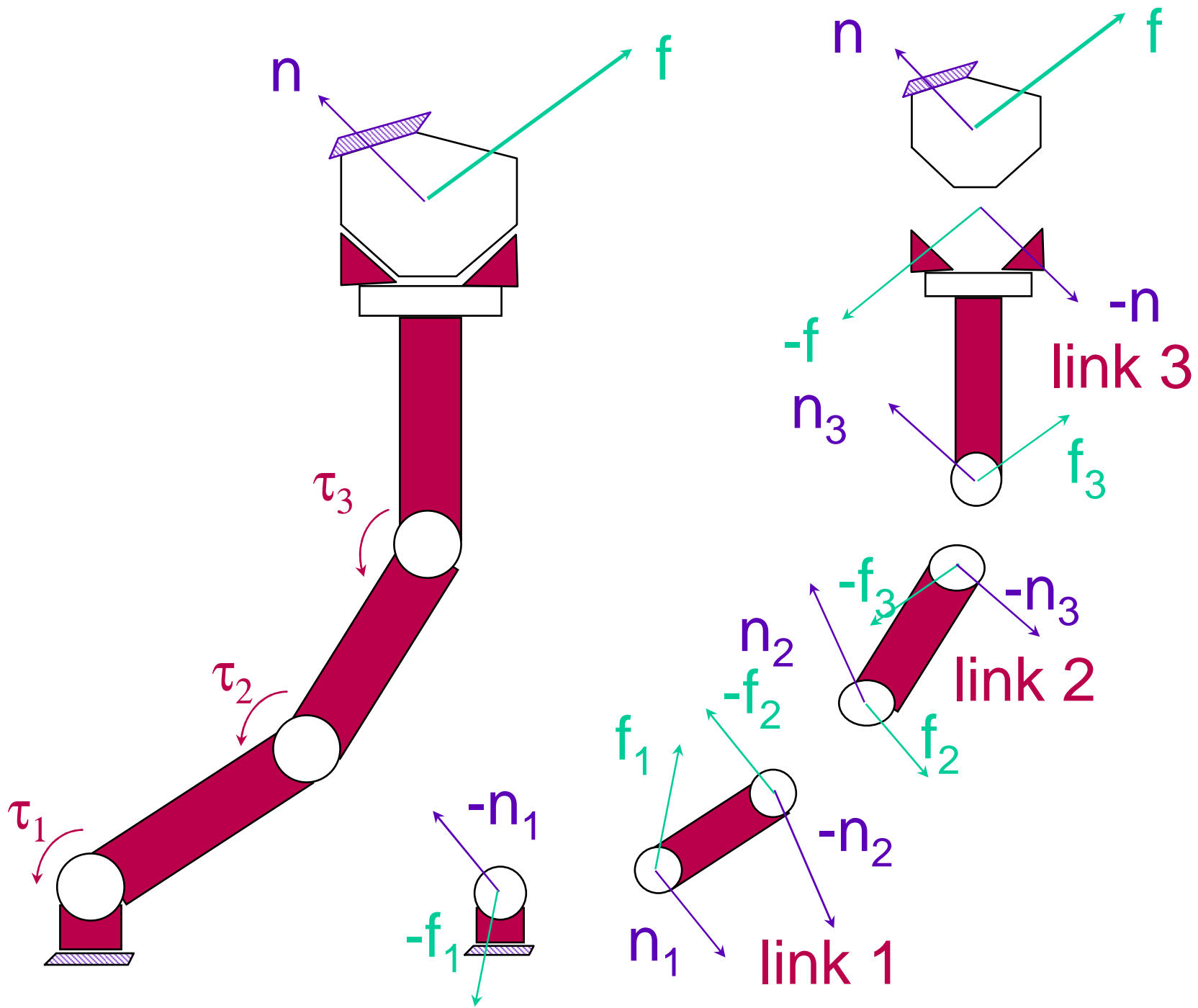
Kinetic Energy: $\sum_i K_i$

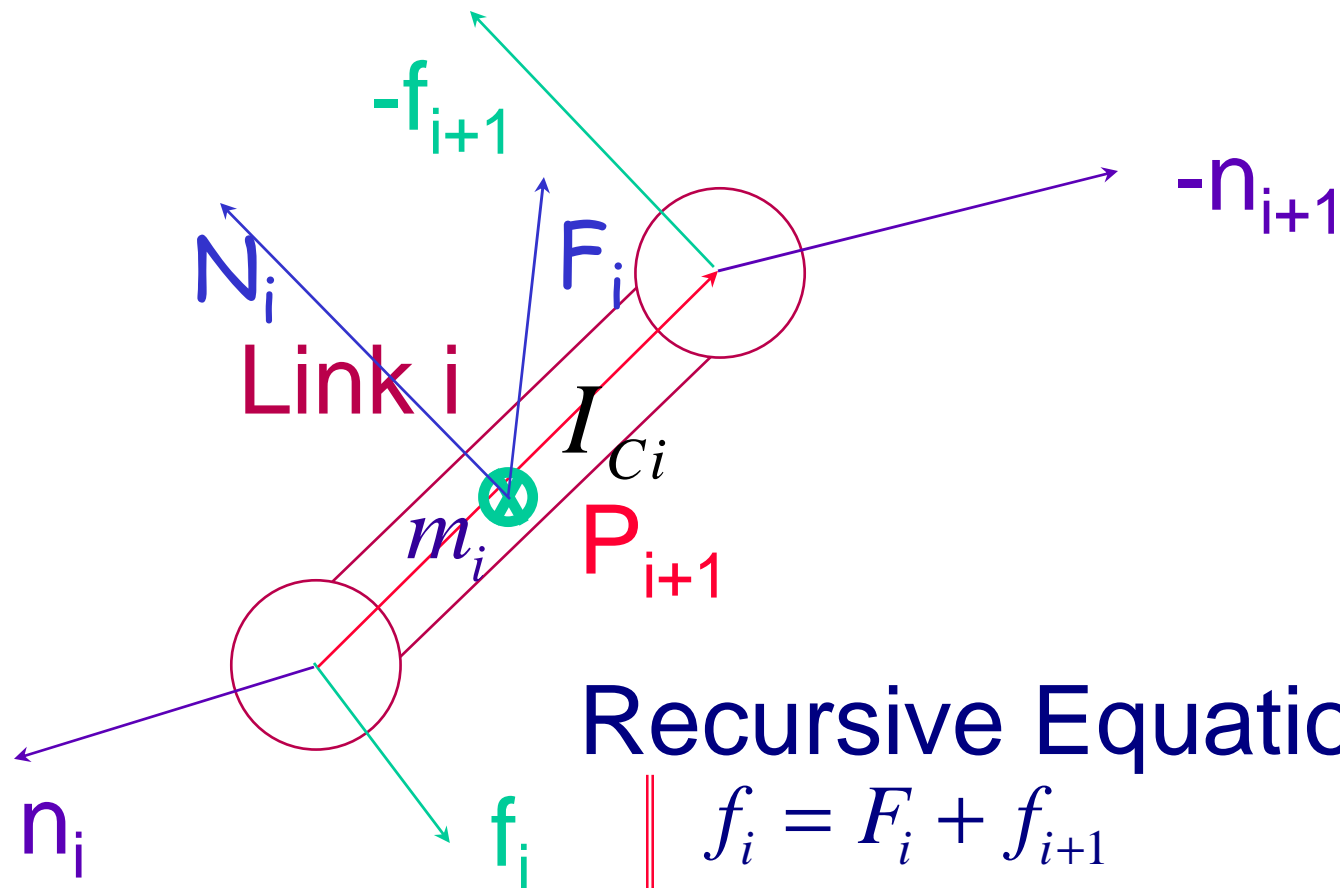
Potential Energy V

Generalized Coordinates

$$K = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}$$

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{V} + \mathbf{G} = \boldsymbol{\tau}$$





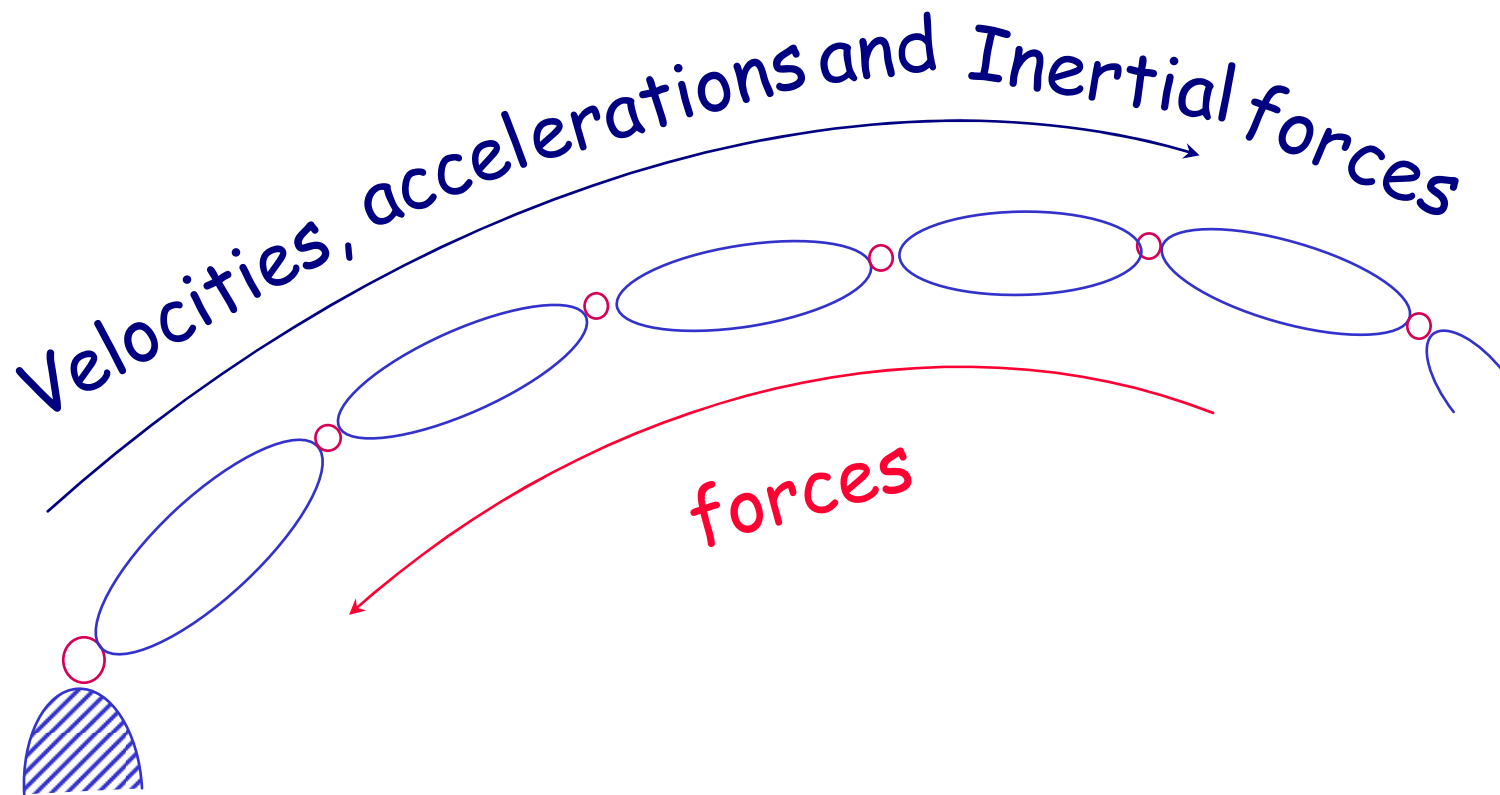
Recursive Equations

$$f_i = F_i + f_{i+1}$$

$$n_i = N_i + n_{i+1} + \mathbf{p}_{C_i} \times F_i + \mathbf{p}_{i+1} \times f_{i+1}$$

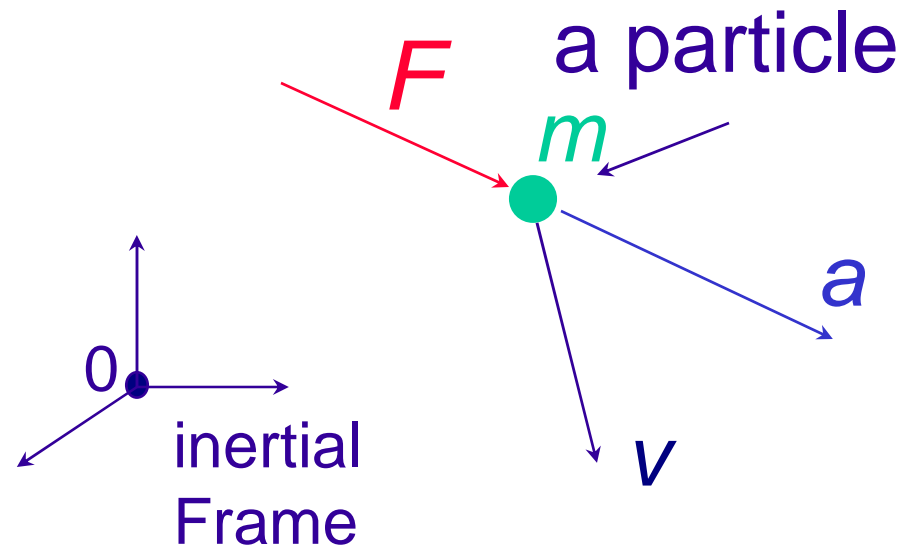
$$\tau_i = \begin{cases} n_i \cdot Z_i & \text{revolute} \\ f_i \cdot Z_i & \text{prismatic} \end{cases}$$

Newton-Euler Algorithm



Newton's Law

$$\underline{F} = m \underline{a}$$



$$\frac{d}{dt}(mv) = F$$

Linear Momentum

$$\underline{\varphi} = m\underline{v}$$

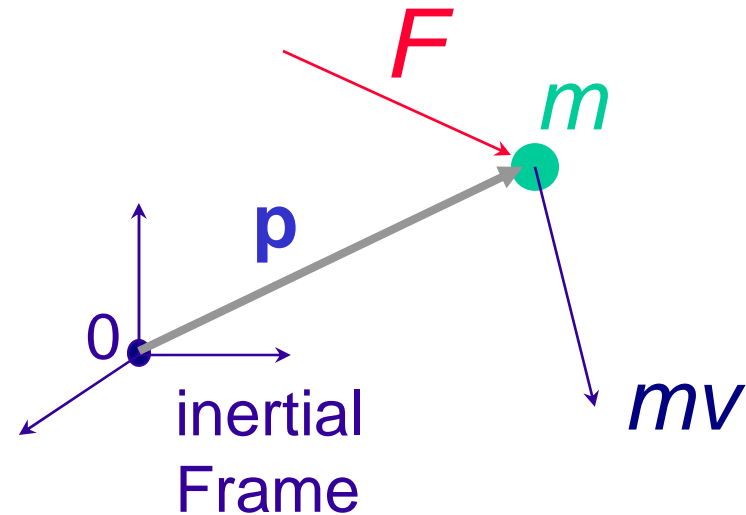
rate of change of the linear momentum is equal to the applied force

Angular Momentum

$$m\dot{\mathbf{v}} = \mathbf{F}$$

take the moment /0

$$\mathbf{p} \times m\dot{\mathbf{v}} = \mathbf{p} \times \mathbf{F}$$



$$\frac{d}{dt}(\mathbf{p} \times m\mathbf{v}) = \mathbf{p} \times m\dot{\mathbf{v}} + \mathbf{v} \times m\dot{\mathbf{v}} = \mathbf{p} \times m\dot{\mathbf{v}}$$

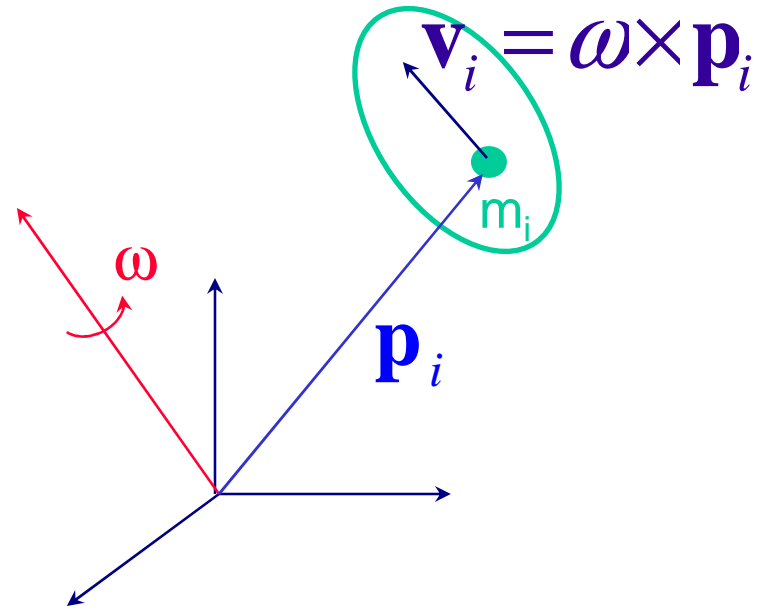
$$\frac{d}{dt}(\mathbf{p} \times m\mathbf{v}) = \mathbf{N}$$

applied moment

angular momentum $\phi = \mathbf{p} \times m\mathbf{v}$

Rigid Body

Rotational Motion



$$\text{Angular Momentum} = \sum_i \mathbf{p}_i \times m_i \mathbf{v}_i$$

$$\phi = \sum_i m_i \mathbf{p}_i \times (\omega \times \mathbf{p}_i)$$

$$m_i \rightarrow \rho dv \quad (\rho: \text{density})$$

$$\phi = \int_V \mathbf{p} \times (\omega \times \mathbf{p}) \rho dv$$

$$\phi = \int p \times (\omega \times p) \rho dv$$

$$\mathbf{p} \times (\omega \times \mathbf{p}) = \hat{\mathbf{p}}(-\hat{\mathbf{p}})\omega$$

$$\phi = \left[\int_{\mathcal{V}} -\hat{\mathbf{p}}\hat{\mathbf{p}}\rho dv \right] \omega$$

Inertia Tensor

$$\underline{\phi = I\omega}$$

Linear Momentum

$$\underline{\phi = mv}$$

Newton Equation

$$\frac{d}{dt}(mv) = F$$

$$\dot{\phi} = F$$

$$ma = F$$

Angular Momentum

$$\underline{\phi = I\omega}$$

Euler Equation

$$\frac{d}{dt}(I\omega) = N$$

$$\dot{\phi} = N$$

$$I\dot{\omega} + \omega \times I\omega = N$$

Inertia Tensor

$$I = \int_V -\hat{\mathbf{p}}\hat{\mathbf{p}}\rho dv \quad (-\hat{\mathbf{p}}\hat{\mathbf{p}}) = (\mathbf{p}^T \mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T$$

$$I = \int_V [(\mathbf{p}^T \mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T] \rho dv$$

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad ; \mathbf{p}^T \mathbf{p} = x^2 + y^2 + z^2$$
$$(\mathbf{p}^T \mathbf{p})I_3 = (x^2 + y^2 + z^2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{p}\mathbf{p}^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix} (x \quad y \quad z) = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}$$

$$(-\hat{\mathbf{p}}\hat{\mathbf{p}}) = \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & z^2 + x^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix}$$

Inertia Tensor

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Moments of
Inertia 

$$I_{xx} = \iiint (y^2 + z^2) \rho dx dy dz$$

$$I_{yy} = \iiint (z^2 + x^2) \rho dx dy dz$$

$$I_{zz} = \iiint (x^2 + y^2) \rho dx dy dz$$

$$I_{xy} = \iiint xy \rho dx dy dz$$

Products of
Inertia 

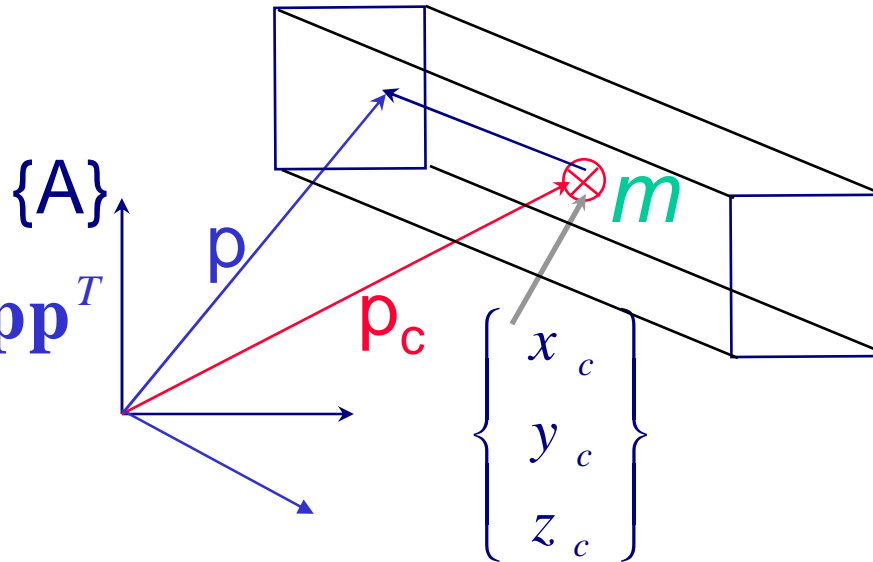
$$I_{xz} = \iiint xz \rho dx dy dz$$

$$I_{yz} = \iiint yz \rho dx dy dz$$

Parallel Axis theorem

$$I = \int_V -\hat{\mathbf{p}}\hat{\mathbf{p}}\rho dv$$

$$(-\hat{\mathbf{p}}\hat{\mathbf{p}}) = (\mathbf{p}^T \mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T$$

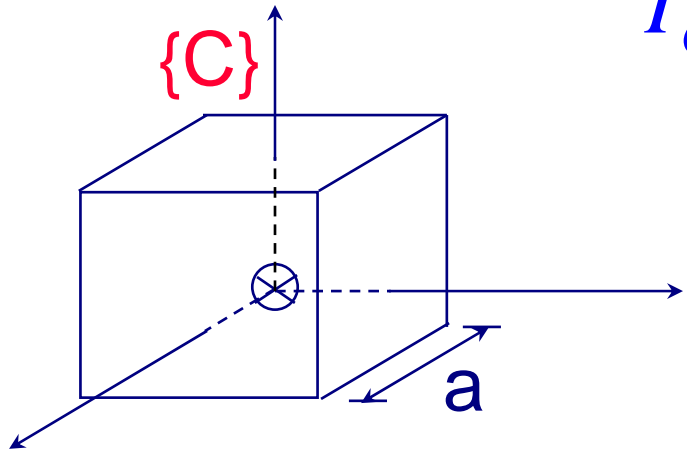


$$I_A = I_C + m [(\mathbf{p}_c^T \mathbf{p}_c)I_3 - \mathbf{p}_c \mathbf{p}_c^T]$$

$$I_{A_{zz}} = I_{C_{zz}} + m(x_c^2 + y_c^2)$$

$$I_{A_{xy}} = I_{C_{xy}} + m x_c y_c$$

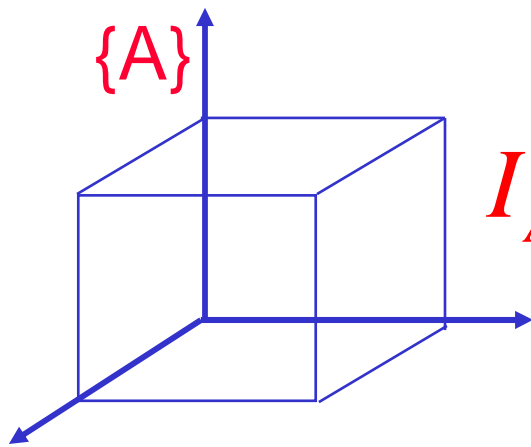
Example



$$I_{Czz} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \iint \rho(x^2 + y^2) dx dy dz$$

$$I_{Czz} = \frac{1}{6} \rho a^5; \quad m = \rho a^3$$

$$I_{Cxx} = I_{Cyy} = I_{Czz} = \frac{ma^2}{6}$$

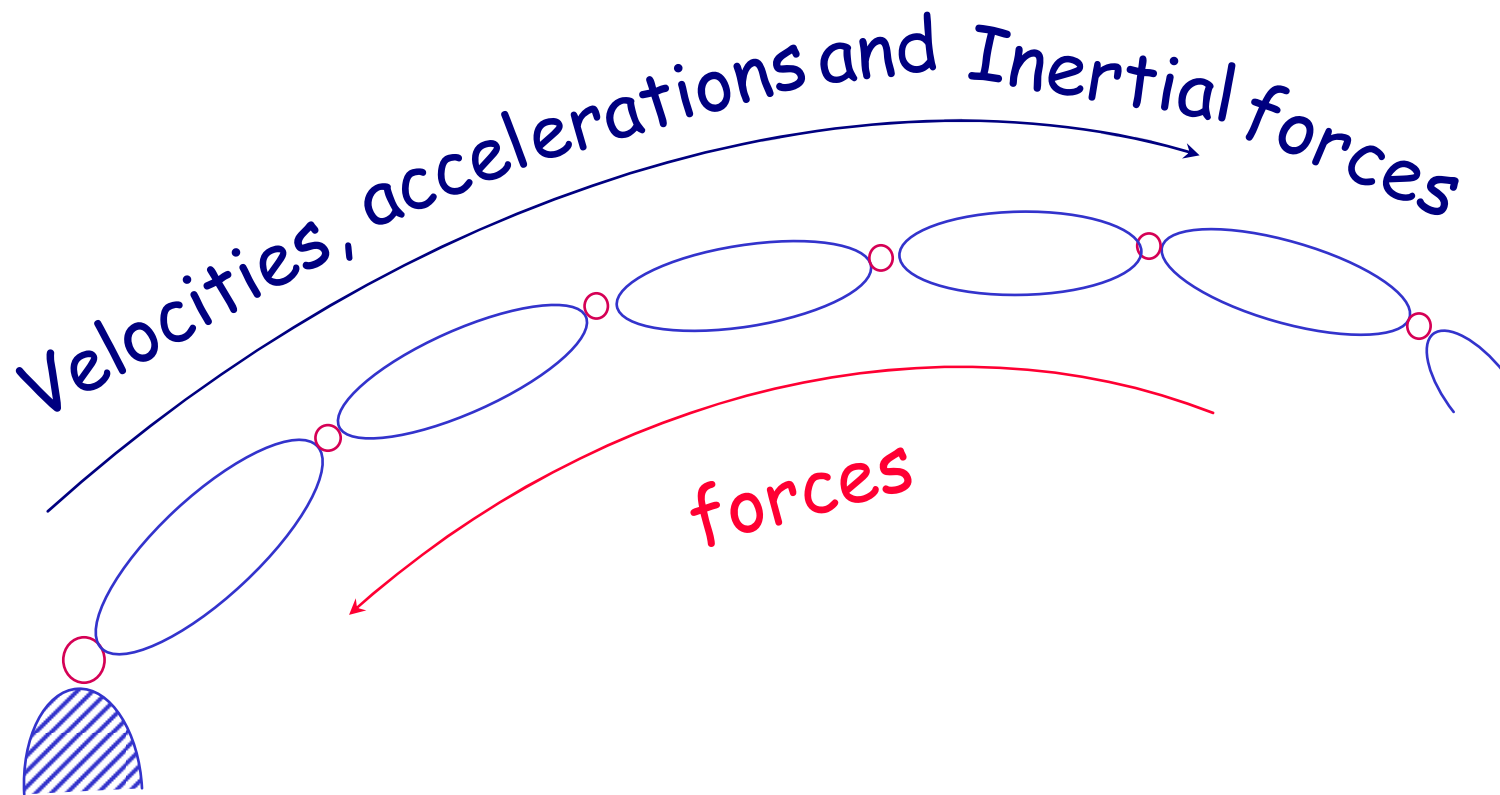


$${}^A x_c = {}^A y_c = {}^A z_c = \frac{a}{2}$$

$$I_{Axx} = I_{Ayy} = I_{Azz} = I_{Czz} + \frac{ma^2}{2} = \frac{2}{3} ma^2$$

$$I_{Axy} = I_{Axz} = I_{Ayz} = \frac{ma^2}{4}$$

Newton-Euler Algorithm



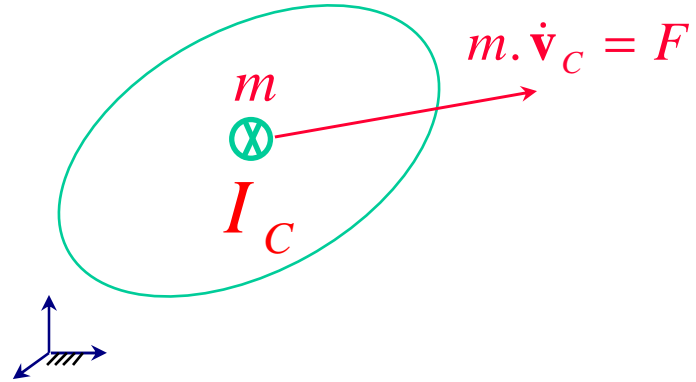
Newton-Euler Equations

Translational Motion

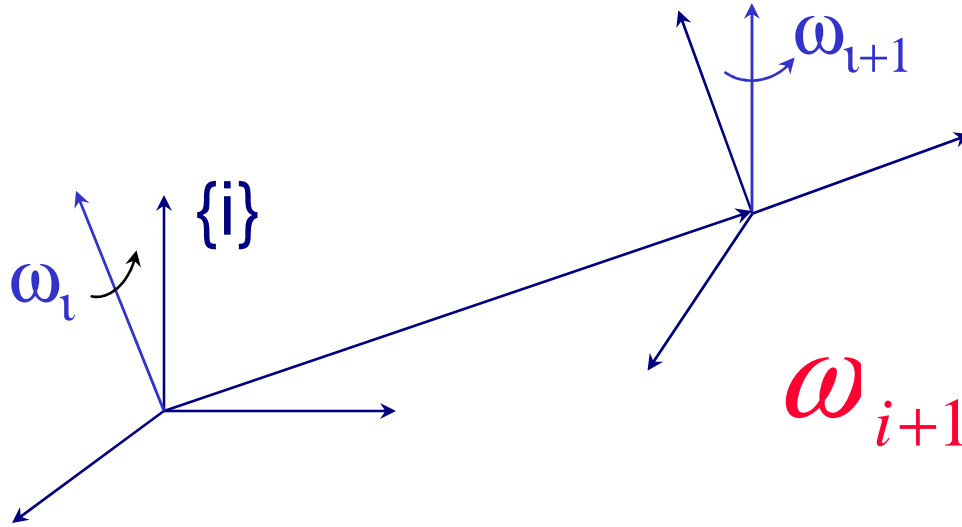
$$m \dot{\mathbf{v}}_C = F$$

Rotational Motion

$$I_C \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times I_C \boldsymbol{\omega} = N$$



Angular Acceleration

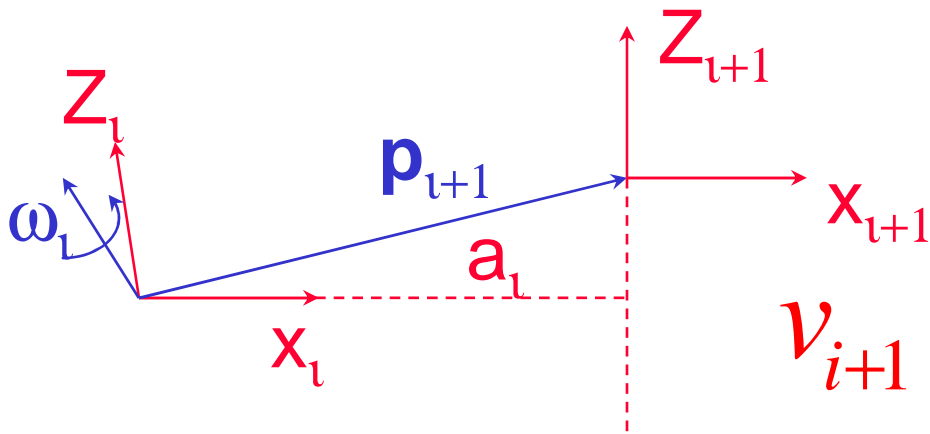


$$\omega_{i+1} = \omega_i + \Omega_{i+1}$$

$$\Omega_{i+1} = \dot{\theta}_{i+1} Z_{i+1}$$

$$\left\| \dot{\omega}_{i+1} = \dot{\omega}_i + \dot{\theta}_{i+1} (\omega_i \times Z_{i+1}) + \ddot{\theta}_{i+1} Z_{i+1} \right.$$

Linear Acceleration



$$v_{i+1} = v_i + \omega_i \times p_{i+1} + V_{i+1}$$

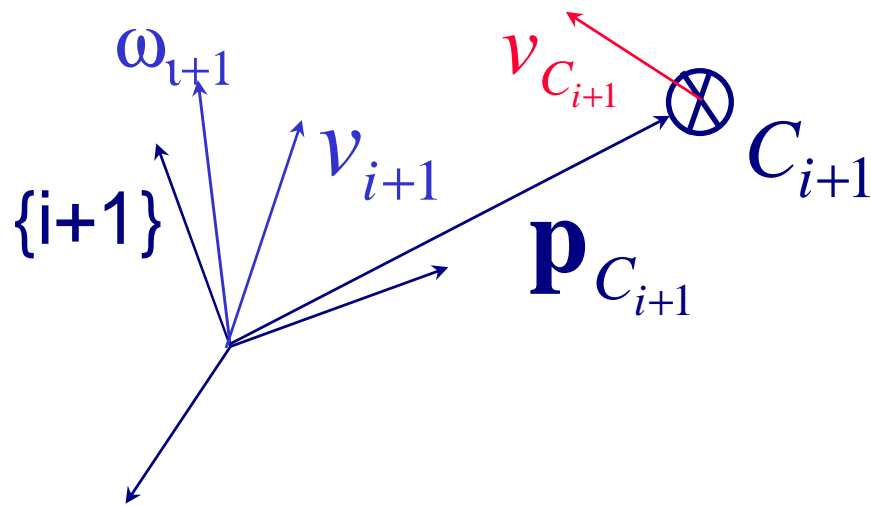
$$V_{i+1} = \dot{d}_{i+1} Z_{i+1}$$

$$P_{i+1} = a_i x_i + d_{i+1} Z_{i+1}$$

$$\dot{v}_{i+1} = \dot{v}_i + \dot{\omega}_i \times p_{i+1} + \omega_i \times \dot{p}_{i+1} + \dot{V}_{i+1}$$

$$\dot{v}_{i+1} = \dot{v}_i + \dot{\omega}_i \times p_{i+1} + \omega_i \times (\omega_i \times p_{i+1}) \\ + 2\dot{d}_{i+1} \omega_i \times Z_{i+1} + \ddot{d}_{i+1} Z_{i+1}$$

Velocity and Acceleration at center of mass



$$\mathbf{v}_{C_{i+1}} = \mathbf{v}_{i+1} + \boldsymbol{\omega}_{i+1} \times \mathbf{p}_{C_{i+1}}$$

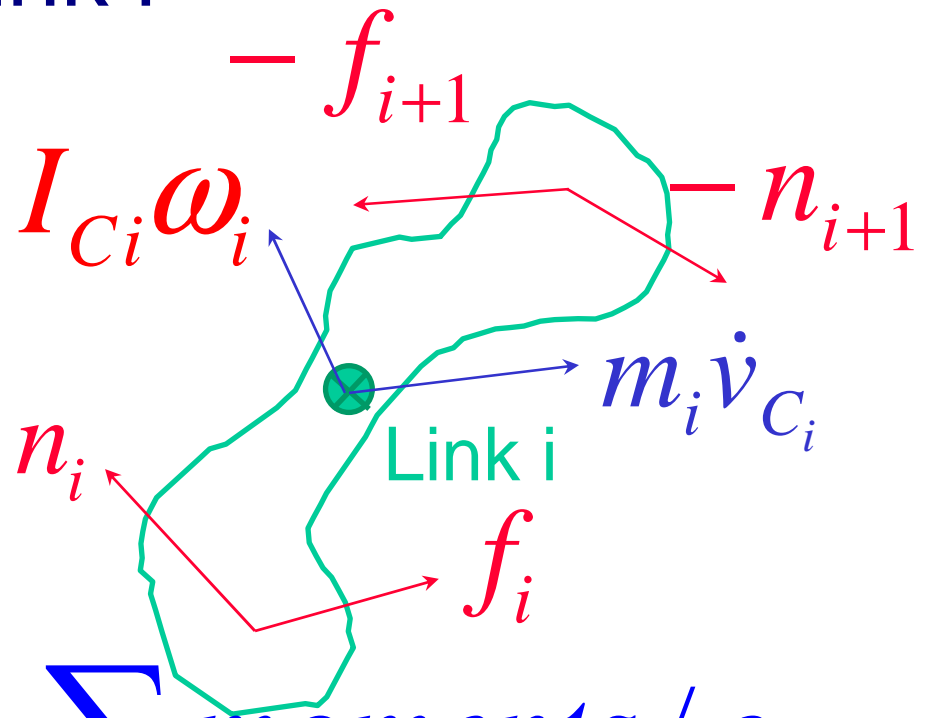
$$\dot{\mathbf{v}}_{C_{i+1}} = \dot{\mathbf{v}}_{i+1} + \dot{\boldsymbol{\omega}}_{i+1} \times \mathbf{p}_{C_{i+1}} + \boldsymbol{\omega}_{i+1} \times (\boldsymbol{\omega}_{i+1} \times \mathbf{p}_{C_{i+1}})$$

Dynamic forces on Link i

$$I_{C_i} \dot{\omega}_i + \omega_i \times I_{C_i} \omega_i$$

$$m_i \dot{v}_{C_i} = \sum \text{forces}$$

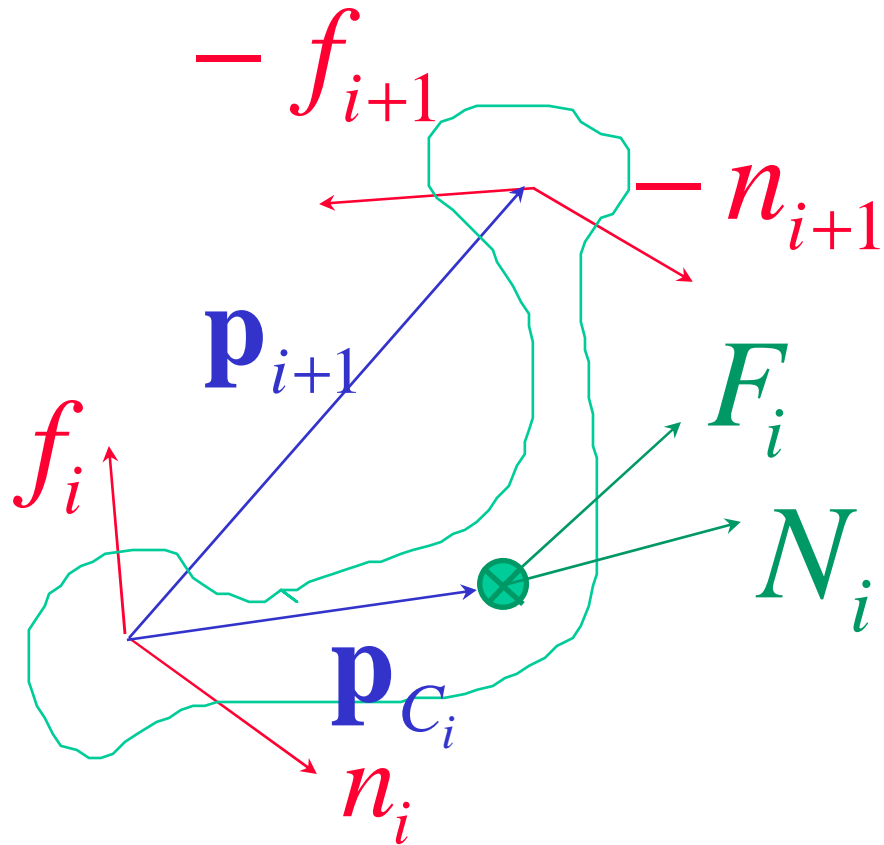
$$I_{C_i} \dot{\omega}_i + \omega_i \times I_{C_i} \omega_i = \sum \text{moments} / c_i$$



Inertial forces/moments

$$F_i = m_i \dot{v}_{C_i}$$

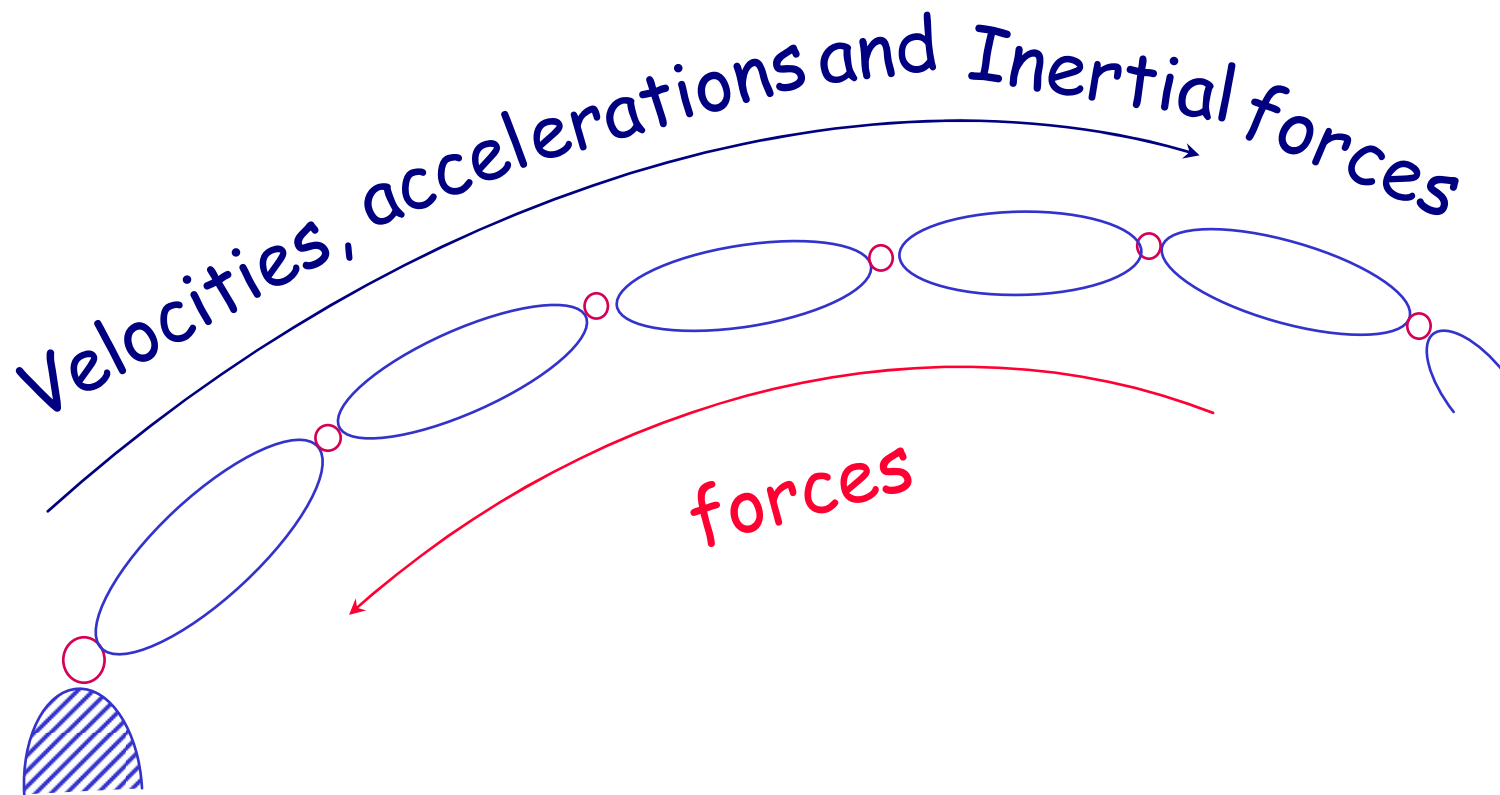
$$N_i = I_{C_i} \dot{\omega}_i + \omega_i \times I_{C_i} \omega_i$$



$$F_i = f_i - f_{i+1}$$

$$N_i = n_i - n_{i+1} + (-\mathbf{p}_{C_i}) \times f_i + (\mathbf{p}_{i+1} - \mathbf{p}_{C_i}) \times (-f_{i+1})$$

Newton-Euler Algorithm



Recursive Equations

$$f_i = F_i + f_{i+1}$$

$$n_i = N_i + n_{i+1} + \mathbf{p}_{C_i} \times F_i + \mathbf{p}_{i+1} \times f_{i+1}$$

$$\tau_i = \begin{cases} n_i \cdot Z_i & \text{revolute} \\ f_i \cdot Z_i & \text{prismatic} \end{cases}$$

with

$$F_i = m_i \dot{\mathbf{v}}_{C_i}$$

$$N_i = I_{C_i} \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times I_{C_i} \boldsymbol{\omega}_i$$

where

$$\boldsymbol{\omega}_{i+1} = \boldsymbol{\omega}_i + \boldsymbol{\Omega}_{i+1} = \boldsymbol{\omega}_i + \dot{\theta}_{i+1} Z_{i+1}$$

$$\dot{\boldsymbol{\omega}}_{i+1} = \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times Z_{i+1} \dot{\theta}_{i+1} + \ddot{\theta}_{i+1} Z_{i+1}$$

$$\dot{\mathbf{v}}_{i+1} = \dot{\mathbf{v}}_i + \dot{\boldsymbol{\omega}}_i \times \mathbf{p}_{i+1} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{p}_{i+1}) + 2\dot{d}_{i+1} \boldsymbol{\omega}_i \times Z_{i+1} + \ddot{d}_{i+1} Z_{i+1}$$

$$\dot{\mathbf{v}}_{C_{i+1}} = \dot{\mathbf{v}}_{i+1} + \dot{\boldsymbol{\omega}}_{i+1} \times \mathbf{p}_{C_{i+1}} + \boldsymbol{\omega}_{i+1} \times (\boldsymbol{\omega}_{i+1} \times \mathbf{p}_{C_{i+1}})$$

Outward iterations: $i : 0 \longrightarrow 5$

$${}^{i+1}\boldsymbol{\omega}_{i+1} = {}^{i+1}_i R^i \boldsymbol{\omega}_i + \dot{\boldsymbol{\theta}}_{i+1} {}^{i+1} \mathbf{Z}_{i+1}$$

$${}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} = {}^{i+1}_i R^i \dot{\boldsymbol{\omega}}_i + {}^{i+1}_i R^i \boldsymbol{\omega}_i \times {}^{i+1} \mathbf{Z}_{i+1} \dot{\boldsymbol{\theta}}_{i+1} + \ddot{\boldsymbol{\theta}}_{i+1} {}^{i+1} \mathbf{Z}_{i+1}$$

$${}^{i+1}\dot{\mathbf{v}}_{i+1} = {}^{i+1}_i R^i (\dot{\boldsymbol{\omega}}_i \times {}^i \mathbf{p}_{i+1} + {}^i \boldsymbol{\omega}_i \times ({}^i \boldsymbol{\omega}_i \times {}^i \mathbf{p}_{i+1}) + {}^i \dot{\mathbf{v}}_i)$$

$${}^{i+1}\dot{\mathbf{v}}_{C_{i+1}} = {}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} \times {}^{i+1} \mathbf{p}_{C_{i+1}} + {}^{i+1} \boldsymbol{\omega}_{i+1} \times ({}^{i+1} \boldsymbol{\omega}_{i+1} \times {}^{i+1} \mathbf{p}_{C_{i+1}}) + {}^{i+1}\dot{\mathbf{v}}_{i+1}$$

$${}^{i+1} \mathbf{F}_{i+1} = m_{i+1} {}^{i+1}\dot{\mathbf{v}}_{C_{i+1}}$$

$${}^{i+1} \mathbf{N}_{i+1} = {}^{C_{i+1}} I_{i+1} {}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} + {}^{i+1} \boldsymbol{\omega}_{i+1} \times {}^{C_{i+1}} I_{i+1} {}^{i+1} \boldsymbol{\omega}_{i+1}$$

Inward iterations: $i : 6 \longrightarrow 1$

$${}^i f_i = {}^i R^{i+1} f_{i+1} + {}^i F_i$$

$${}^i n_i = {}^i N_i + {}^i R^{i+1} n_{i+1} + {}^i \mathbf{p}_{C_i} \times {}^i F_i + {}^i \mathbf{p}_{i+1} \times {}^i R^{i+1} f_{i+1}$$

$$\boldsymbol{\tau}_i = {}^i n_i^T {}^i \mathbf{Z}_i \quad \text{Gravity: set } {}^0 \dot{\mathbf{v}}_0 = 1G$$