Movie Segment

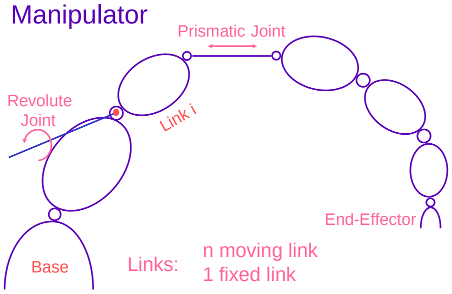
Passive Walking, Tad McGeer,
Fraser Univ, British Columbia,
ICRA 1991 video proceedings

Kinematics

Spatial Descriptions

- Task Description
- Transformations
- Representations

Manipulator

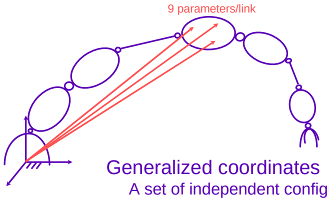


Links: n moving link
1 fixed link

Joints: Revolute (1 DOF)
Prismatic (1 DOF)

Configuration Parameters

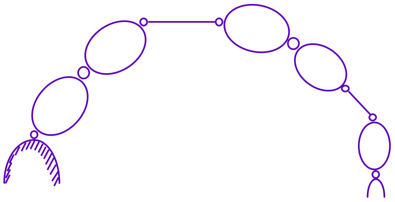
A set of position parameters that describes the full configuration of the system.



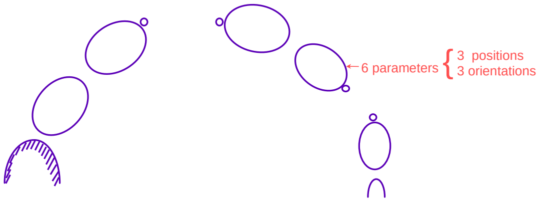
Generalized coordinates
A set of independent configuration parameters

Degrees of Freedom
Number of generalized coordinates

Generalized Coordinates

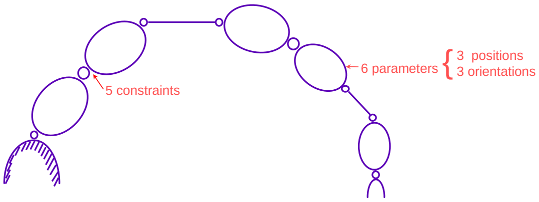


Generalized Coordinates



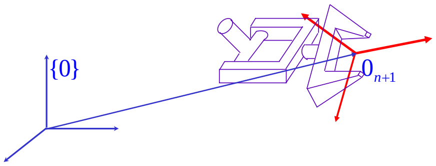
n moving links: 6n parameters

Generalized Coordinates



n moving links: 6n parameters
n 1 d.o.f. joints: 5n constraints
d.o.f. (system): 6n - 5n = n

End-Efector Configuration Parameters



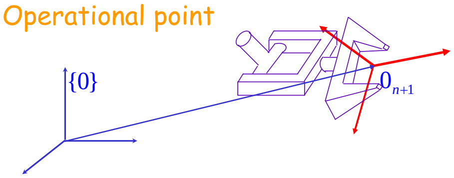
A set of m parameters:

$$(x_1, x_2, x_3, \ldots, x_m)$$

that completely specifies the end-effector position and orientation with respect to $\{0\}$

Operational Coordinates

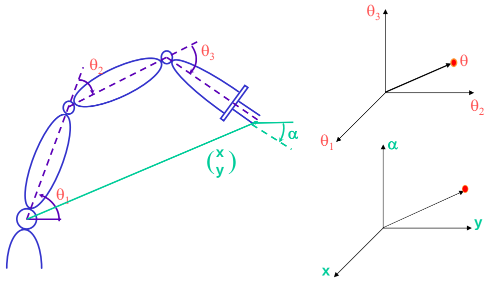
O_{n+1} : Operational point



A set $x_1, x_2, \ldots, x_{m_0}$
of m_0 independent configuration parameters

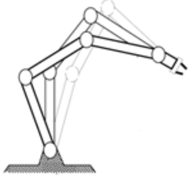
m_0 : number of degrees of freedom of the end-effector.

Joint Coordinates → Joint Space



Operational Coordinates → Operational Space

Redundancy

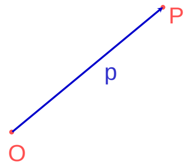


A robot is said to be redundant if

$$\boxed{n > m_0}$$

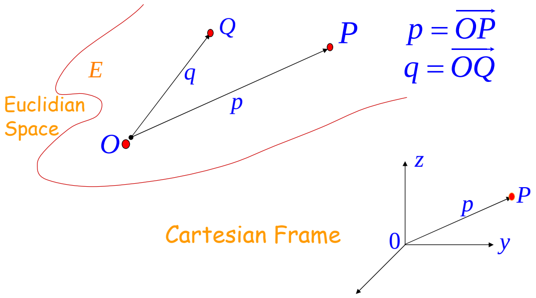
Degrees of redundancy: $n - m_0$

Position of a Point



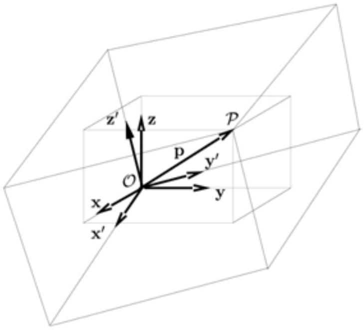
With respect to a fixed origin O, the position of a point P is described by the vector OP or simply by p.

Rigid Body Configuration

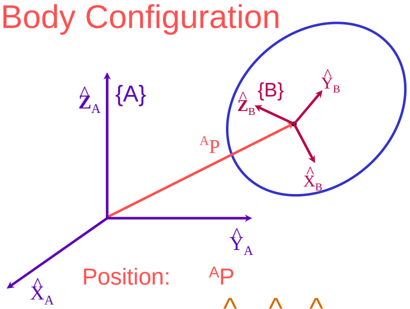


$$p = \overrightarrow{OP}$$
$$q = \overrightarrow{OQ}$$

Coordinate Frames



Rigid Body Configuration



Position: ^AP

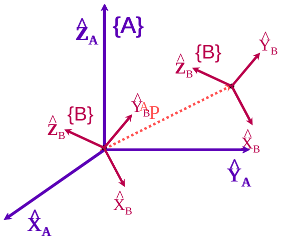
Orientation: $\{^A\hat{X}_B, {}^A\hat{Y}_B, {}^A\hat{Z}_B\}$

Describes rotations of {B} with respect to {A}

➡ **Rotation Matrix**

Rotation Matrix

$$^A_BR = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

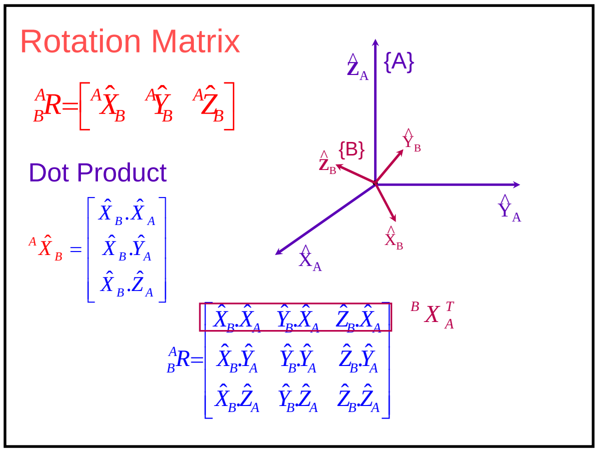


$$^A\hat{X}_B = {}^A_BR\, {}^B\hat{X}_B$$

$$^A\hat{X}_B = {}^A_BR \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad {}^A\hat{Y}_B = {}^A_BR \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad {}^A\hat{Z}_B = {}^A_BR \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

➡

$$^A_BR = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix}$$



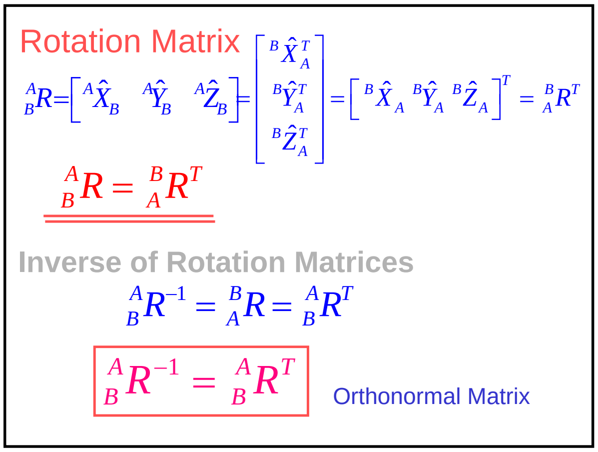
Rotation Matrix

$${}^A_BR=\begin{bmatrix}{}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B\end{bmatrix}$$

Dot Product

$${}^A\hat{X}_B=\begin{bmatrix}\hat{X}_B.\hat{X}_A\\ \hat{X}_B.\hat{Y}_A\\ \hat{X}_B.\hat{Z}_A\end{bmatrix}$$

$${}^A_BR=\begin{bmatrix}\boxed{\hat{X}_B.\hat{X}_A \quad \hat{Y}_B.\hat{X}_A \quad \hat{Z}_B.\hat{X}_A}\\ \hat{X}_B.\hat{Y}_A \quad \hat{Y}_B.\hat{Y}_A \quad \hat{Z}_B.\hat{Y}_A\\ \hat{X}_B.\hat{Z}_A \quad \hat{Y}_B.\hat{Z}_A \quad \hat{Z}_B.\hat{Z}_A\end{bmatrix} \quad {}^BX_A^T$$



Rotation Matrix

$${}^A_BR=\begin{bmatrix}{}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B\end{bmatrix}=\begin{bmatrix}{}^B\hat{X}_A^T\\ {}^B\hat{Y}_A^T\\ {}^B\hat{Z}_A^T\end{bmatrix}=\begin{bmatrix}{}^B\hat{X}_A & {}^B\hat{Y}_A & {}^B\hat{Z}_A\end{bmatrix}^T={}^B_AR^T$$

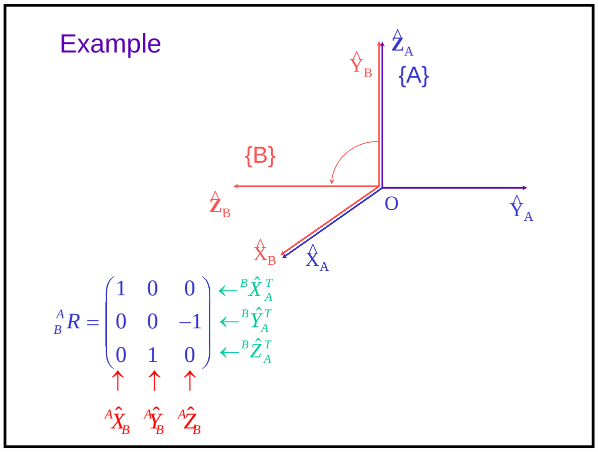
$$\underline{\underline{{}^A_BR={}^B_AR^T}}$$

Inverse of Rotation Matrices

$${}^A_BR^{-1}={}^B_AR={}^A_BR^T$$

$$\boxed{{}^A_BR^{-1}={}^A_BR^T}$$

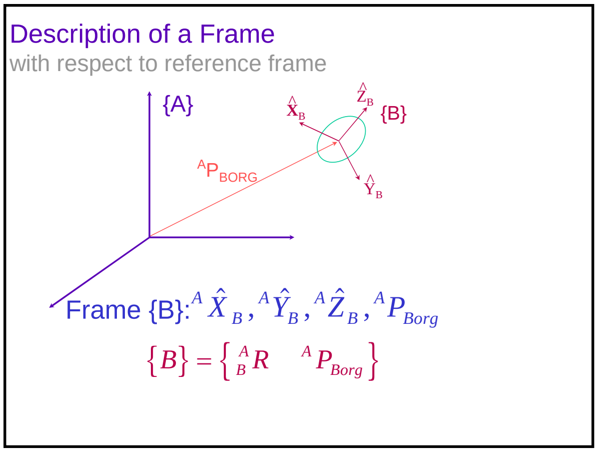
Orthonormal Matrix



Example

$${}^A_BR=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\begin{matrix}\leftarrow {}^B\hat{X}_A^T\\ \leftarrow {}^B\hat{Y}_A^T\\ \leftarrow {}^B\hat{Z}_A^T\end{matrix}$$

$$\begin{matrix}\uparrow & \uparrow & \uparrow\\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B\end{matrix}$$

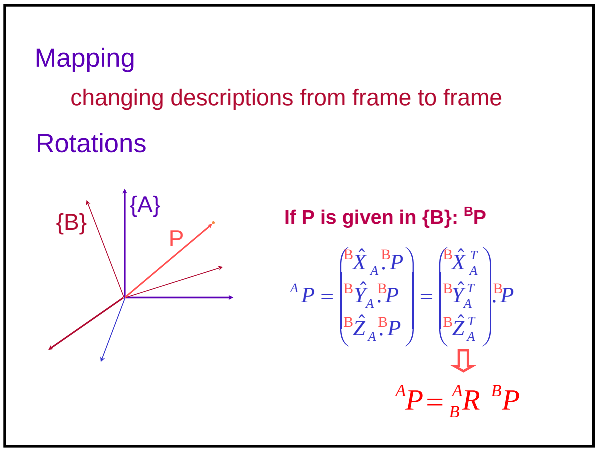


Description of a Frame

with respect to reference frame

Frame {B}: ${}^A\hat{X}_B, {}^A\hat{Y}_B, {}^A\hat{Z}_B, {}^AP_{Borg}$

$$\{B\}=\left\{{}^A_BR \quad {}^AP_{Borg}\right\}$$



Mapping

changing descriptions from frame to frame

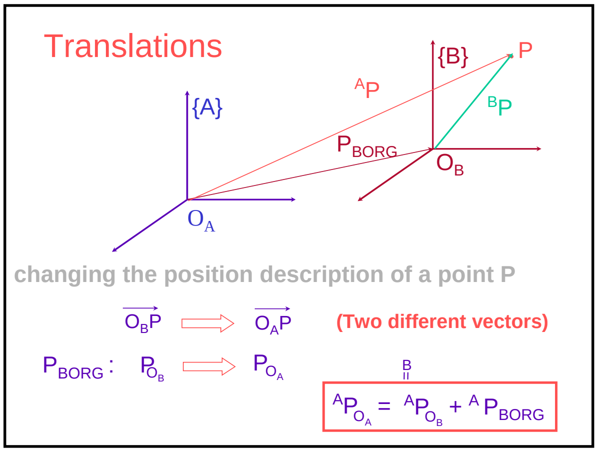
Rotations

If P is given in {B}: ${}^BP$

$${}^AP=\begin{pmatrix}{}^B\hat{X}_A.{}^BP\\ {}^B\hat{Y}_A.{}^BP\\ {}^B\hat{Z}_A.{}^BP\end{pmatrix}=\begin{pmatrix}{}^B\hat{X}_A^T\\ {}^B\hat{Y}_A^T\\ {}^B\hat{Z}_A^T\end{pmatrix}.{}^BP$$

⇩

$${}^AP={}^A_BR\ {}^BP$$



Translations

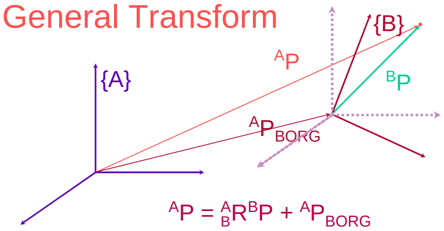
changing the position description of a point P

$\overrightarrow{O_BP} \Rightarrow \overrightarrow{O_AP}$ **(Two different vectors)**

P_{BORG} : $P_{O_B} \Rightarrow P_{O_A}$

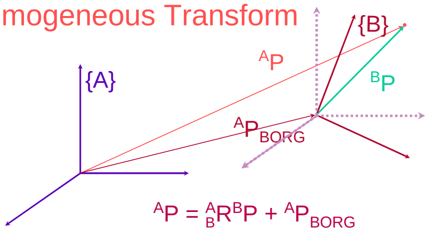
$$\boxed{{}^AP_{O_A}={}^A\overset{\overset{B}{\shortparallel}}{P}_{O_B}+{}^AP_{BORG}}$$

General Transform



$^AP = {}^A_BR\,{}^BP + {}^AP_{BORG}$

Homogeneous Transform

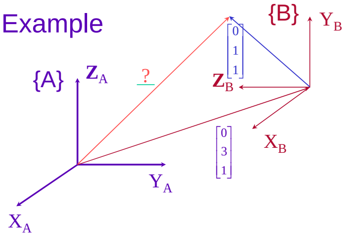


$^AP = {}^A_BR\,{}^BP + {}^AP_{BORG}$

$$\begin{bmatrix} {}^AP \\ \hline 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^A_BR & & {}^AP_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^BP \\ \hline 1 \end{bmatrix}$$

$$\underline{\underline{\underset{(4x1)}{{}^AP} = \underset{(4x4)}{{}^A_BT}\ \underset{(4x1)}{{}^BP}}}$$

Example



Homogeneous Transform

$$^A_BT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^BP = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

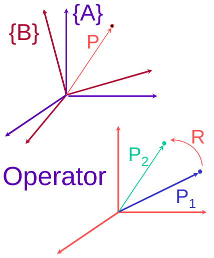
$$^AP = {}^A_BT.{}^BP \Rightarrow {}^AP = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Operators

Mapping: changing descriptions from frame to frame

Operators: moving points (within the same frame)

Mapping



$^AP = {}^A_BR\ {}^BP$

Rotational Operator

R: $P_1 \longrightarrow P_2$

$$\boxed{P_2 = R\ P_1}$$

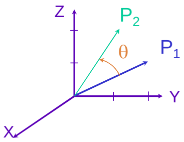
Rotational Operators

$R_K(\theta)$: $P_1 \longrightarrow P_2$

$P_2 = R_K(\theta)\ P_1$

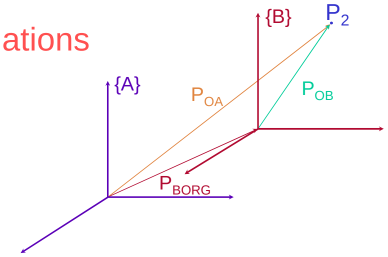
Example

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



$$P_2 = R_X(\theta)P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Translations

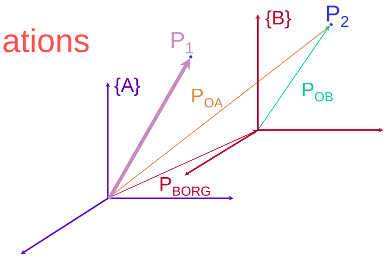


Mapping: P_{BORG} : $P_{OB} \longrightarrow P_{OA}$ (same point)

2 diff. vectors

$P_{OA} = P_{OB} + P_{BORG}$

Translations



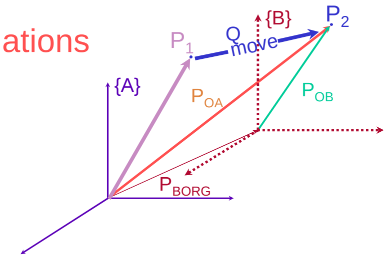
Mapping: $P_{BORG}: \; P_{OB} \longrightarrow P_{OA}$ (same point)

2 diff. vectors

$P_{OA} = P_{OB} + P_{BORG}$

Translational Operator:

Translations



Mapping: $P_{BORG}: \; P_{OB} \longrightarrow P_{OA}$ (same point)

2 diff. vectors

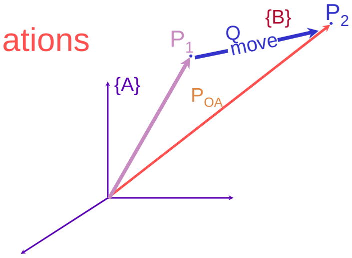
$P_{OA} = P_{OB} + P_{BORG}$

Translational Operator:

$Q: \quad P_1 \longrightarrow P_2$ (2 points, 2 diff vectors)

$P_2 = P_1 + Q$

Translations



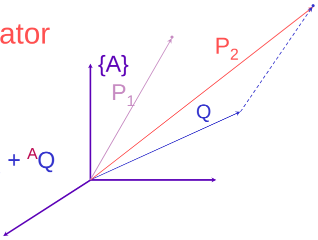
Translational Operator:

$Q: \quad P_1 \longrightarrow P_2$ (2 points, 2 diff vectors)

$P_2 = P_1 + Q$

Translation Operator

Operator: ${}^{A}P_2 = {}^{A}P_1 + {}^{A}Q$



Homogeneous Transform:

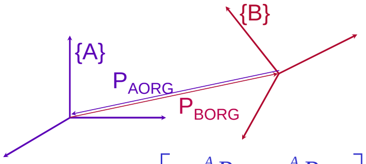
$$D_Q = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow {}^{A}P_2 = {}^{A}D_Q\,{}^{A}P_1$$

General Operators

$$P_2 = \left(\begin{array}{c|c} R_K(\theta) & Q \\ \hline 0\;\;0\;\;0 & 1 \end{array} \right) P_1$$

$$\boxed{P_2 = \; T\,P_1}$$

Inverse Transform



$${}^{A}_{B}T = \begin{bmatrix} & {}^{A}_{B}R & & {}^{A}P_{Borg} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R^{-1} = R^{T} \quad (T^{-1} \neq T^{T})$

$${}^{A}_{B}T^{-1} = {}^{B}_{A}T = \begin{bmatrix} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T} \cdot {}^{A}P_{Borg} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boxed{{}^{B}P_{AORG}}$

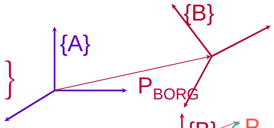
Movie Segment

Flexible Microactuator, Toshiba,
ICRA 1991 video proceedings

Homogeneous Transform Interpretations

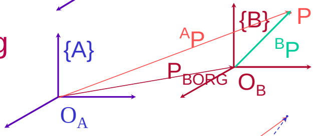
Description of a frame

${}^A_BT$: $\{B\} = \{ {}^A_BR \quad {}^AP_{Borg} \}$



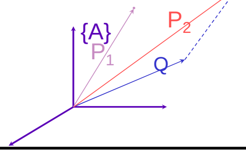
Transform mapping

${}^A_BT : {}^BP \rightarrow {}^AP$

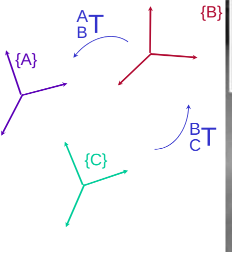


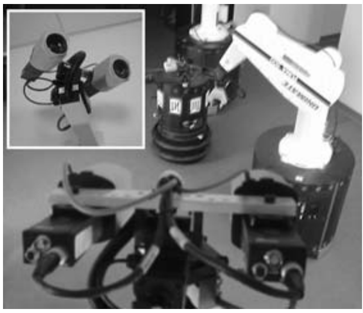
Transform operator

$T : P_1 \rightarrow P_2$

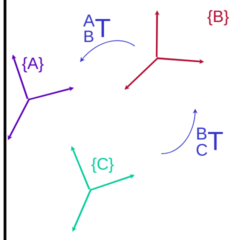


Transform Equation

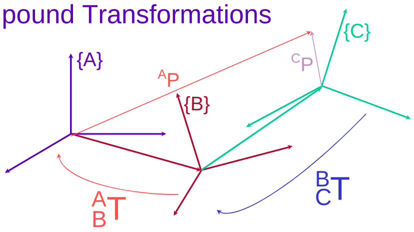




Transform Equation



Compound Transformations



$${}^BP = {}^B_CT\ {}^CP$$

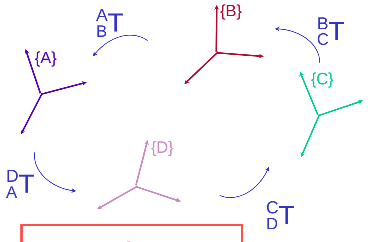
$${}^AP = {}^A_BT\ {}^BP$$

$${}^AP = {}^A_BT\ {}^B_CT\ {}^CP \Rightarrow {}^A_CT = {}^A_BT\ {}^B_CT$$

$${}^A_C T = {}^A_B T \; {}^B_C T$$

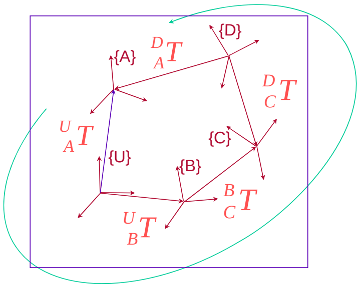
$${}^A_C T = \begin{bmatrix} {}^A_B R \, {}^B_C R & {}^A_B R \, {}^B P_{Corg} + {}^A P_{Borg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

Transform Equation



$$\boxed{{}^A_B T \; {}^B_C T \; {}^C_D T \; {}^D_A T \equiv I}$$

$$\Rightarrow \quad {}^B_A T = {}^B_C T \; {}^C_D T \; {}^D_A T$$



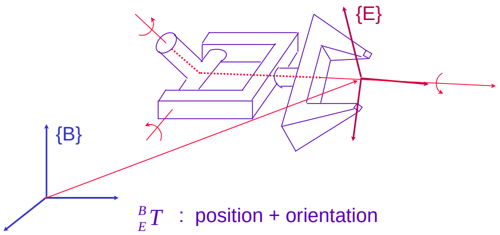
$${}^D_A T^{-1} \cdot {}^D_C T \cdot {}^B_C T^{-1} \cdot {}^U_B T^{-1} \cdot {}^U_A T \equiv I$$

$${}^U_A T = {}^U_B T \cdot {}^B_C T \cdot {}^D_C T^{-1} \cdot {}^D_A T$$

Spatial Descriptions

- Task Description
- Transformations
- Representations ⇐

End-Effector Configuration



${}^B_E T$: position + orientation

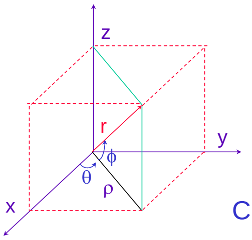
End-Effector Configuration Parameters

$$X = \begin{bmatrix} X_P \\ X_R \end{bmatrix}$$

position

orientation

Position Representations



Cartesian:	(x, y, z)
Cylindrical:	(ρ, θ, z)
Spherical:	(r, θ, ϕ)

Rotation Representations

Rotation Matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix}$$

Direction Cosines

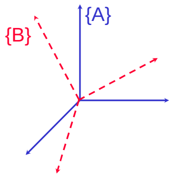
$$x_r = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}_{(9x1)}$$

Constraints

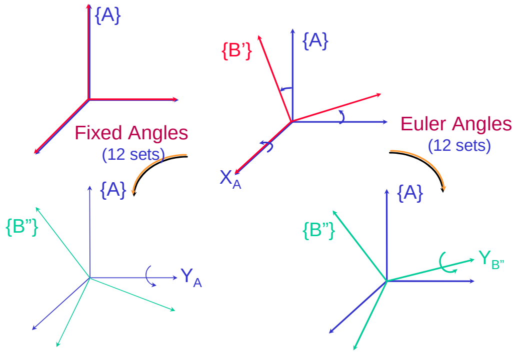
$$|\mathbf{r}_1| = |\mathbf{r}_2| = |\mathbf{r}_3| = 1$$

$$\mathbf{r}_1.\mathbf{r}_2 = \mathbf{r}_1.\mathbf{r}_3 = \mathbf{r}_2.\mathbf{r}_3 = 0$$

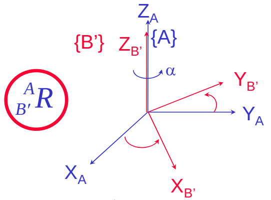
Three Angle Representations

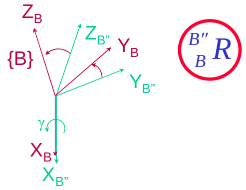


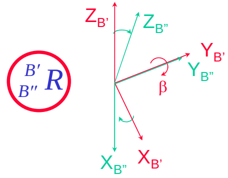
Three Angle Representations



Euler Angles (Z-Y-X)







$${}^{A}_{B'}R$$

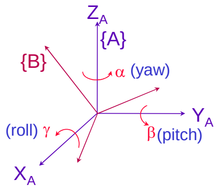
$${}^{B''}_{B}R$$

$${}^{B'}_{B''}R$$

$${}^{A}_{B}R = {}^{A}_{B'}R.{}^{B'}_{B''}R.{}^{B''}_{B}R$$

$$\boxed{{}^{A}_{B}R = R_Z(\alpha).R_Y(\beta).R_X(\gamma)}$$

X-Y-Z Fixed Angles



$R_X(\gamma)$: $v \to R_X(\gamma).v$

$R_Y(\beta)$: $(R_X(\gamma).v) \to R_Y(\beta).(R_X(\gamma).v)$

$R_Z(\alpha)$: $(R_Y(\beta).R_X(\gamma).v) \to R_Z(\alpha).(R_Y(\beta).R_X(\gamma).v)$

$$\boxed{{}^{A}_{B}R = {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)}$$

Z-Y-X Euler Angles

$${}^{A}_{B}R = R_{Z'}(\alpha).R_{Y'}(\beta).R_{X'}(\gamma)$$

$$\begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.\begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}.\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

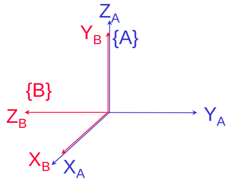
$${}^{A}_{B}R = {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha.c\beta & X & X \\ s\alpha.c\beta & X & X \\ -s\beta & c\beta.s\gamma & c\beta.c\gamma \end{bmatrix}$$

Z-Y-Z Euler Angles

$${}^{A}_{B}R = R_{Z'}(\alpha).R_{Y'}(\beta).R_{Z'}(\gamma)$$

$${}^{A}_{B}R = {}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} X & X & c\alpha.s\beta \\ X & X & s\alpha.s\beta \\ -s\beta.c\gamma & s\beta.s\gamma & c\beta \end{bmatrix}$$

Example



$R_{Z'Y'X'}(\alpha,\beta,\gamma)$: $\alpha = 0$

$\beta = 0$

$\gamma = 90^o$

Fixed & Euler Angles

X-Y-Z Fixed Angles

$$R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

Z-Y-X Euler Angles

$$R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

$$\boxed{R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_{XYZ}(\gamma,\beta,\alpha)}$$

Inverse Problem

Given ${}^{A}_{B}R$ find (α,β,γ)

$$\xrightarrow{} R_{Z'Y'X'}$$

$${}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha.c\beta & c\alpha.s\beta.s\gamma - s\alpha.c\gamma & c\alpha.s\beta.c\gamma + s\alpha.s\gamma \\ s\alpha.c\beta & s\alpha.s\beta.s\gamma + c\alpha.c\gamma & s\alpha.s\beta.c\gamma - c\alpha.s\gamma \\ -s\beta & c\beta.s\gamma & c\beta.c\gamma \end{bmatrix}$$

$$\left.\begin{aligned} \cos\beta = c\beta = \sqrt{r_{11}^2 + r_{21}^2} \\ \sin\beta = s\beta = -r_{31} \end{aligned}\right\} \rightarrow \beta = A\tan 2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})$$

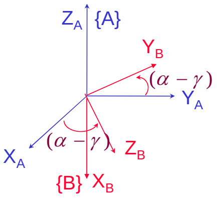
if $c\beta = 0$ $(\beta = \pm 90^o)$ ⇨ Singularity of the representation

⇨ Only $(\alpha + \gamma)$ or $(\alpha - \gamma)$ is defined

Singularities - Example ($R_{Z'Y'X'}$)

$c\beta = 0,\ s\beta = +1$

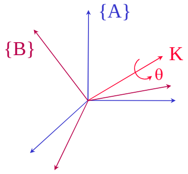
$${}^{A}_{B}R = \begin{pmatrix} 0 & -s(\alpha-\gamma) & c(\alpha-\gamma) \\ 0 & c(\alpha-\gamma) & s(\alpha-\gamma) \\ -1 & 0 & 0 \end{pmatrix}$$



$c\beta = 0,\ s\beta = -1$

$${}^{A}_{B}R = \begin{pmatrix} 0 & -s(\alpha+\gamma) & -c(\alpha+\gamma) \\ 0 & c(\alpha+\gamma) & -s(\alpha+\gamma) \\ 1 & 0 & 0 \end{pmatrix}$$

Equivalent angle-axis representation, $R_K(\theta)$



$$X_r = \theta.K = \begin{bmatrix} \theta.k_X \\ \theta.k_Y \\ \theta.k_Z \end{bmatrix}$$

$$R_K(\theta) = \begin{bmatrix} k_x.k_x.v\theta + c\theta & k_x.k_y.v\theta - k_z.s\theta & k_x.k_z.v\theta + k_y.s\theta \\ k_x.k_y.v\theta + k_z.s\theta & k_y.k_y.v\theta + c\theta & k_y.k_z.v\theta - k_x.s\theta \\ k_x.k_z.v\theta - k_y.s\theta & k_y.k_z.v\theta + k_x.s\theta & k_z.k_z.v\theta + c\theta \end{bmatrix}$$

with $v\theta = 1 - c\theta$

$$R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\theta = Ar\cos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$${}^{A}K = \frac{1}{2.\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix},$$ singularity for $\sin\theta = 0$

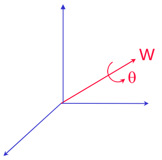
Euler Parameters

$$\varepsilon_1 = W_x . \sin\frac{\theta}{2}$$

$$\varepsilon_2 = W_y . \sin\frac{\theta}{2}$$

$$\varepsilon_3 = W_z . \sin\frac{\theta}{2}$$

$$\varepsilon_4 = \cos\frac{\theta}{2}$$



Normality Condition

$$|W| = 1, \quad \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$$

ε : point on a unit hypershpere
in four-dimensional space

Inverse Problem Given ${}^A_BR$ find ε

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \equiv \begin{bmatrix} 1-2\varepsilon_2^2-2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_2^2 \end{bmatrix}$$

$$r_{11} + r_{22} + r_{33} = 3 - 4\underbrace{(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)}_{(1-\varepsilon_4^2)}$$

$$\varepsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

$$\varepsilon_1 = \frac{r_{32} - r_{23}}{4\varepsilon_4}, \quad \varepsilon_2 = \frac{r_{13} - r_{31}}{4\varepsilon_4}, \quad \varepsilon_3 = \frac{r_{21} - r_{12}}{4\varepsilon_4}$$

$\underline{\underline{\varepsilon_4 = 0?}}$

Lemma For all rotations one of the Euler Parameters is greater than or equal to 1/2

$$\left(\sum_1^4 \varepsilon_i^2 = 1\right)$$

Algorithm Solve with respect to $\max_i\{\varepsilon_i\}$

- $\varepsilon_1 = \max_i\{\varepsilon_i\}$

$$\varepsilon_1 = \frac{1}{2}\sqrt{r_{11} - r_{22} - r_{33} + 1}$$

$$\varepsilon_2 = \frac{(r_{21} + r_{12})}{4\varepsilon_1}, \quad \varepsilon_3 = \frac{(r_{31} + r_{13})}{4\varepsilon_1}, \quad \varepsilon_4 = \frac{(r_{32} - r_{23})}{4\varepsilon_1}$$

- $\varepsilon_1 = \max_i\{\varepsilon_i\}$

$$\varepsilon_1 = \frac{1}{2}\sqrt{r_{11} - r_{22} - r_{33} + 1}$$

- $\varepsilon_2 = \max_i\{\varepsilon_i\}$

$$\varepsilon_2 = \frac{1}{2}\sqrt{-r_{11} + r_{22} - r_{33} + 1}$$

- $\varepsilon_3 = \max_i\{\varepsilon_i\}$

$$\varepsilon_3 = \frac{1}{2}\sqrt{-r_{11} - r_{22} + r_{33} + 1}$$

- $\varepsilon_4 = \max_i\{\varepsilon_i\}$

$$\varepsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

Euler Parameters / Euler Angles

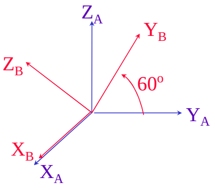
$$\varepsilon_1 = \sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2}$$

$$\varepsilon_2 = \sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2}$$

$$\varepsilon_3 = \cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}$$

$$\varepsilon_4 = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}$$

Quiz



Direction Cosines

$$x_r = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ \sqrt{3}/2 \\ 0 \\ -\sqrt{3}/2 \\ 1/2 \end{bmatrix} \begin{matrix} \\ r_1 \\ \\ \\ r_2 \\ \\ \\ r_3 \\ \\ \end{matrix}$$

Euler Parameters

$$x_r = \begin{bmatrix} \boxed{1/2} \\ \boxed{0} \\ \boxed{0} \\ \boxed{\sqrt{3}/2} \end{bmatrix}$$