Homework #1

CS222 / PHIL358

Due: Friday, April 24

1. (a) Show that any relation R that is Euclidean and reflexive is also symmetric and transitive.
   (b) Give a derivation in the logic **KT** of the **D** axiom, $\neg\Box(p \wedge \neg p)$. In other words, show that it follows in **K** from the axiom **T** alone.
2. Show that axiom **5**, $\Diamond\phi \rightarrow \Box\Diamond\phi$ is valid on a frame (W, R), if and only if R is Euclidean.
3. Consider the following logic $\mathcal{L}$ in the language with two operators, K and B. We take all **S5** axioms for K, all **KD45** axioms for B, and following two "bridge axioms":

   $K\phi \rightarrow B\phi$

   $B\phi \rightarrow BK\phi$

   We take as rules for $\mathcal{L}$ both *modus ponens* and the necessitation rule for K and B (from $\mathcal{L} \vdash \phi$, infer $\mathcal{L} \vdash K\phi$ and $\mathcal{L} \vdash B\phi$). Show that $\mathcal{L} \vdash K\phi \leftrightarrow B\phi$.
4. Show that $\neg K\neg K\phi \rightarrow K\neg K\neg K\phi$ is valid in all (single agent) KB-models. This can either be done directly by giving a model-theoretic argument, or by providing a derivation from the sound and complete proof system given on p.435-436 of *Multiagent Systems* (p.418 of the hard copy).
5. In this problem we consider a possible definition of *common belief*, analogous to the definition of common knowledge. Suppose we have just two agents, 1 and 2. Given a frame (W, R_1, R_2), define $\mathfrak{R} := (R_1 \cup R_2)^*$, i.e. the transitive closure of $R_1 \cup R_2$. Then we define our common belief operator $\mathfrak{C}$ as follows:

   $$\mathcal{M}, w \vDash \mathfrak{C}\phi \quad :\Leftrightarrow \quad \forall x \in W,\ \text{if } w\mathfrak{R}x \text{ then } \mathcal{M}, x \vDash \phi$$

   Provide a **KD45** structure $\mathcal{M}$, a state w, and a formula ϕ, such that $\mathcal{M}, w \vDash B_1\mathfrak{C}\phi$, but $\mathcal{M}, w \nvDash \mathfrak{C}\phi$.