CS 205a Fall 2008 Midterm 1

Multiple Choice (6 x 1 pt each)

For each of the following questions, circle all answers which are correct. You must circle all of the answers for a given question correctly to receive credit.

1. Which of the following are true about condition numbers?
   (a) Squaring a matrix doubles its condition number.
   (b) The condition number for a real-valued eigenvalue problem is $\frac{1}{cos\theta}$, where θ is the angle between the right and left eigenvectors.
   (c) The condition number of a matrix is the ratio of the largest and smallest singular values.
   (d) The condition number of solving $Ax = b$ is independent of the choice of b.
   (e) The cost of solving the system decreases as the condition number goes to zero.
2. Which of the following are true about optimization and root-finding?
   (a) Steepest descent is guaranteed to converge in n iterations, where n is the dimension of your space.
   (b) The secant method and Newton's method have the same rate of convergence as long as there exist x_1, x_2 such that $f(x_1) > 0$ and $f(x_2) < 0$.
   (c) Steepest descent will converge if the second derivative of $f(x)$ is less than zero in the limit as the step size goes to zero.
   (d) Golden section search will find a minimum value even if a function is not unimodal on a given interval.
3. Which of the following are properties of symmetric matrices?
   (a) The left and right eigenvectors are the same.
   (b) They are guaranteed to not be defective.
   (c) The condition number is equal to the largest eigenvalue.
   (d) All projection matrices are symmetric.
4. Which of the following matrices are necessarily orthogonal?
   (a) Permutation.
   (b) Symmetric positive definite.
   (c) Householder.
   (d) Nonsingular.
   (e) Diagonal.

5. Suppose that a square matrix A is perfectly well-conditioned. Which of the following matrices share this property?

   (a) cA, where c is a nonzero scalar.

   (b) DA, where D is a nonsingular diagonal matrix.

   (c) PA, where P is any permutation matrix.

   (d) BA, where B is any nonsingular matrix.

   (e) A^{-1}.

6. Which of the following methods will always provide a least-squares solution to the matrix equation $Ax = b$, where A is an $m \times n$ matrix?

   (a) Perform LU factorization then back and forward substitution.

   (b) Perform Cholesky factorization then back and forward substitution.

   (c) Use the method of normal equations.

   (d) Perform QR factorization to transform into a triangular least-squares problem and then solve it.

   (e) Use n steps of Newton's method.

Error (2 pts)

1. Describe three sources of error in computer simulations of real phenomena. (1 pt)

2. Explain why it may be reasonable to solve a system numerically which is only approximately the same as the equations you want to solve. (1 pt)

QR (6 pts)

1. What are the eigenvalues of the Householder transformation? (1 pt)

2. Suppose you are computing the QR factorization of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}$$

   How many Householder transformations are required? (1 pt)

3. What does the first column of A become as a result of applying the first Householder transformation? (1 pts)

4. What does the first column then become as a result of applying the second Householder transformation? (2 pts)

5. Show how you find the least-squares solution to $Ax = b$ using the QR factorization and prove that it minimizes the two-norm of the residual. (1 pts)

Optimization (4 pts)

1. Show that the iterative method $x_{k+1} = \frac{x_{k-1}f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}$ is mathematically equivalent to the secant method for solving a scalar nonlinear equation $f(x) = 0$ for x. (1 pt)

2. Let $f : \mathbb{R}^n \Rightarrow \mathbb{R}$ be given by $f(x) = \frac{1}{2}x^T Ax - x^T b + c$ where A is an $n \times n$ symmetric positive definite matrix, b and x are vectors in $\mathbb{R}^n$, and c is a scalar. Show that Newton's method for minimizing this function converges in one iteration from any starting point x_0. (2 pts)

3. If the steepest descent method is used on this problem, what happens if the starting value x_0 is an eigenvector of A? (1 pt)

Properties of matrices (8 pts)

1. Prove that the eigenvalues of a positive definite matrix are positive. (2 pts)

2. If A is idempotent, what are its potential eigenvalues? (1 pt)

3. If A is a matrix with linearly independent columns, show that $A(A^TA)^{-1}A^T$ is an orthogonal projector onto the span of A. (2 pts)

4. How does the result from (3) relate to the least-squares problem? (1 pt)

5. Prove that all projection matrices are symmetric. (2 pts)