CS205 – Class 17

Covered in class: All
Reading: 9.3.9

1. **Multivalue methods** – efficiently use lower accuracy on higher derivatives
    a. Consider the Taylor expansion $x^{n+1} = x^n + \Delta t x_t^n + \frac{\Delta t^2}{2} x_{tt}^n + O(\Delta t^3)$ i.e. consider case where we have $x_t = v,\ x_{tt} = a\ or\ v_t = a$.
        i. If x^n has $O(\Delta t^r)$ errors than x_t^n can have $O(\Delta t^{r-1})$ errors without ruining the accuracy, similarly x_{tt}^n can have $O(\Delta t^{r-2})$ errors
        ii. e.g., 3rd order accurate $\vec{x}$ can be obtained with a 2nd order accurate $\vec{v}$ and 1st order accurate $\vec{a}$
        iii. Solving $\begin{pmatrix} \vec{x} \\ \vec{v} \end{pmatrix}$ as a standard system is overkill on $\vec{v}$
    b. Standard **constant acceleration** equations
        i. $\vec{x}^{n+1} = \vec{x}^n + \Delta t \vec{v}^n + \frac{\Delta t^2}{2} \vec{a}^n$ *quadratic* position
        ii. $\vec{v}^{n+1} = \vec{v}^n + \Delta t \vec{a}^n$ *linear* velocity
        iii. $\vec{a}^{n+1} = \vec{a}^n$ *constant* acceleration (that is constant from time n to just before time n+1)
2. **Newmark Method** – most famous multivalue method in *computational mechanics*
    a. Actually a lot of methods in disguise
    b. $\vec{x}^{n+1} = \vec{x}^n + \Delta t \vec{v}^n + \frac{\Delta t^2}{2} \left[(1-2\beta)\vec{a}^n + 2\beta \vec{a}^{n+1} \right]$
    c. $\vec{v}^{n+1} = \vec{v}^n + \Delta t \left[(1-\gamma)\vec{a}^n + \gamma \vec{a}^{n+1} \right]$
    d. Choice of β, γ parameters makes a specific method.
        i. $\beta = \gamma = 0$ - standard *constant acceleration* case (above)
        ii. $\beta = 1/2$, $\gamma = 1$ - piecewise constant, implicit acceleration
            1. $\vec{x}^{n+1} = \vec{x}^n + \Delta t \vec{v}^n + \frac{\Delta t^2}{2} \vec{a}^{n+1}$
            2. $\vec{v}^{n+1} = \vec{v}^n + \Delta t \vec{a}^{n+1}$
            3. Second equation is the same as 1st order accurate backward Euler
            4. First equation is $\vec{x}^{n+1} = \vec{x}^n + \Delta t \left(\frac{\vec{v}^n + \vec{v}^{n+1}}{2} \right)$ which is the 2nd order accurate trapezoidal rule
            5. Overall still 1st order accurate

iii. Exists a theorem states: 2nd order accuracy is obtained *if and only if* $\gamma = 1/2$

iv. $\beta = 1/4$, $\gamma = 1/2$ - *Trapezoidal rule* – 2nd order accurate

1. Again, constant acceleration, but this time using the midpoint acceleration
2. $\vec{x}^{n+1} = \vec{x}^n + \Delta t \vec{v}^n + \frac{\Delta t^2}{2}\left(\frac{\vec{a}^n + \vec{a}^{n+1}}{2}\right)$
3. $\vec{v}^{n+1} = \vec{v}^n + \Delta t\left(\frac{\vec{a}^n + \vec{a}^{n+1}}{2}\right)$
4. first equation is equivaluent to $\vec{x}^{n+1} = \vec{x}^n + \Delta t\left(\frac{\vec{v}^n + \vec{v}^{n+1}}{2}\right)$

v. $\beta = 0$, $\gamma = 1/2$ *central differencing*

1. $\vec{x}^{n+1} = \vec{x}^n + \Delta t \vec{v}^n + \frac{\Delta t^2}{2}\vec{a}^n$
2. $\vec{v}^{n+1} = \vec{v}^n + \Delta t\left(\frac{\vec{a}^n + \vec{a}^{n+1}}{2}\right)$
3. Called central differencing because both the acceleration and the velocity can be expressed as centered finite differences
4. $\vec{v}^{n+1} = \frac{\vec{x}^{n+2} - \vec{x}^n}{2\Delta t}$ can be derived by adding $\vec{x}^{n+1} = \vec{x}^n + \Delta t \vec{v}^n + \frac{\Delta t^2}{2}\vec{a}^n$ to $\vec{x}^{n+2} = \vec{x}^{n+1} + \Delta t \vec{v}^{n+1} + \frac{\Delta t^2}{2}\vec{a}^{n+1}$ and rearranging to obtain $\frac{\vec{x}^{n+2} - \vec{x}^n}{2\Delta t} = \frac{\vec{v}^{n+1}}{2} + \frac{1}{2}\left(\vec{v}^n + \Delta t\left(\frac{\vec{a}^n + \vec{a}^{n+1}}{2}\right)\right)$ and then realizing that the last term is identical to $\vec{v}^{n+1}$
5. $\vec{a}^{n+1} = \frac{\vec{x}^{n+2} - 2\vec{x}^{n+1} + \vec{x}^n}{\Delta t^2}$ can be derived by subtracting $\vec{x}^{n+1} = \vec{x}^n + \Delta t \vec{v}^n + \frac{\Delta t^2}{2}\vec{a}^n$ from $\vec{x}^{n+2} = \vec{x}^{n+1} + \Delta t \vec{v}^{n+1} + \frac{\Delta t^2}{2}\vec{a}^{n+1}$ and reorganizing to obtain $\frac{\vec{x}^{n+2} - 2\vec{x}^{n+1} + \vec{x}^n}{\Delta t^2} = \left(\frac{\vec{v}^{n+1} - \vec{v}^n}{\Delta t}\right) + \left(\frac{\vec{a}^{n+1} - \vec{a}^n}{2}\right)$ and then noting that $\frac{\vec{v}^{n+1} - \vec{v}^n}{\Delta t} = \frac{\vec{a}^n + \vec{a}^{n+1}}{2}$

3. **<u>Staggering the position and velocity</u>**

a. Define the velocity at the half grid points so that $\vec{v}^{n+1/2} = \frac{\vec{x}^{n+1} - \vec{x}^n}{\Delta t}$ is second order accurae

i. Note that $\vec{x}^n$ is still at the grid points

ii. If we define $\vec{v}^{n+1} = \frac{\vec{v}^{n+1/2} + \vec{v}^{n+3/2}}{2} = \frac{\vec{x}^{n+2} - \vec{x}^{n}}{2\Delta t}$, then this is exactly central differencing for velocity

iii. We can rewrite this as an update formula for position $\vec{x}^{n+1} = \vec{x}^{n} + \Delta t \vec{v}^{n+1/2}$

b. The acceleration is also at the grid points, and $\frac{\vec{v}^{n+3/2} - \vec{v}^{n+1/2}}{\Delta t} = \vec{a}^{n+1}$ is second order accurate

i. Note that this is also equivalent to $\frac{\left(\frac{\vec{x}^{n+2} - \vec{x}^{n+1}}{\Delta t}\right) - \left(\frac{\vec{x}^{n+1} - \vec{x}^{n}}{\Delta t}\right)}{\Delta t} = \vec{a}^{n+1}$ or $\frac{\vec{x}^{n+2} - 2\vec{x}^{n+1} + \vec{x}^{n}}{\Delta t^2} = \vec{a}^{n+1}$ which is second order accurate for a second derivative

ii. Again, as in the velocity case, this is just central differencing

iii. We can rewrite this as an update formula for acceleration $\vec{v}^{n+3/2} = \vec{v}^{n+1/2} + \Delta t \vec{a}^{n+1}$

c. The acceleration is evaluated at the grid points using

$$\vec{a}^{n+1} = \vec{a}(\vec{x}^{n+1}, \vec{v}^{n+1}) = \vec{a}\left(\vec{x}^{n+1}, \frac{\vec{v}^{n+1/2} + \vec{v}^{n+3/2}}{2}\right)$$

d. Summary, given $\vec{x}^{n}$ and $\vec{v}^{n}$:

i. By definition $\vec{v}^{n} = \frac{\vec{v}^{n-1/2} + \vec{v}^{n+1/2}}{2}$

1. So $\vec{v}^{n+1/2} = 2\vec{v}^{n} - \vec{v}^{n-1/2} = 2\vec{v}^{n} - (\vec{v}^{n+1/2} - \Delta t \vec{a}^{n})$ using $\vec{v}^{n+1/2} = \vec{v}^{n-1/2} + \Delta t \vec{a}^{n}$
2. This can be rearranged to $\vec{v}^{n+1/2} = \vec{v}^{n} + \frac{\Delta t}{2}\vec{a}^{n}$ to get the half step velocity

   $\vec{x}^{n+1} = \vec{x}^{n} + \Delta t \vec{v}^{n+1/2}$
3. This is identical to $\vec{x}^{n+1} = \vec{x}^{n} + \Delta t \vec{v}^{n} + \frac{\Delta t^2}{2}\vec{a}^{n}$

ii. Then $\vec{v}^{n+3/2} = \vec{v}^{n+1/2} + \Delta t \vec{a}\left(\vec{x}^{n+1}, \frac{\vec{v}^{n+1/2} + \vec{v}^{n+3/2}}{2}\right)$

1. Using $\vec{v}^{n+1} = \frac{\vec{v}^{n+1/2} + \vec{v}^{n+3/2}}{2}$ leads to $2\vec{v}^{n+1} - \vec{v}^{n+1/2} = \vec{v}^{n+1/2} + \Delta t \vec{a}\left(\vec{x}^{n+1}, \vec{v}^{n+1}\right)$ or

   $\vec{v}^{n+1} = \vec{v}^{n+1/2} + \frac{\Delta t}{2}\vec{a}(\vec{x}^{n+1}, \vec{v}^{n+1})$

e. This last equation is implicit in the velocity

i. Fully explicit if acceleration doesn't depend on velocity

ii. Otherwise iterate $\vec{u}^{k+1} = \vec{v}^{n+1/2} + \frac{\Delta t}{2}\vec{a}(\vec{x}^{n+1}, \vec{u}^{k})$

1. Starting with $\vec{u}^{0} = \vec{v}^{n+1/2}$
2. $\vec{u}^{1} = \vec{v}^{n+1/2} + \frac{\Delta t}{2}\vec{a}(\vec{x}^{n+1}, \vec{v}^{n+1/2})$ which is an explicit time step

iii. Often the dependence of acceleration on velocity is symmetric (e.g. damping forces), so one can use a fast Ax=b solver, e.g. PCG

iv. The overall velocity update looks like $\vec{v}^{n+1} = \vec{v}^{n+1/2} + \frac{\Delta t}{2}\vec{a}(\vec{x}^{n+1}, \vec{v}^{n+1}) = \vec{v}^{n} + \frac{\Delta t}{2}\vec{a}(\vec{x}^{n}, \vec{v}^{n}) + \frac{\Delta t}{2}\vec{a}(\vec{x}^{n+1}, \vec{v}^{n+1})$ which is the trapezoidal rule.

v. There is no stability restriction on the time step from the accelerations dependence on velocity
   1. The only time step stability restriction comes from the accelerations dependence on position
   2. Can take a slightly bigger time step
   3. Of course, bigger time steps are bad for the trapezoidal rule, so one could switch to backward Euler for the velocity
      a. $\vec{v}^{n+1} = \vec{v}^{n} + \Delta t \vec{a}(\vec{x}^{n+1}, \vec{v}^{n+1})$
      b. Still use $\vec{x}^{n+1} = \vec{x}^{n} + \Delta t \vec{v}^{n} + \frac{\Delta t^2}{2}\vec{a}^{n}$ for position
      c. Only 1st order accurate overall