CS 205 – Class 12

Readings: Same as last
Covered in class: All

1. Convergence and the Error
    a. The error looks like $e_0 = \sum_j a_j s_j$ where the s_j are the search directions and a_j are numerical coefficients
    b. $s_k \cdot Ae_0 = s_k \cdot A\sum_j a_j s_j = \sum_j a_j s_k \cdot As_j = a_k s_k \cdot As_k$ since the search directions are orthogonal in A space
        i. thus $a_k = \frac{s_k \cdot Ae_0}{s_k \cdot As_k} = \frac{s_k \cdot A(e_0 + \sum_{j=1}^{k-1} \alpha_j s_j)}{s_k \cdot As_k}$ where the summation can be added since it is identically zero when multiplied by $s_k \cdot A$
        ii. now $e_k = e_{k-1} + s_{k-1}\alpha_{k-1} = e_{k-2} + s_{k-2}\alpha_{k-2} + s_{k-1}\alpha_{k-1} = \ldots$
            1. i.e. $e_k = e_0 + \sum_{j=1}^{k-1} \alpha_j s_j$
        iii. thus $a_k = \frac{s_k \cdot Ae_k}{s_k \cdot As_k}$ and (from above) $a_k = -\alpha_k$
    c. so the error is $e_0 = -\sum_j \alpha_j s_j$ and $e_k = -\sum_j \alpha_j s_j + \sum_{j=1}^{k-1} \alpha_j s_j$
        i. after n steps the second term is equal to the first term and the error is zero
    d. multiplying this error equation by $s_i \cdot A$ gives
    $s_i \cdot Ae_k = -\sum_j \alpha_j s_i \cdot As_j + \sum_{j=1}^{k-1} \alpha_j s_i \cdot As_j$
        i. for i < k, there is exactly one nonzero term in each sum, and these terms cancel
        ii. thus for i < k, $s_i \cdot Ae_k = 0$ and thus $s_i \cdot r_k = 0$ (we'll use this below)
        iii. this means that the current residual at step k is orthogonal to all the previous search directions
2. finding the A-orthogonal directions with Gram-Schmidt
    a. given a vector V, construct s_k by subtracting out the "A-overlap" of V with s_1 to s_{k-1} so that $s_k \cdot As_i = 0$ for i=1,k-1
    b. we define $s_k = V - \sum_{j=1}^{k-1} \frac{V \cdot As_j}{s_j \cdot As_j} s_j$
        i. note that $s_k \cdot As_i = V \cdot As_i - \sum_{j=1}^{k-1} \frac{V \cdot As_j}{s_j \cdot As_j} s_j \cdot As_i$ and then all the terms in the sum vanish except for one leaving
        $s_k \cdot As_i = V \cdot As_i - \frac{V \cdot As_i}{s_i \cdot As_i} s_i \cdot As_i = 0$ as desired

c. for $i \ge k$, $s_k \cdot r_i = V_k \cdot r_i - \sum_{j=1}^{k-1} \frac{V_k \cdot As_j}{s_j \cdot As_j} s_j \cdot r_i = V_k \cdot r_i$, where the summation vanishes because the residual at step i is orthogonal to all the previous search directions

i. when k=i this leads to $s_k \cdot r_k = V_k \cdot r_k$ and $\alpha_k = \frac{s_k \cdot r_k}{s_k \cdot As_k} = \frac{V_k \cdot r_k}{s_k \cdot As_k}$ (we'll use this below)

ii. when k < i, $0 = V_k \cdot r_i$, i.e. the residual is orthogonal to all the previous V_k as well (we'll use this below)