CS205 Mathematical Methods for Computer Vision, Robotics, and Graphics

Midterm 2

The following is a statement of the Stanford University Honor Code:

1. The honor Code is an undertaking of the students, individually and collectively:
   (a) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;
   (b) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.
2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.
3. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

By writing my name below, I certify that I acknowledge and accept the Honor Code.

Name ______________________________

Stanford ID ______________________________

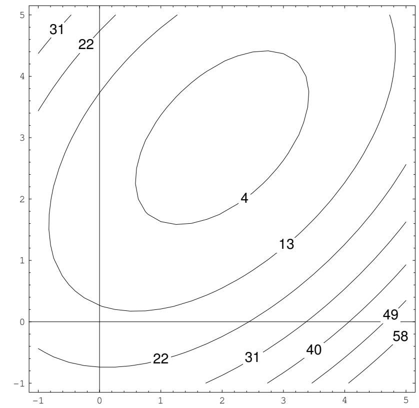
Section	**Score**	**Maximum**
Multiple Choice		8
Problem 1		4
Problem 2		4
Problem 3		4
Total		20

Multiple choice questions [8×1pt=8pts]

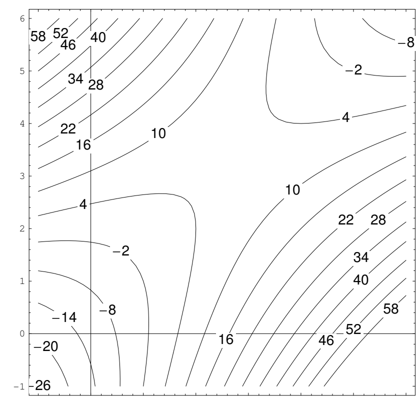
(Questions 1-3) In order to solve the linear system $\mathbf{Ax} = \mathbf{b}$ we may alternatively consider the unconstrained minimization problem for the function:

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{b}^T\mathbf{x} + c$$

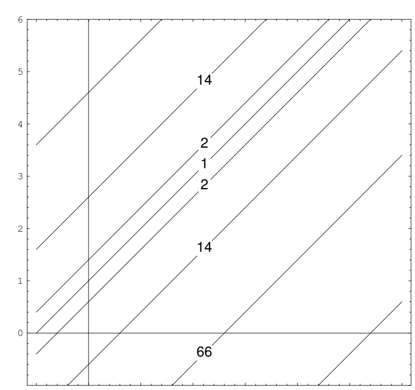
The figures below depict the isocontours of the function $f(\mathbf{x})$ for different choices of the matrix $\mathbf{A}$, the vector $\mathbf{b}$ and the scalar c. You may assume that the matrix $\mathbf{A}$ is *symmetric*.



1. Which of the following algorithms could be used to solve $\mathbf{Ax} = \mathbf{b}$ for the choice of $\mathbf{A}$ and $\mathbf{b}$ corresponding to the top plot on the right? [circle all that apply]
   (a) Conjugate Gradient
   (b) Steepest descent
   (c) Cholesky factorization
   (d) LU factorization



2. Which of the following algorithms could be used to solve $\mathbf{Ax} = \mathbf{b}$ for the choice of $\mathbf{A}$ and $\mathbf{b}$ corresponding to the middle plot on the right? [circle all that apply]
   (a) Conjugate Gradient
   (b) Steepest descent
   (c) Cholesky factorization
   (d) LU factorization



3. Which of the following algorithms could be used to solve $\mathbf{Ax} = \mathbf{b}$ for the choice of $\mathbf{A}$ and $\mathbf{b}$ corresponding to the bottom plot on the right? [circle the best answer]
   (a) Conjugate Gradient
   (b) LU factorization
   (c) Normal equations
   (d) None of the above

4. Which of the following statements about the Forward Euler method for solving ODEs are *true*? [circle all that apply]
   (a) It is first-order accurate, which implies that the local truncation error incurred at each step is $O(h)$.
   (b) It is significantly more efficient than implicit methods like the Backward Euler method since it does not require a linear system solve at every step.
   (c) It can be derived from the Taylor series expansion of the ODE.
   (d) It is much more accurate than higher-order methods on ill-posed problems.

5. Which of the following statements about boundary conditions on the discretized Laplace equation are *true*? [circle all that apply]
   (a) The imposition of each Dirichlet boundary condition necessarily adds a row to the matrix.
   (b) Non-zero Dirichlet boundary conditions make the matrix non-symmetric.
   (c) Neumann boundary conditions can cause the matrix to have a non-trivial nullspace.
   (d) Dirichlet boundary conditions must be carefully chosen to satisfy the compatibility condition.

6. Given a sparse linear system, which of the following are *true*? [circle all that apply]
   (a) Iterative methods are more appropriate than direct methods for solving the system.
   (b) Incomplete Cholesky preconditioning should not be used, since it would require creating and storing a dense matrix.
   (c) Off-diagonal elements can be dropped, since the matrix is necessarily nearly diagonal.
   (d) The Conjugate Gradient algorithm should only be used if the matrix is skew-symmetric.

7. Which of the following statements about the heat equation are *true*? [circle all that apply]
   (a) It is an example of a hyperbolic partial differential equation
   (b) Unlike the Laplace equation, the heat equation is not time-dependent.
   (c) If the conduction coefficient is spatially varying, we can simplify the heat equation to $T_t = \nabla \cdot (\rho \nabla T) = \rho \nabla^2 T$
   (d) Standard central differencing of the spatial domain coupled with a Backward Euler discretization of the temporal domain yields a discretization that is second-order accurate in time and first-order accurate in space.

8. Which of the following statements about interpolation are *false*? [circle all that apply]

   (a) The development of advanced discretization methods like finite differencing has eliminated the need for most interpolation techniques.

   (b) In practice, piecewise linear interpolation is often the method of choice.

   (c) Higher order polynomials are oscillatory and expensive to compute.

   (d) Newton interpolation produces a more accurate interpolating polynomial than Lagrange interpolation.

Problem 1

[4pts] Apply the Conjugate Gradient algorithm to find the solution of the following symmetric positive definite linear system

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Use the initial guess $\vec{\mathbf{x}}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

Problem 2

[4pts] Consider the first-order ordinary differential equation:

$$y' = \lambda y$$

1. Derive and graph the stability region of the Forward Euler method as applied to this equation.

2. Derive and graph the stability region of the Backward Euler method as applied to this equation.

Hint: Write each formula as a function of $h\lambda$ (where h is the stepsize), and then treat this quantity as a point in the complex plane.

Problem 3

[4pts] Consider the following set of sampled point data:

$$\{(1,1),(2,2),(3,4)\}$$

1. Construct algebraically the Lagrange interpolating polynomial $L(x)$ of degree 2 for this data set.

2. Construct algebraically the Newton interpolating polynomial $N(x)$ of degree 2 for this data set.

3. Suppose that we have already computed $L(x)$ and $N(x)$ when we realize in horror that we forgot to include the sample point $(4, 6)$ in our data set. Discuss the relative merits of adapting the existing Newton and Lagrange polynomials to account for this new point.