CS 205A Fall 2007 Midterm

(8 points) Multiple Choice [circle the best answer]

1. When solving $Ax = b$, which of the following are true?
   (a) If the condition number is bad, Steepest Descent performs well.
   (b) When A is sparse, large and positive definite, Cholesky Factorization is the most space-efficient solver.
   (c) If A is negative semi-definite, we can solve this system using a modified form of Conjugate Gradients.
   (d) An incomplete cholesky preconditioner is useful when all of the eigenvalues of A are similar in magnitude.

2. When solving the characteristic ODE $y' = \lambda y$,
   (a) The problem is **ill-posed** if $\lambda > 0$.
   (b) When discretized using forward euler, the method is **unstable** when $-2 < h\lambda < 0$.
   (c) When discretized using backward euler, the method is **unstable** when $h = \lambda$.
   (d) The Trapezoidal rule is 3^{rd} order accurate.

3. When solving the ODE $y' = y$, which of the following methods will converge to the correct solution?
   (a) Forward Euler.
   (b) Backward Euler.
   (c) Trapezoidal Rule.
   (d) none of the above.

4. When solving $Ax = b$ using the Conjugate Gradients algorithm, which of the following is true?
   (a) The search directions are orthogonal, ($s_i^T s_j = 0$ for $i \neq j$).
   (b) The residual $r_i = b - Ax_i$ is A-orthogonal to every previous search direction ($r_i^T A s_j = 0$ for $i < j$).
   (c) The step length is given by $\alpha_k = \frac{r_k^T A r_k}{r_k^T r_k}$
   (d) The first step of CG, s_0, is *not* equal to the first step of steepest descent, r_0.

5. When interpolating, why might we chose to do piecewise interpolation using a lower-order polynomial rather than a higher-order polynomial.
   (a) We're interpolating over a tiny region, so it doesn't matter if we make errors at that scale.
   (b) Often the lower-order polynomial interpolations overly smooth discontinuous phenomena (which plays havoc with the solvers).
   (c) Higher-order polynomial interpolations tend to oscillate more than lower-order interpolations.

6. With respect to boundary conditions for solving the Laplace equation, which of the following is true?
   (a) If all boundary conditions are Neumann, the solution is guaranteed to exist, but is not unique.
   (b) If all boundary conditions are Dirichlet, the solution is guaranteed to exist, but is not unique.
   (c) If the boundary conditions are a mix of Neumann and Dirichlet, the solution is not guaranteed to exist.
   (d) If any of the boundary conditions are Dirichlet, the solution is unique.
7. The Heat Equation is an example of which of the following types of PDE?
   (a) Hyperbolic
   (b) Parabolic
   (c) Elliptic
   (d) All of the above :-)
8. When we solve the heat equation $T_t = \nabla \cdot (k\nabla T)$, which statement regarding the various schema available to us is true?
   (a) Forward Euler is stable, provided we chose a steplength $\Delta t = \Delta x$.
   (b) Backward Euler converges to the steady state solution as $\Delta t \to \infty$.
   (c) Crank-Nicholson (below) converges to the steady state solution as $\Delta t \to \infty$.

$$\frac{T^{n+1} - T^n}{\Delta t} = \frac{1}{2}\nabla \cdot \left(k\nabla T^{n+1}\right) + \frac{1}{2}\nabla \cdot (k\nabla T^n)$$

   (d) Crank-Nicholson is *always* preferred over Forward or Backward Euler, since it is 2^{nd} order accurate and has no timestep restrictions.

Conjugate Gradients (4 points)

1. (2 points) Write the algorithm for CG.

2. (1 point) What does the CG algorithm rely on to prove that the iterates converge in at most n steps when solving $\min x^T Ax - b^T x$, where A is $n \times n$.

3. (1 point) We proved in homework 7 that we could make small changes to the CG algorithm to solve the least squares problem $\min \|b - Ax\|$ for a positive or negative semi-definite matrix A. Can CG be used to solve indefinite matrices? Why or why not?

Discretizing PDE's (4 points)

1. (1 point) We often choose to solve a PDE implicitly (using Backward Euler) rather than explicitly (using Forward Euler). What is the benefit to doing this, and what price do we pay?

2. (1 point) When solving a stiff problem, which method should we use, and why?

3. (2 points) Please derive the local truncation error for the Trapezoidal Rule, which is given by:

$$\frac{y^{k+1} - y^k}{h} = \frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1})}{2}$$

and state the order of accuracy of this method.
HINT: use the taylor series expansion $f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + ...$

The Heat Equation & The Laplacian (4 points)

1. (2 points) If we discretize a domain into three internal grid nodes (so you should be solving for 3 unknowns), please write down the system of equations that results from the following PDE:

$$\begin{cases} p_{xx} = 0 \text{ with } x \in (0,1) \\ p(0) = 5 \\ p_x(1) = -1 \end{cases}$$

2. (1 point) If we change the left boundary condition from $p(0) = 5$ to $p_x(0) = 2$, does a solution exist? If so, what is it?

3. (1 point) If we're given $T_t = \nabla \cdot (k \nabla T)$ (in one dimension), write down a symmetric discretization of this equation for an internal grid node. Please note that in this case, k is *not* constant.