CS 205A Fall 2007 Midterm

Multiple Choice (8 x 1 pt)

1. For a symmetric ($A = A^T$) $n \times n$ matrix A,
   (a) $A^T A$ is invertible
   (b) The eigenvalues are all real-valued
   (c) For any b, there exists an x such that $Ax = b$
   (d) There exist non-singular matrices L and U such that $A = LU$

2. If we're given a matrix A, the following can be said:
   (a) If A is positive definite, then it has Cholesky Factors (ie $A = LL^T$)
   (b) If A is over-determined, we use QR decomposition with Householder to solve $Ax = b$
   (c) The pseudo-inverse of A, $A^+ = V\Sigma^+U^T$ can be used to solve least-squares problems
   (d) Solving the minimization problem $\min_x \|b - Ax\|_2$ is equivalent to solving $Ax = b$

3. The following can be said about a Householder Matrix $H = I - 2\frac{vv^T}{v^Tv}$
   (a) The condition number for H is ∞
   (b) It is a projection matrix onto the hyperplane orthogonal to v
   (c) It preserves the 2-norm of a vector (ie $\|x\|_2 = \|Hx\|_2$ for all x)
   (d) The eigenvalues of H are 1 with multiplicity n

4. Consider the multi-variate optimization problem $\min_{\vec{x}} f(\vec{x})$.
   (a) When the Hessian is **negative definite**, we are at a local maximizer
   (b) Solving this minimization problem is exactly equivalent to finding some $\vec{x}$ such that $\nabla f(\vec{x}) = 0$
   (c) If f is continuous and twice differentiable, then the Hessian is guaranteed not to be singular
   (d) Steepest Descent performs poorly when the Hessian is poorly conditioned

5. Which of the following statement is true about the solution to the system of linear equations $Ax = b$, where A is an $m \times n$ matrix with $m > n$.
   (a) It has no solutions
   (b) It has infinite solutions
   (c) It has unique solutions
   (d) Can not say in general

6. Let x_0 be a least squares solution to $Ax = b$. Which of the following statement is true in general about the residual $r = Ax_0 - b$
   (a) r is the projection of b onto the null space of A^T
   (b) r is the projection of b onto the column space of A
   (c) r is perpendicular to b
   (d) r lies in the null space of A

7. Consider a symmetric matrix A with an eigenvalue λ and the corresponding unit length eigenvector q. The following is true about $B = A - \lambda q q^T$

   (a) The set of eigenvalues of B can not contain λ

   (b) q lies in the null space of B

   (c) The characteristic polynomials of A and B have the same roots

   (d) If $q_2 \neq q$ is an eigenvector of A with the same eigenvalue λ, then q_2 cannot be an eigenvector of B

8. Given that $f(x)$ is continuous and has a root in the interval $[a, b]$, which of the following methods are guaranteed to converge to the root

   (a) Newton method

   (b) Secant method

   (c) Bisection method

   (d) All of the above

Solving $Ax = b$ (4 pts)

1. Often in applications we are asked to solve the system of equations $Ax_i = b_i$ (where A **does not change**) for many b_i. Please discuss three options available for solving this problem **quickly and repeatedly**, and what properties A must have for each to work. (3 pts)

2. Why do we try to solve the least squares problem $\min_x \|b - Ax\|$ even when we know that A is full-ranked? (1 pt)

Singular Value Decomposition (4 pts)

1. State (without proof) the SVD of the following matrices (2 pts)

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \qquad \begin{pmatrix}-3\\ 5\end{pmatrix} \qquad \begin{pmatrix}0 & 0 & 1\\ 0 & 10 & 0\end{pmatrix} \qquad \begin{pmatrix}-5\end{pmatrix}$$

2. State the properties satisfied by U, Σ and V (1 pt)

3. In **Principle Component Analysis**, how is the SVD modified? How is this justified?

Optimization (4 pts)

1. Reduce the minimization of the function $f(x)$ into a root finding problem. What additional information is needed to ensure that the provided solution in fact *minimizes* the function? (2 pts)

2. Given that $f(x) \geq 0, \forall x$, $\exists x$ s.t. $f(x) = 0$ and a numerical method to find the root of a function, would it be a good idea to use this numerical method to find the value of x which mimimizes the function $f(x)$. Why or why not? (1 pt)

3. In order for the golden section search to work on a function $f(x)$, what properties should $f(x)$ hold? Give a brief reason. (1 pt)