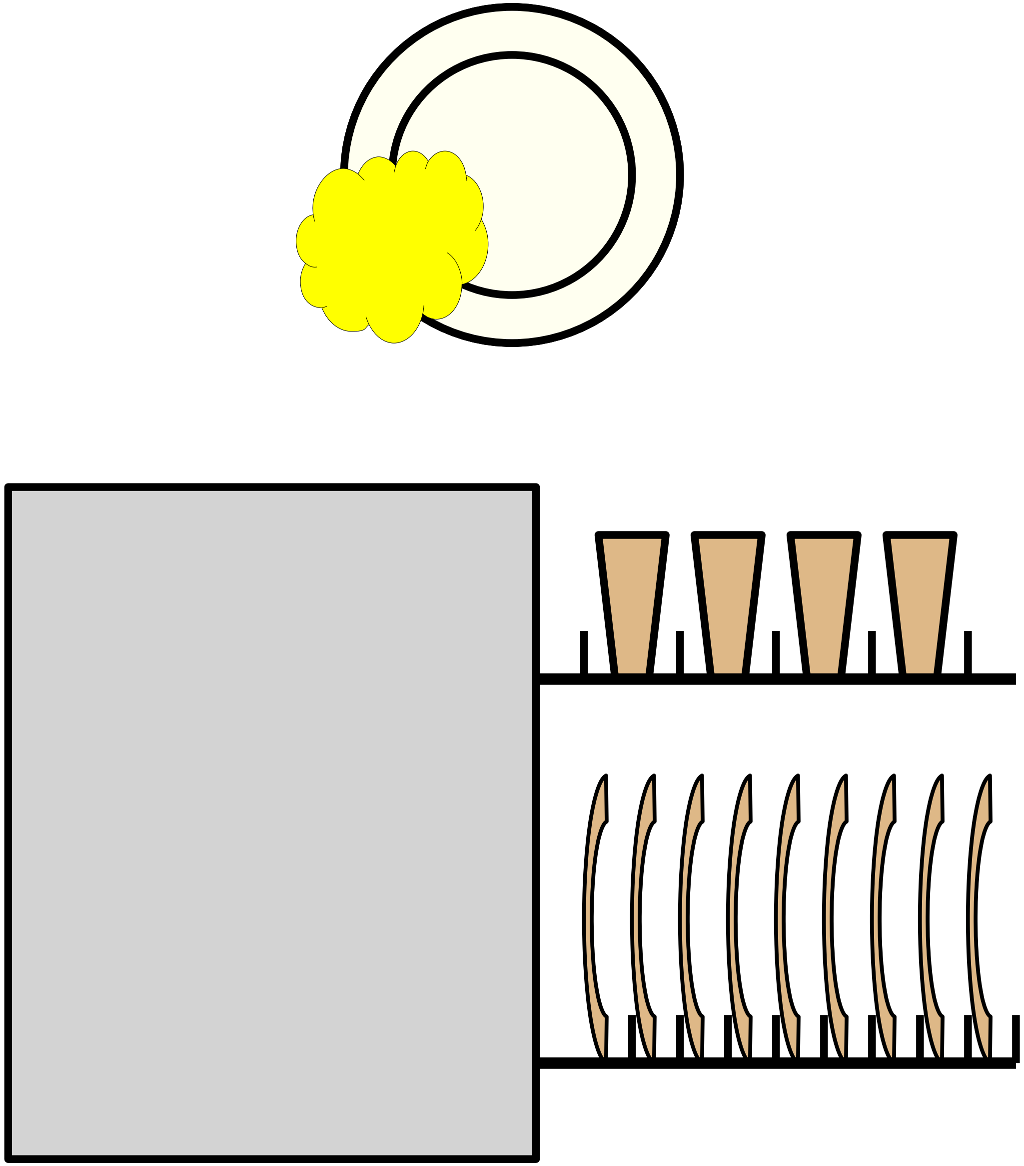
Amortized Analysis

A Motivating Analogy

Doing the Dishes

- What do I do with a dirty dish or kitchen utensil?
- ***Option 1:*** Wash it by hand.
- ***Option 2:*** Put it in the dishwasher rack, then run the dishwasher if it's full.



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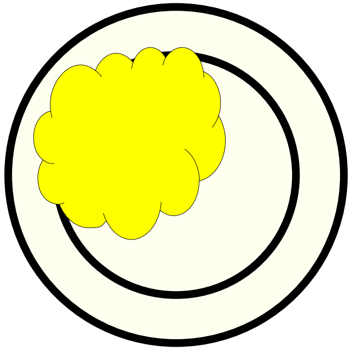

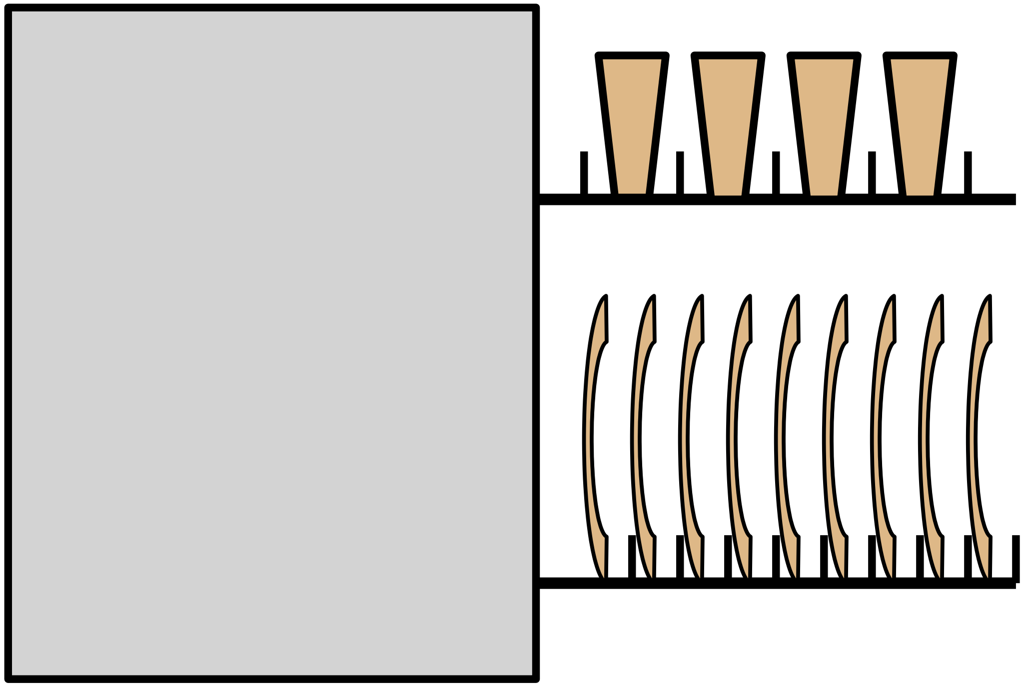

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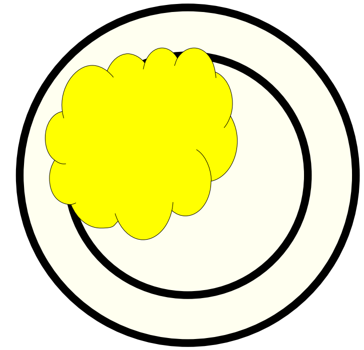
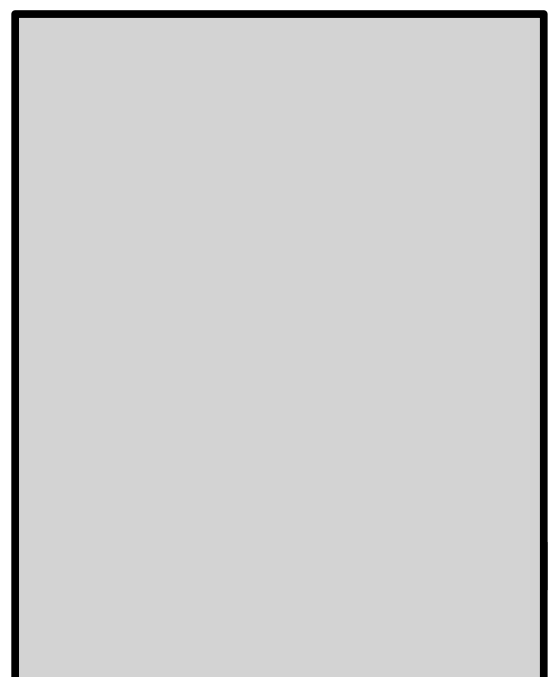
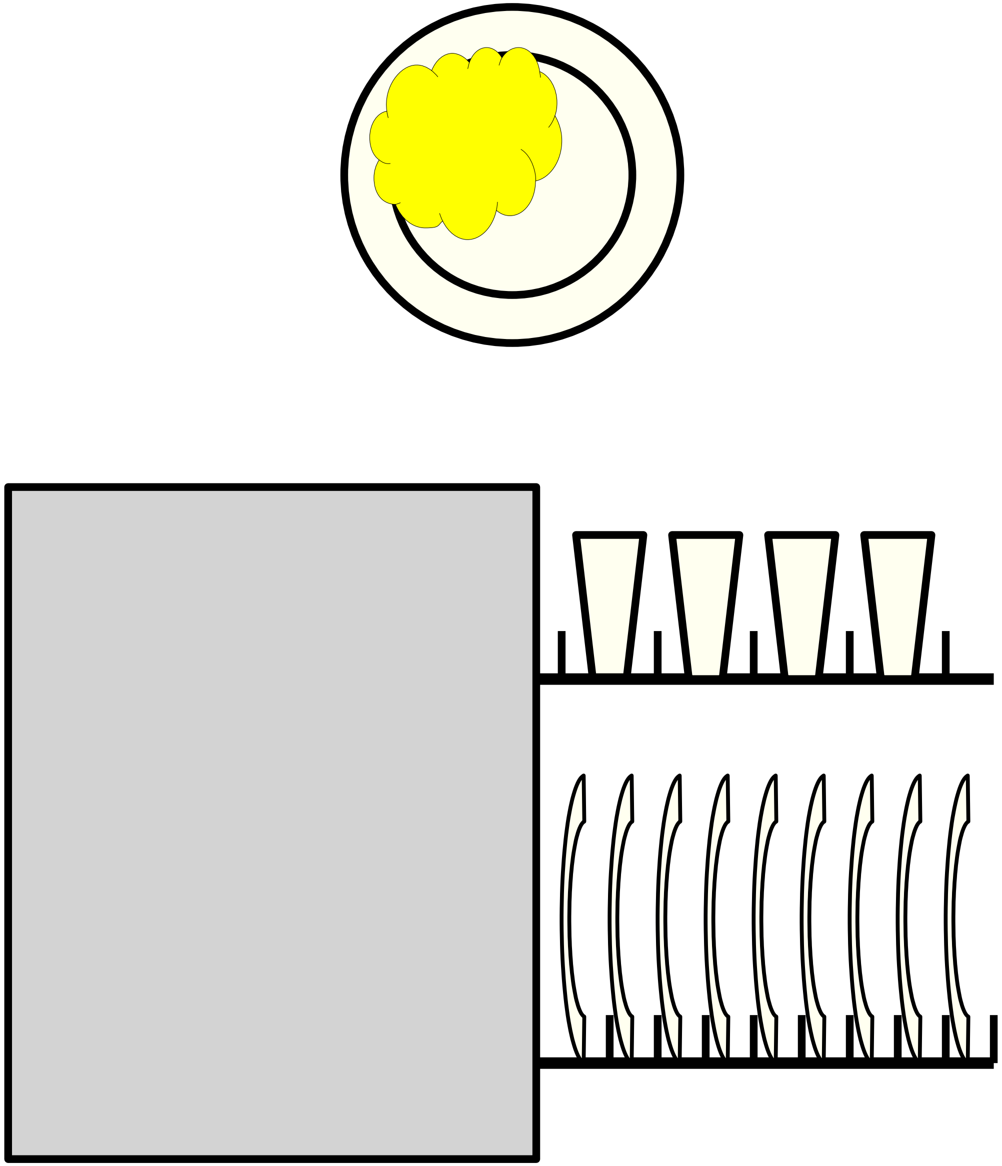




Doing the Dishes

- Washing every individual dish and utensil by hand is *way* slower than using the dishwasher, but I always have access to my plates and kitchen utensils.
- Running the dishwasher is faster in aggregate, but means I may have to wait a bit for dishes to be ready.
- (This is an example of a tradeoff between ***throughput*** and ***latency***.)



Key Idea: Design data structures that trade *per-operation efficiency* for *overall efficiency*.

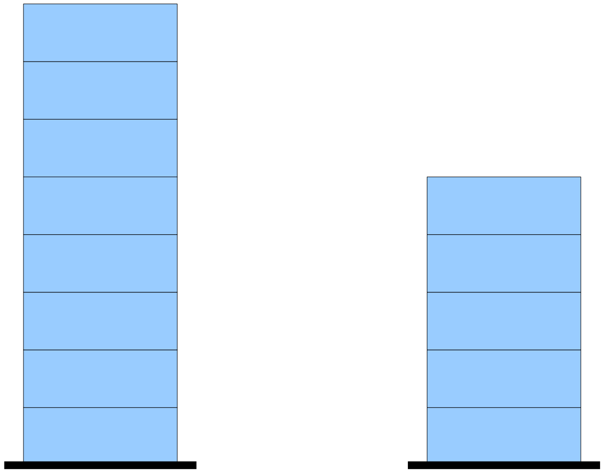
Where We're Going

- ***Amortized Analysis (Today)***
  - A little accounting trickery never hurt anyone, right?
- ***Binomial Heaps (Thursday)***
  - A fast, flexible priority queue that's a great building block for more complicated structures.
- ***Fibonacci Heaps (Next Tuesday)***
  - A priority queue optimized for graph algorithms that, at least in theory, leads to optimal implementations.

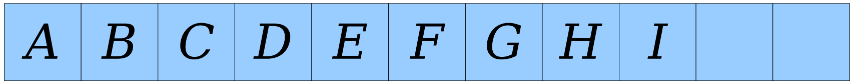
Outline for Today

- ***Amortized Analysis***
  - Trading worst-case efficiency for aggregate efficiency.
- ***Examples of Amortization***
  - Three motivating data structures and algorithms.
- ***Potential Functions***
  - Quantifying messiness and formalizing costs.
- ***Performing Amortized Analyses***
  - How to show our examples are indeed fast.

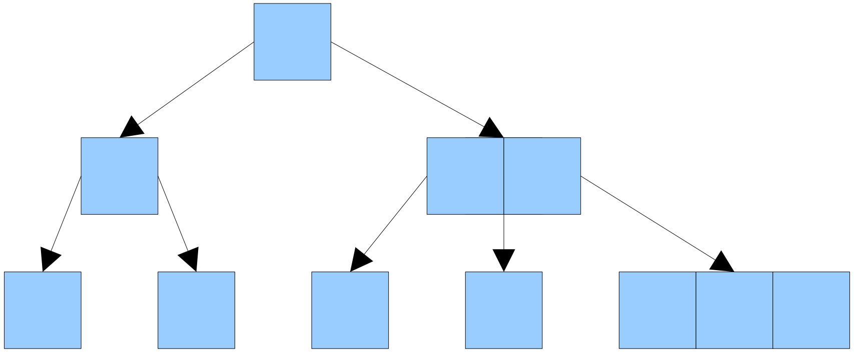
Three Examples



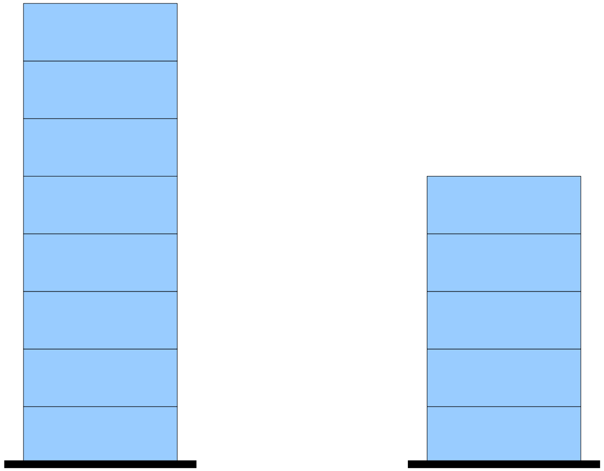
Two-Stack Queues



Dynamic Arrays



Building B-Trees



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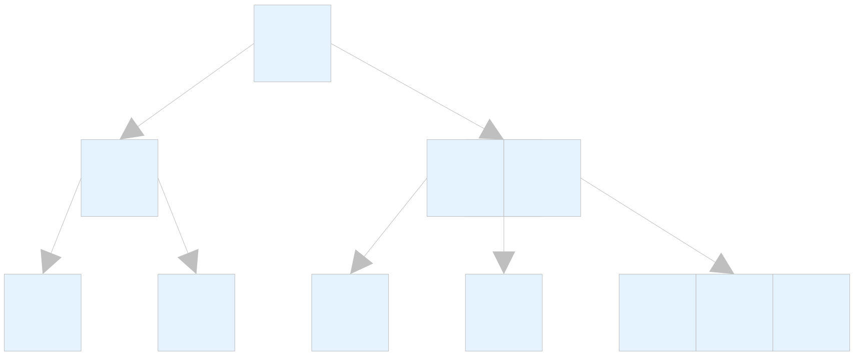

Building B-Trees

The Two-Stack Queue

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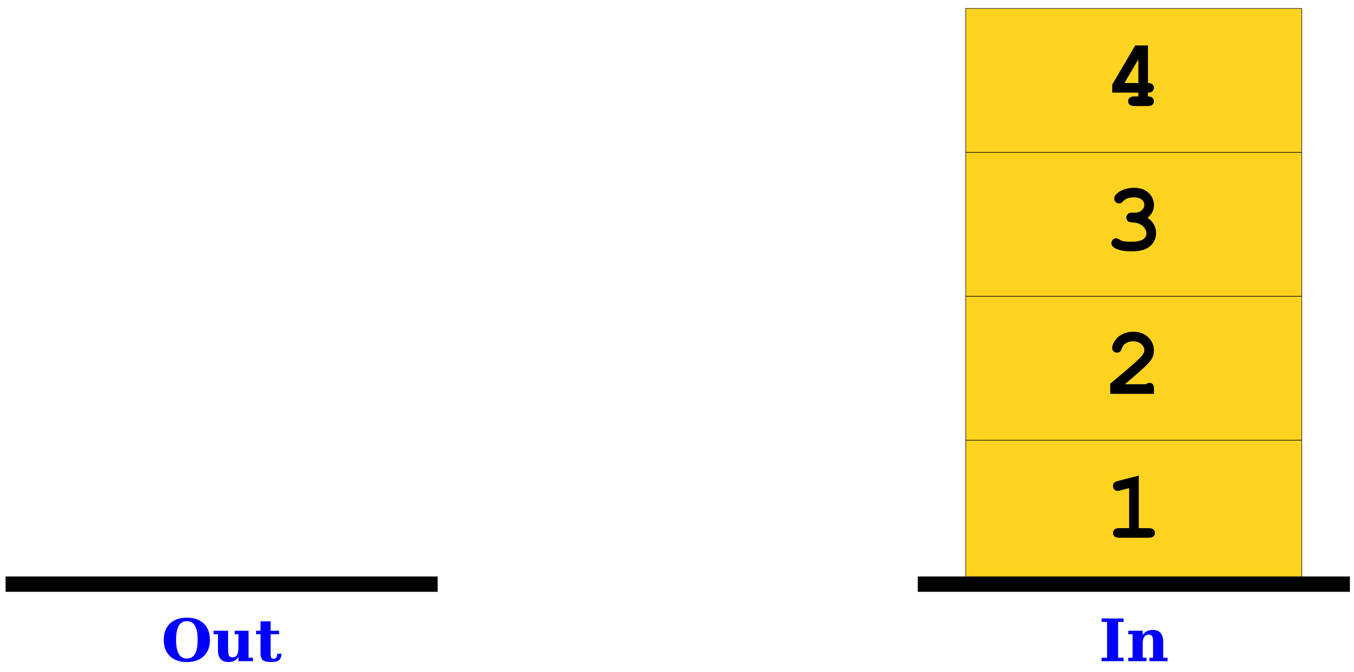

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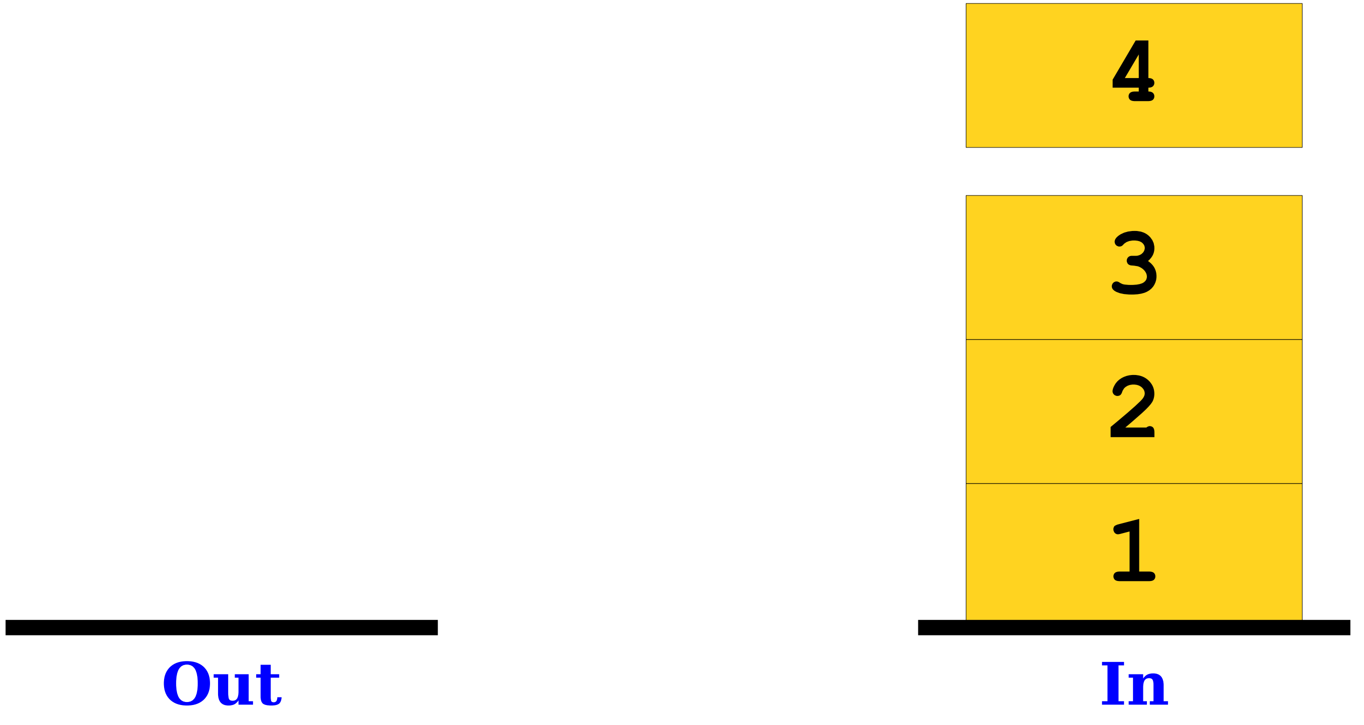

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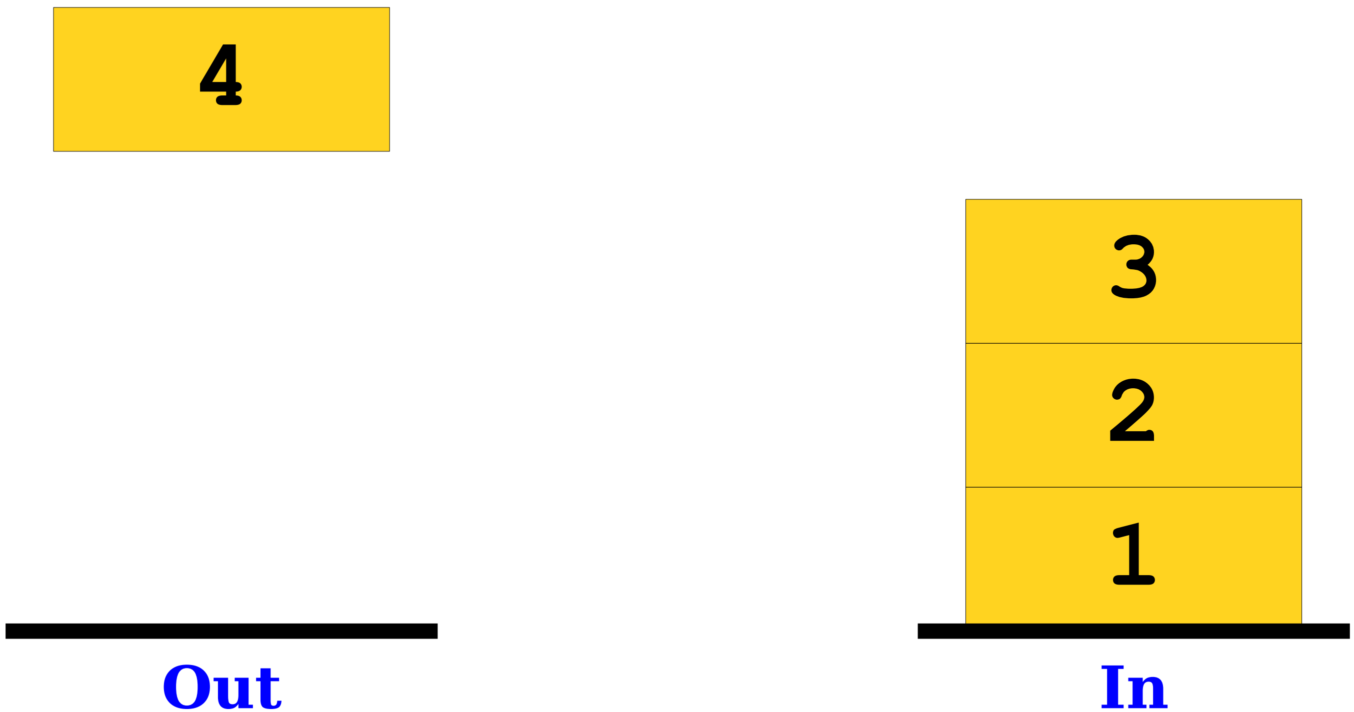

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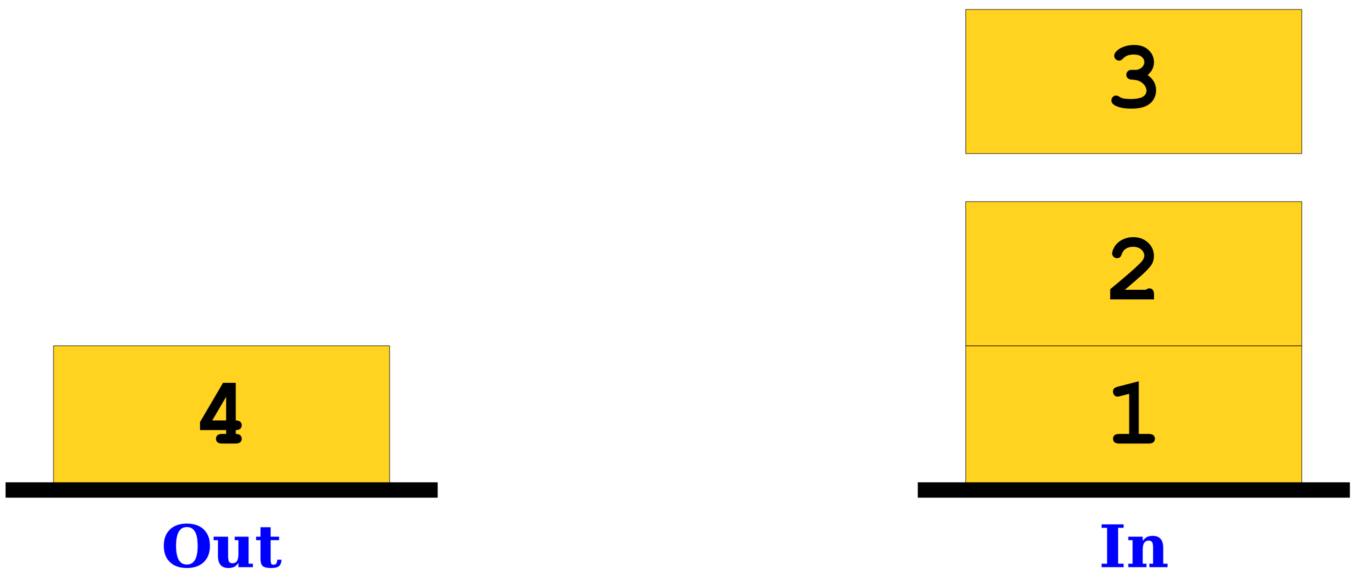

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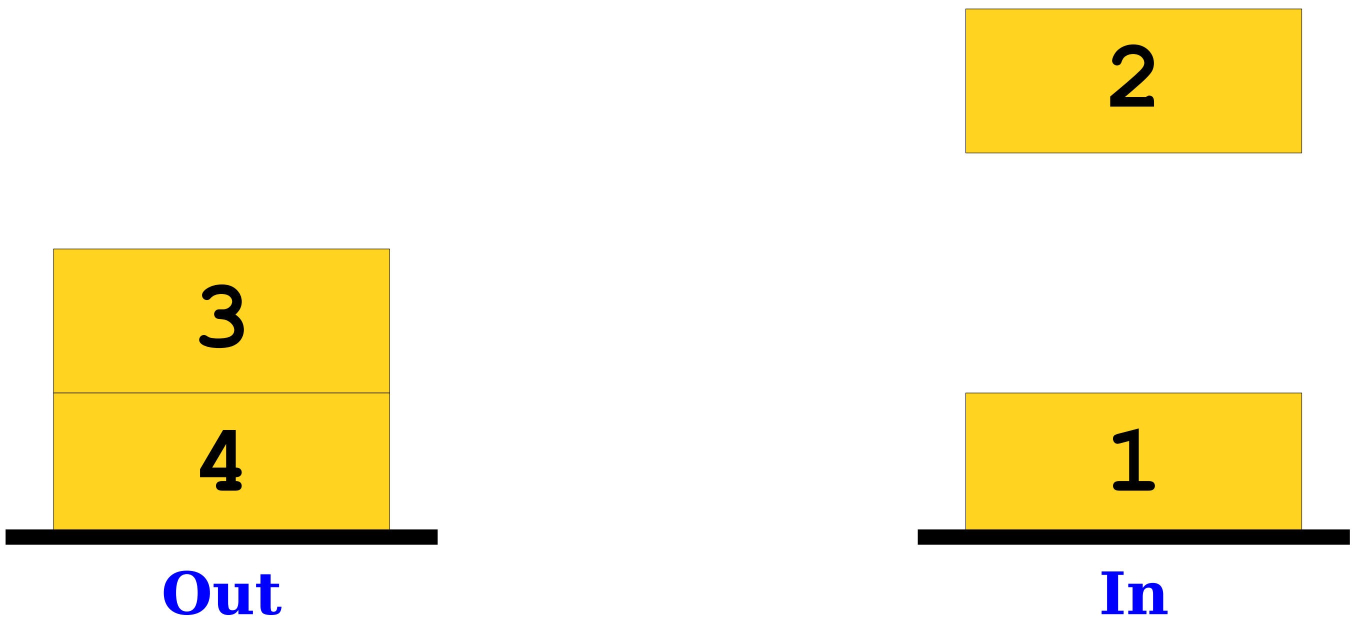

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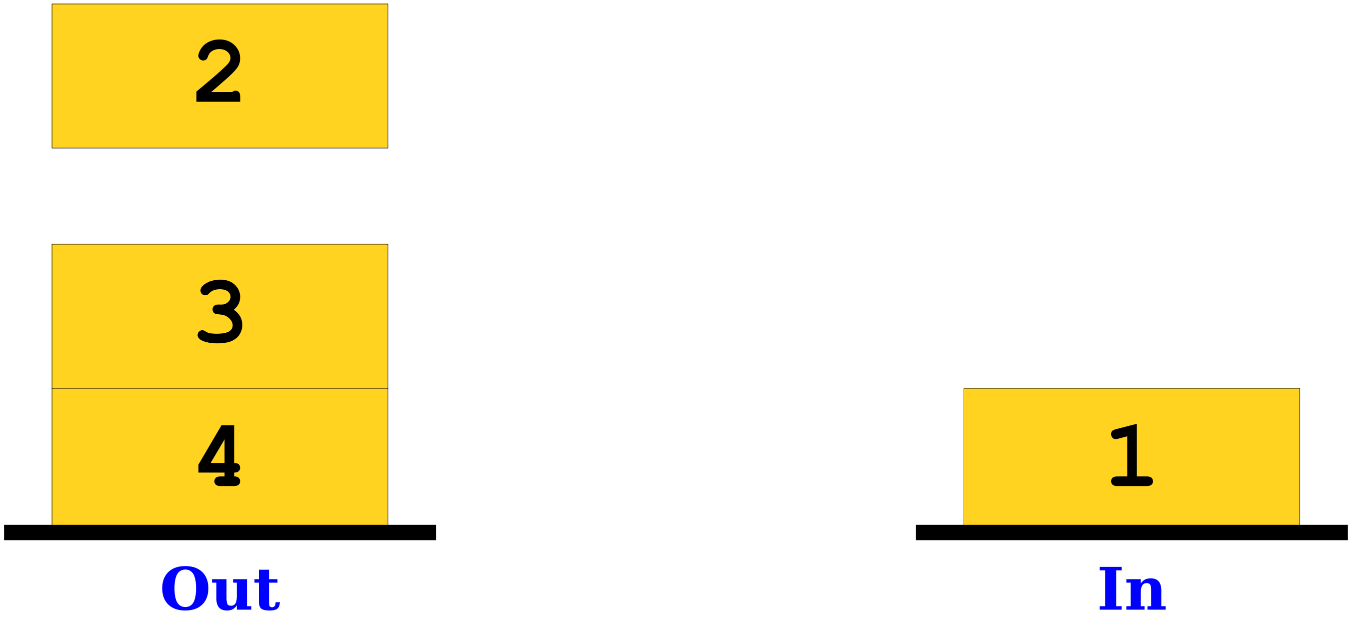

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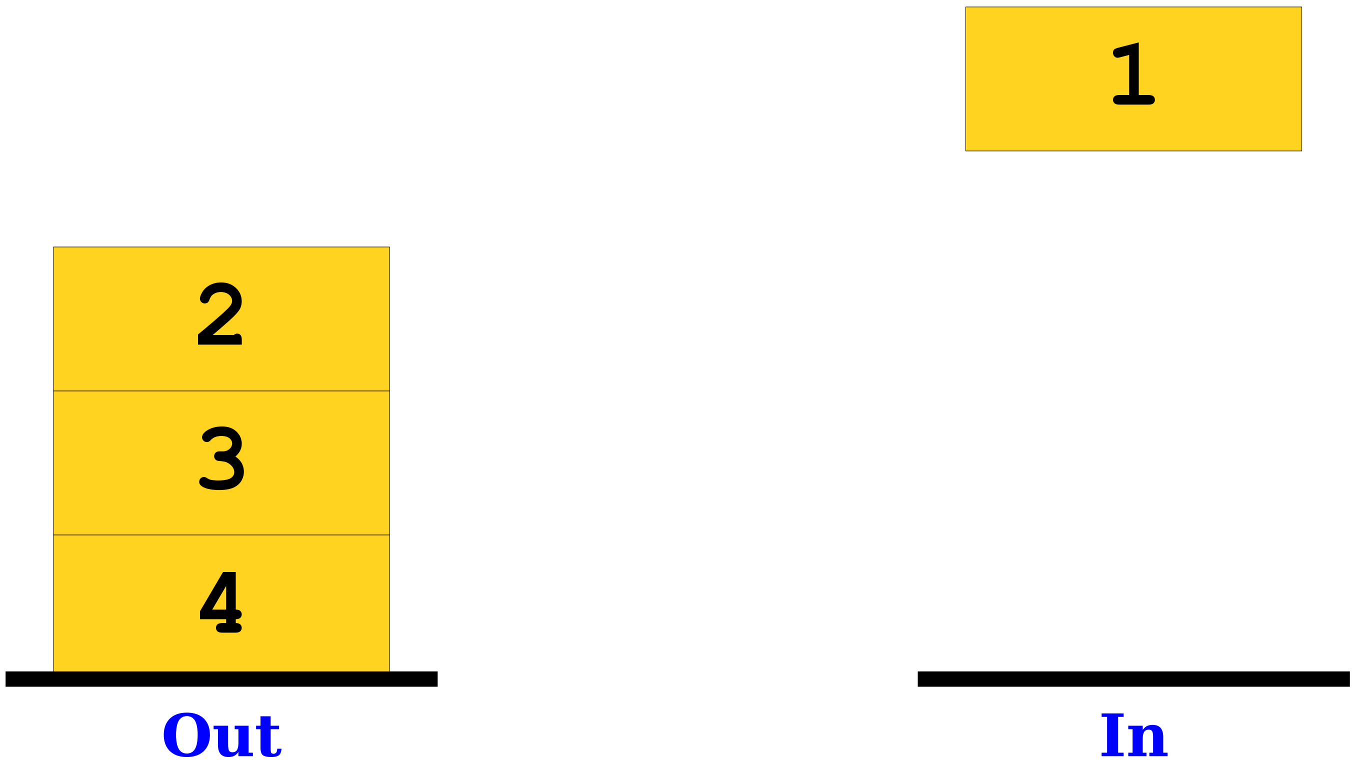

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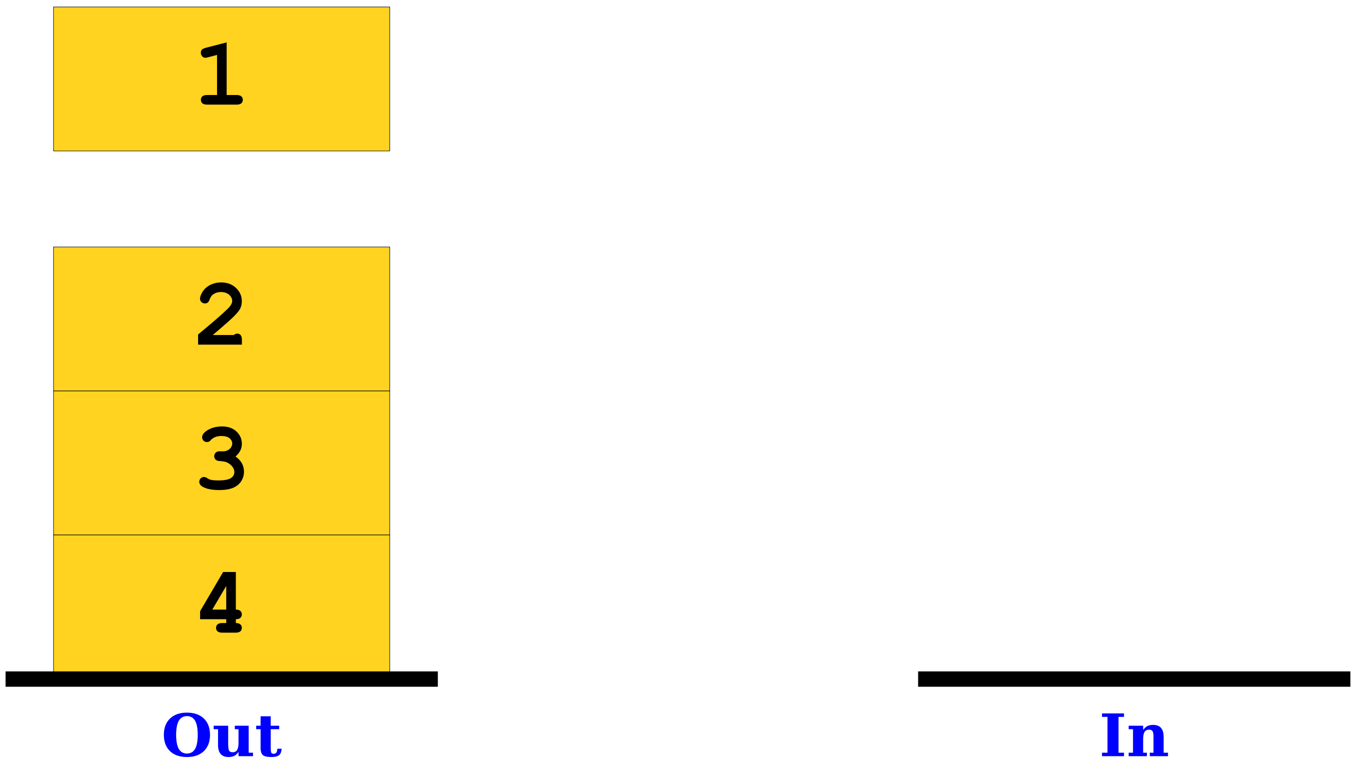

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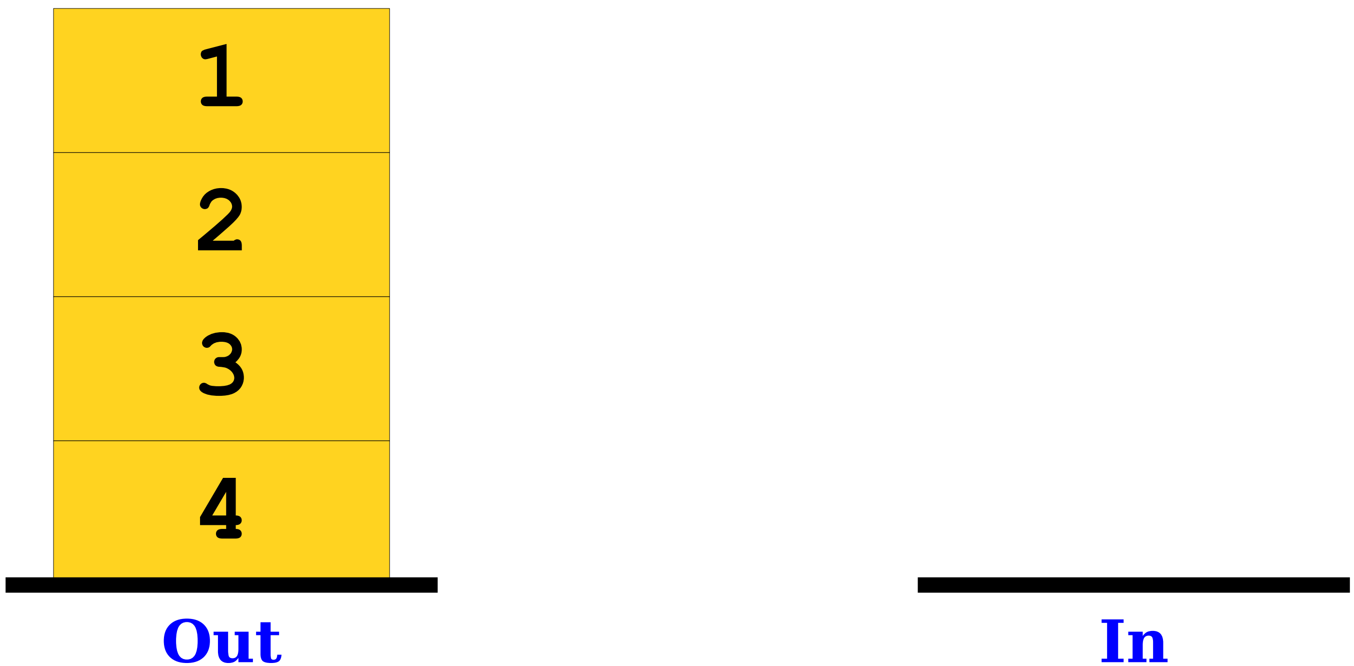

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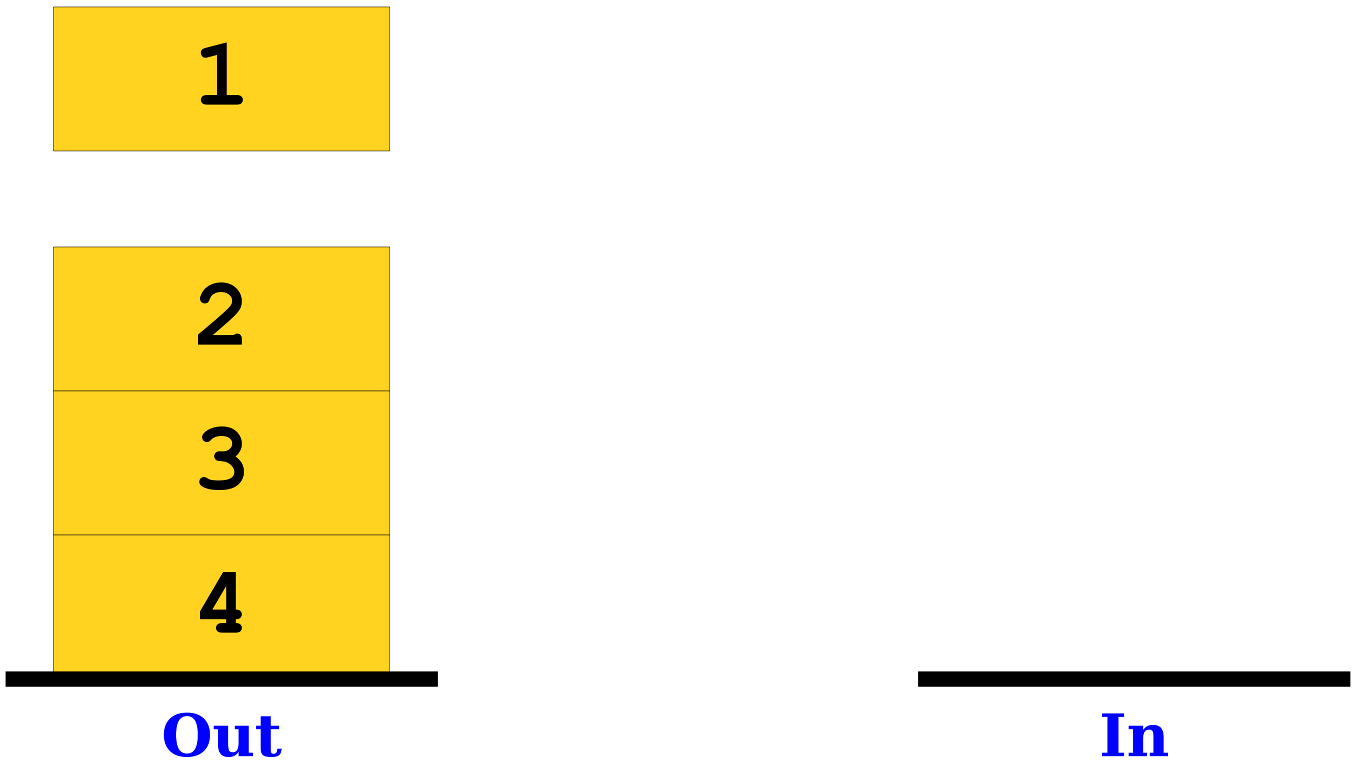

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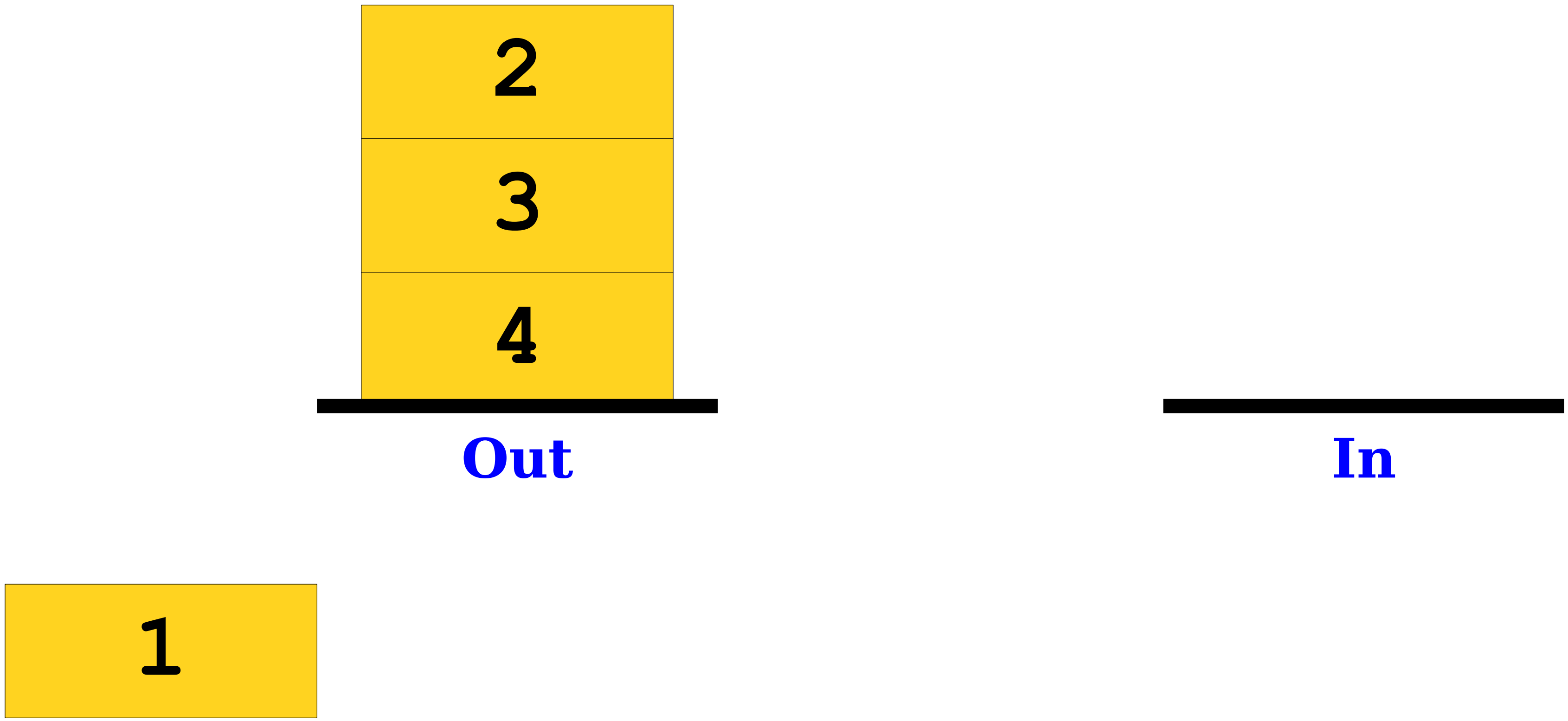

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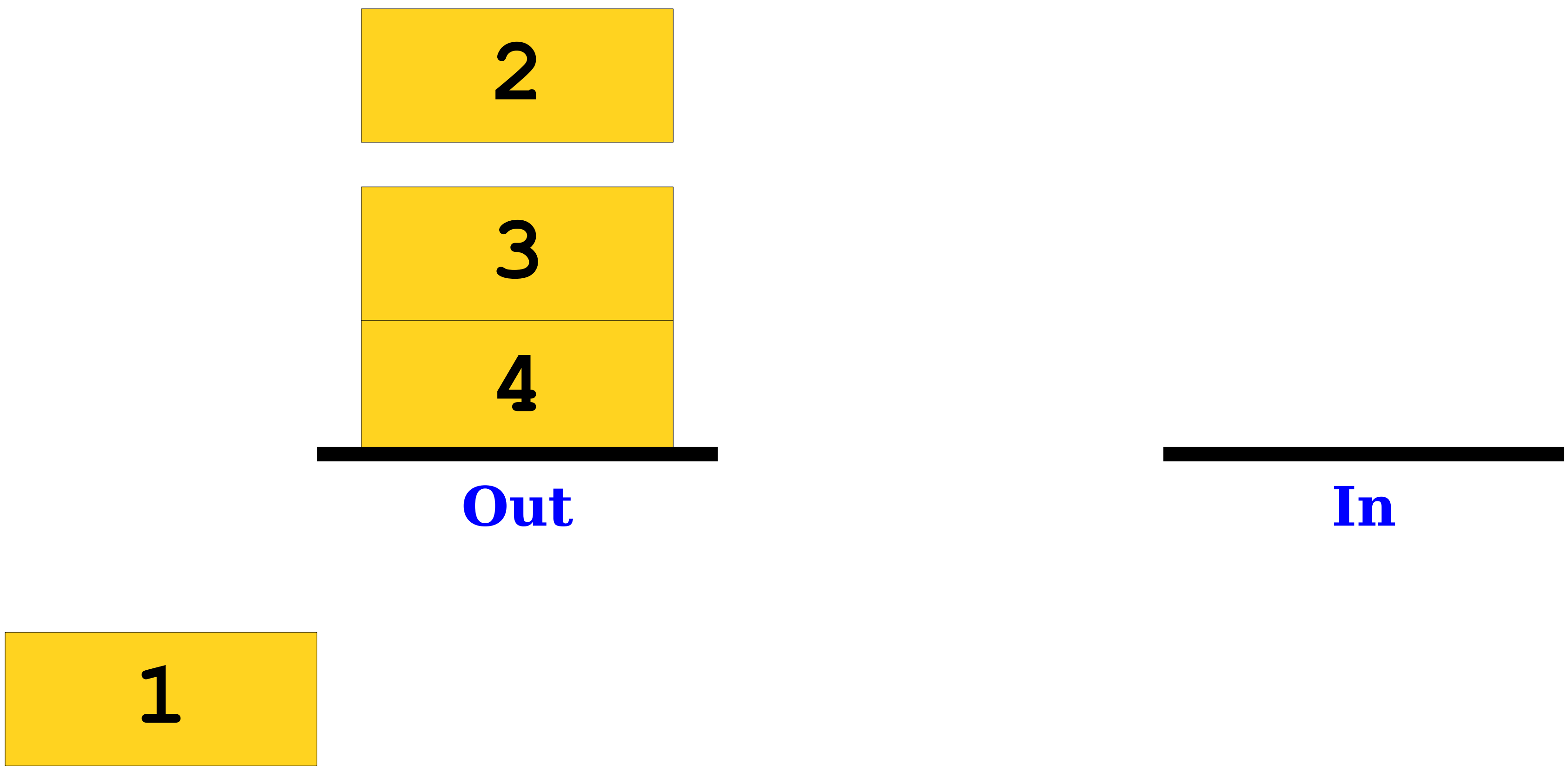

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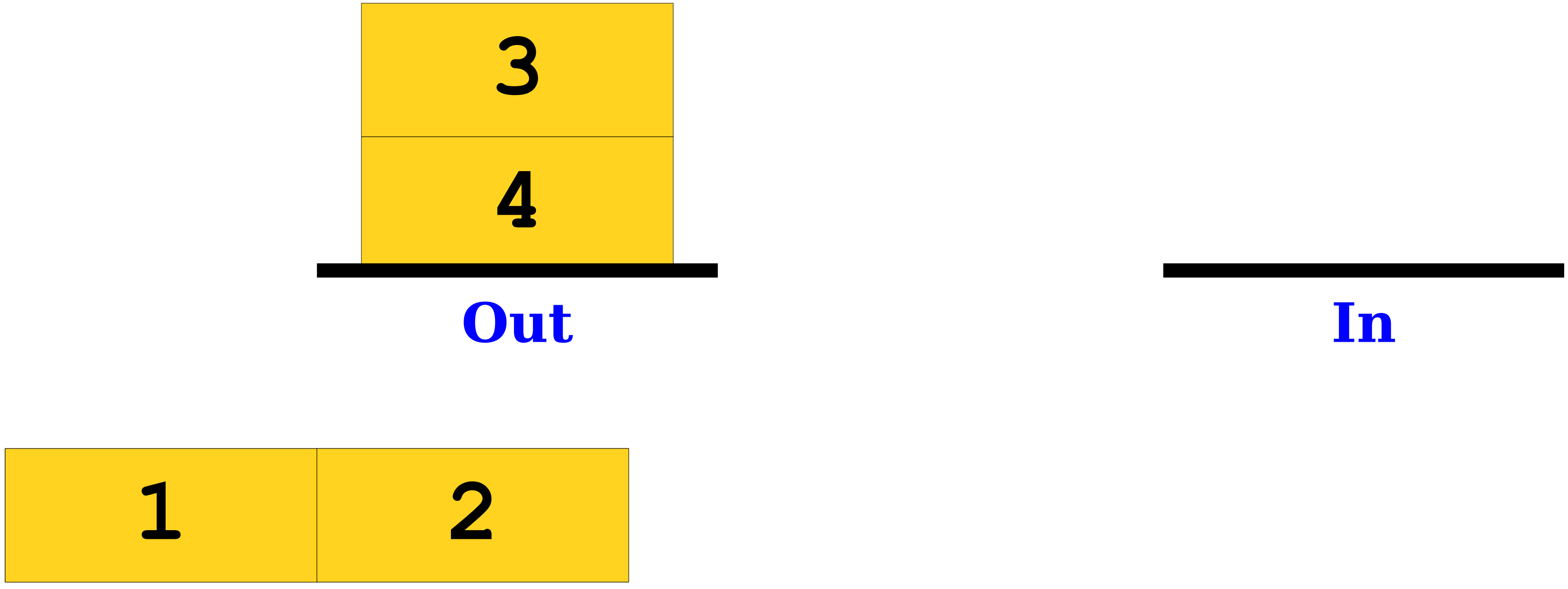

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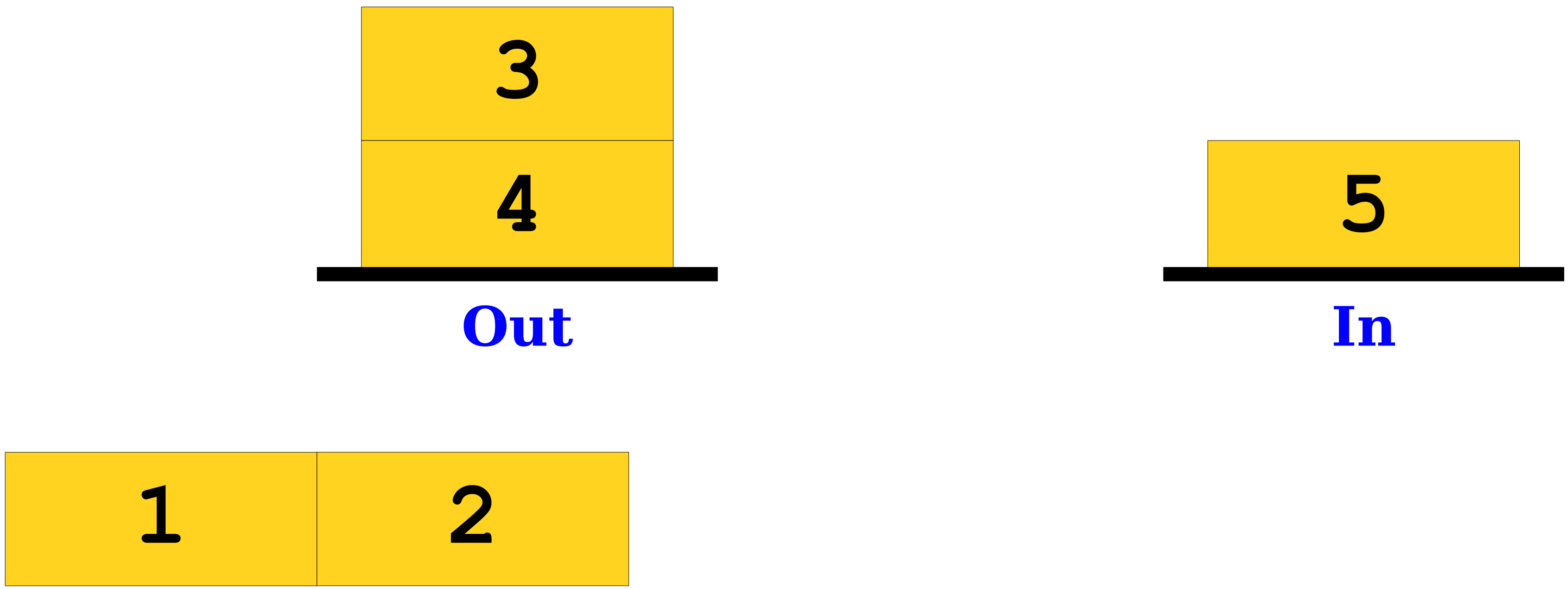

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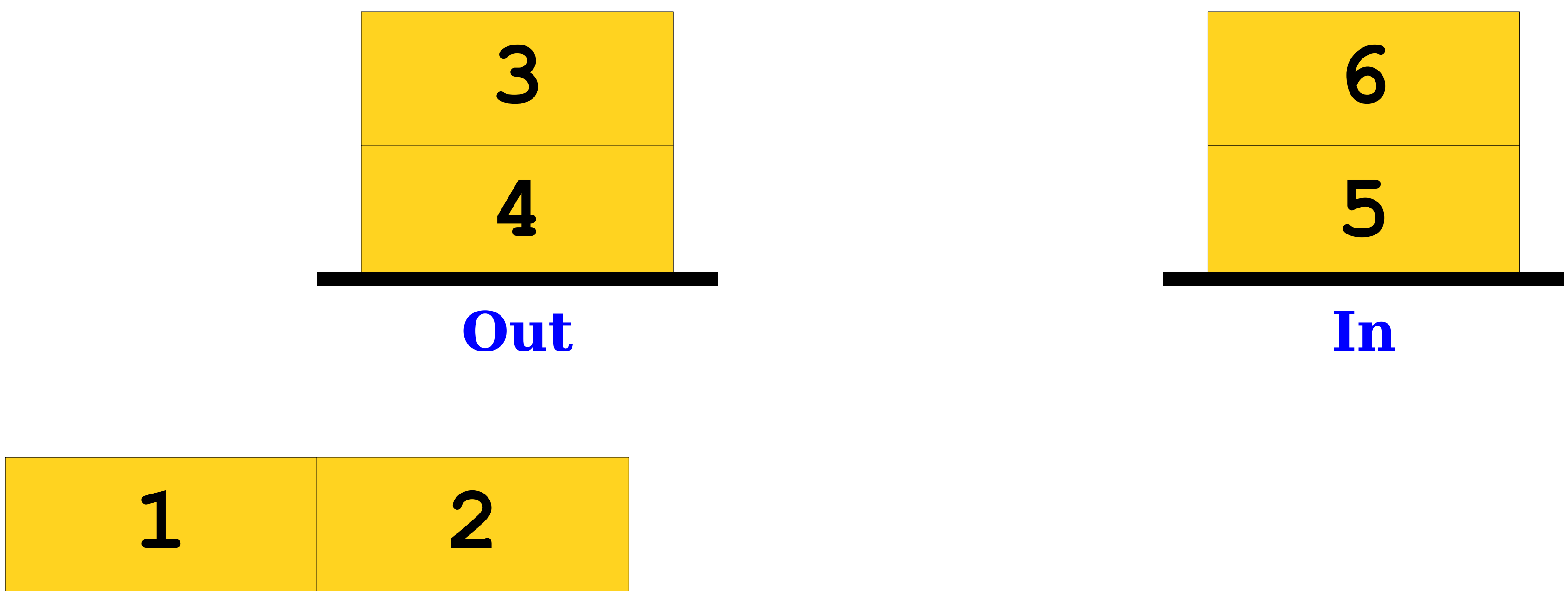

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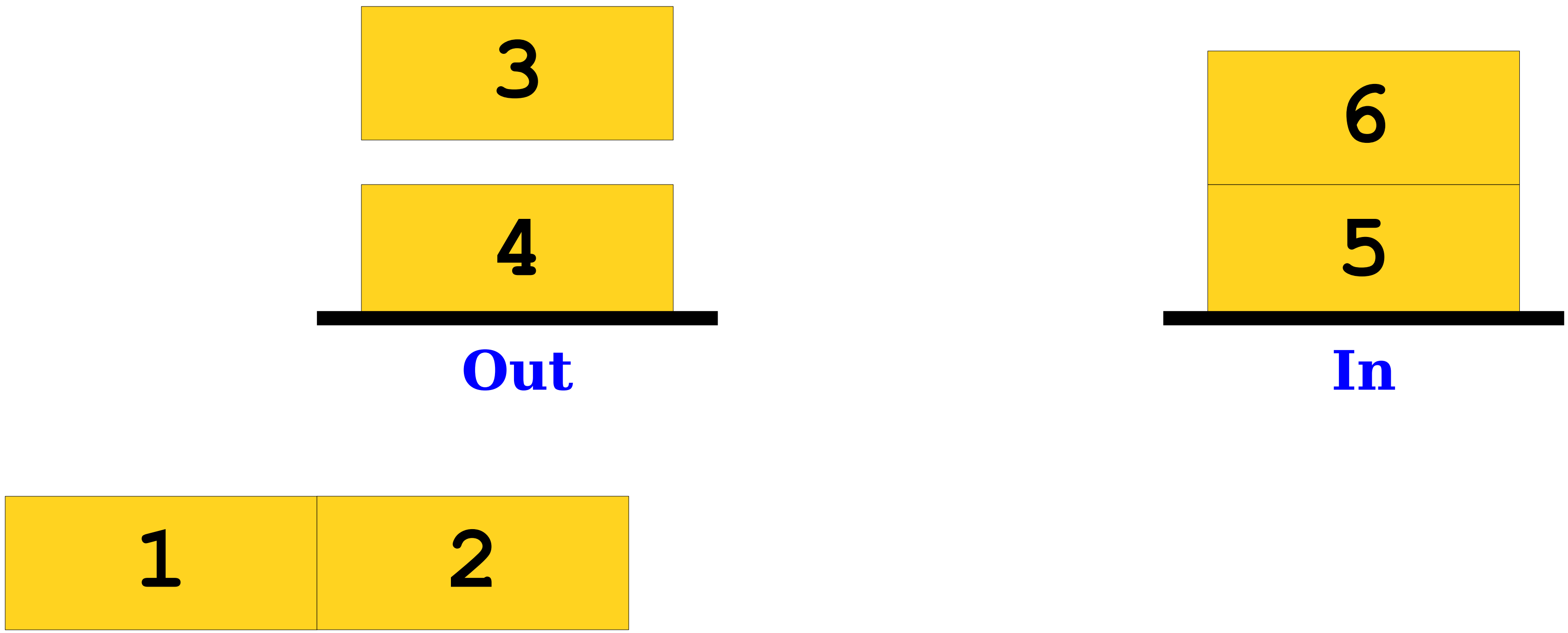

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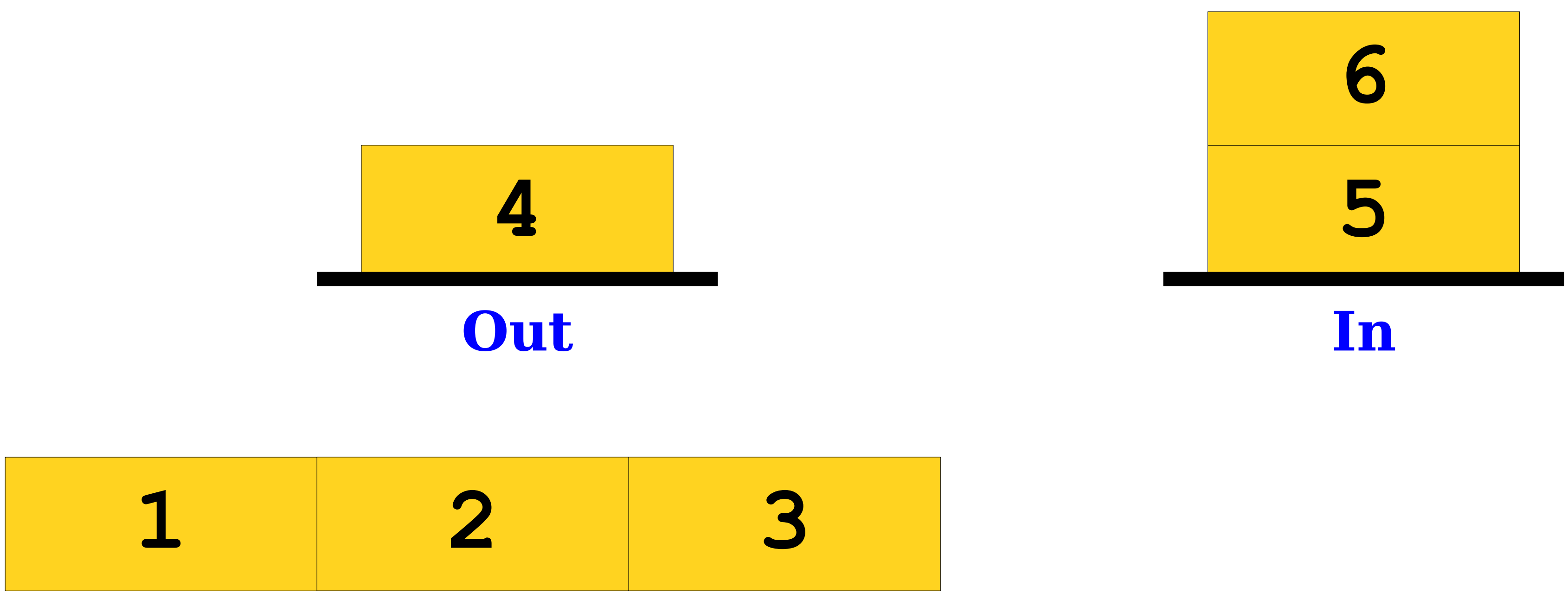

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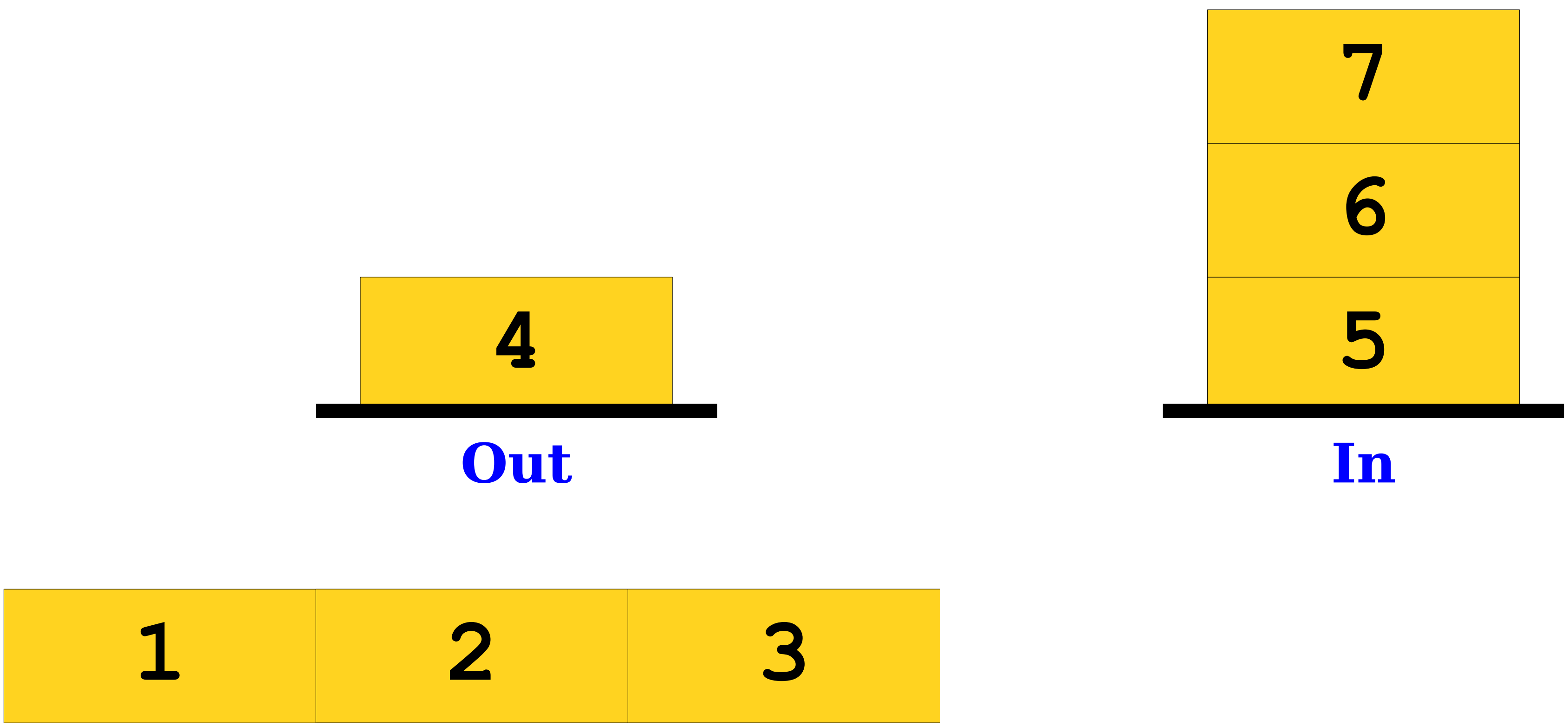

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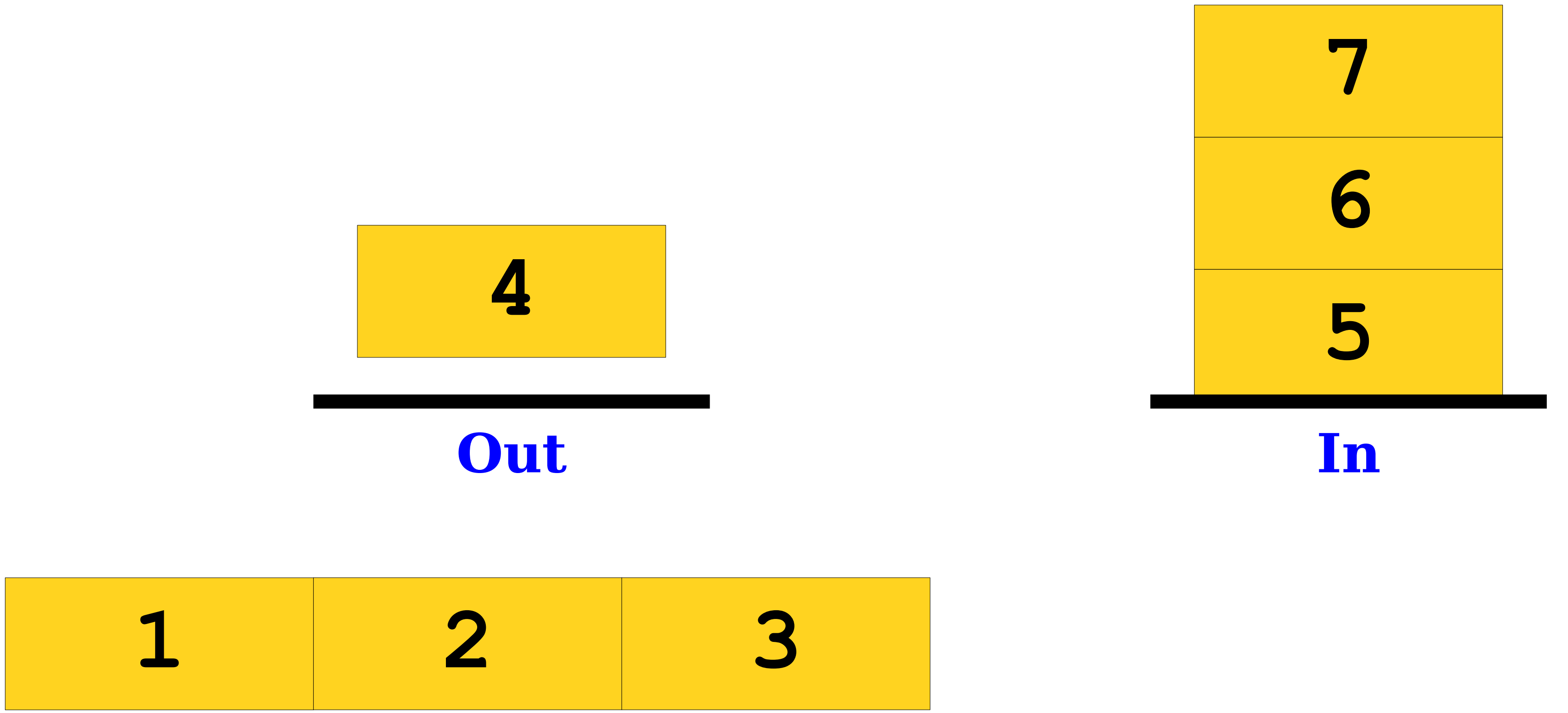

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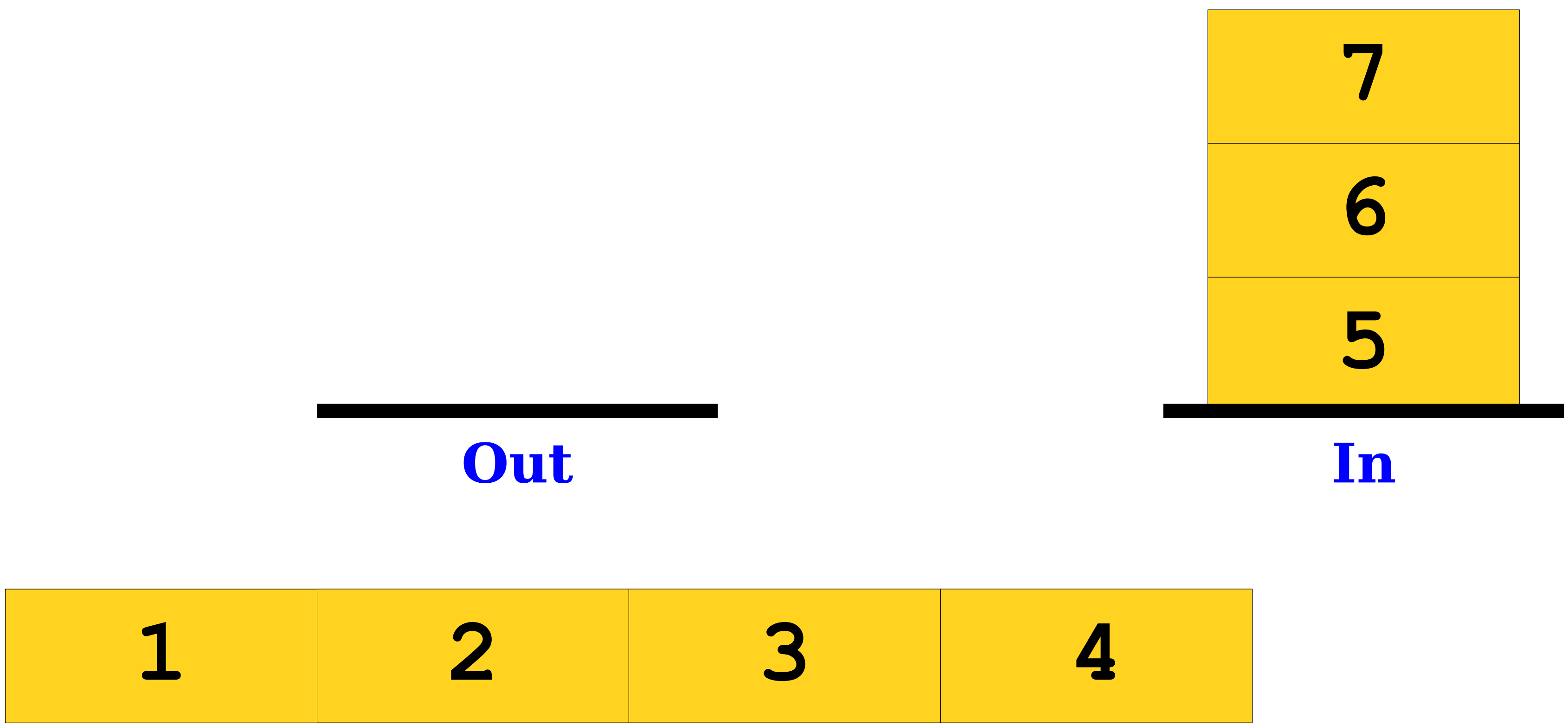

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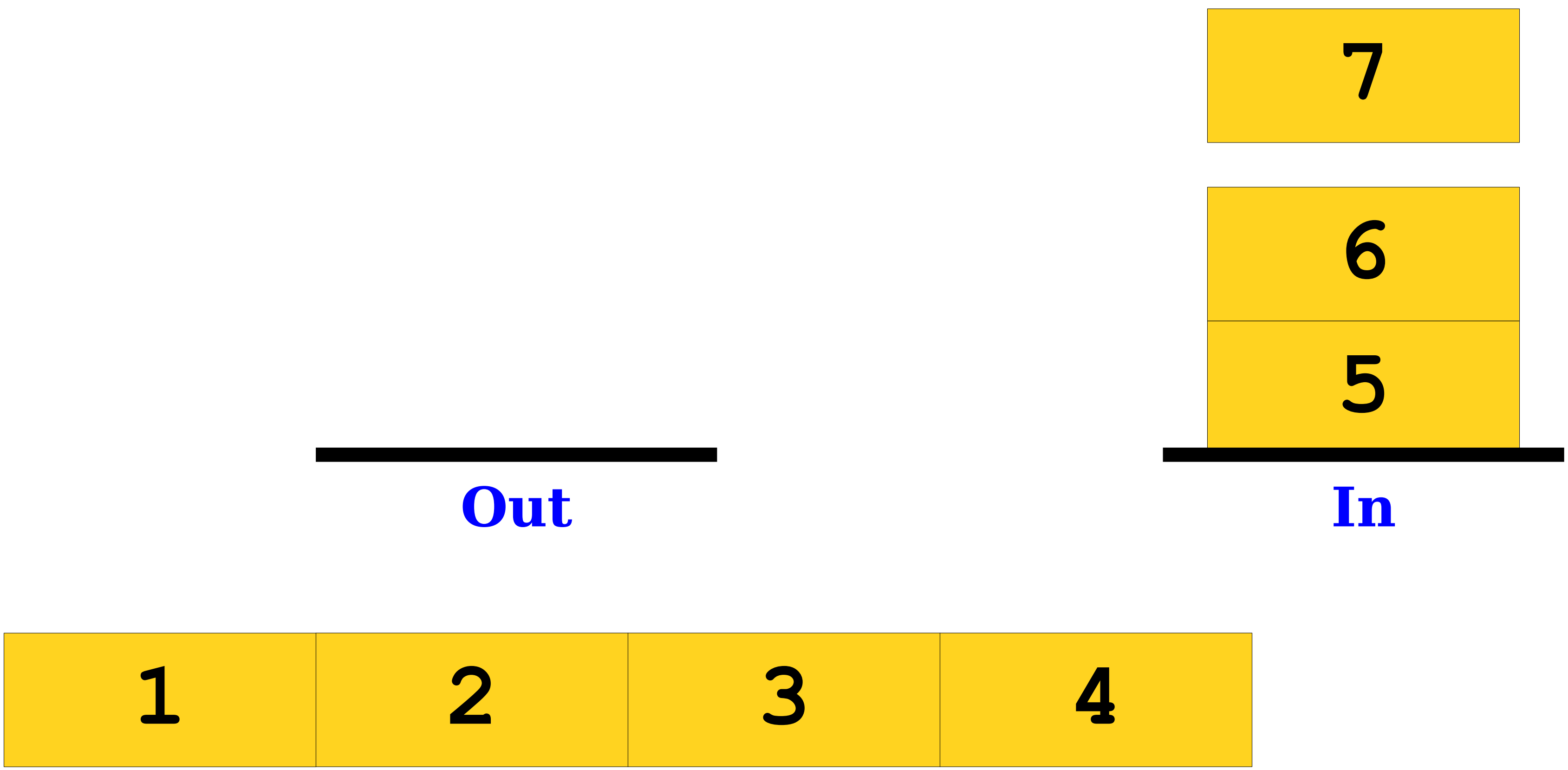

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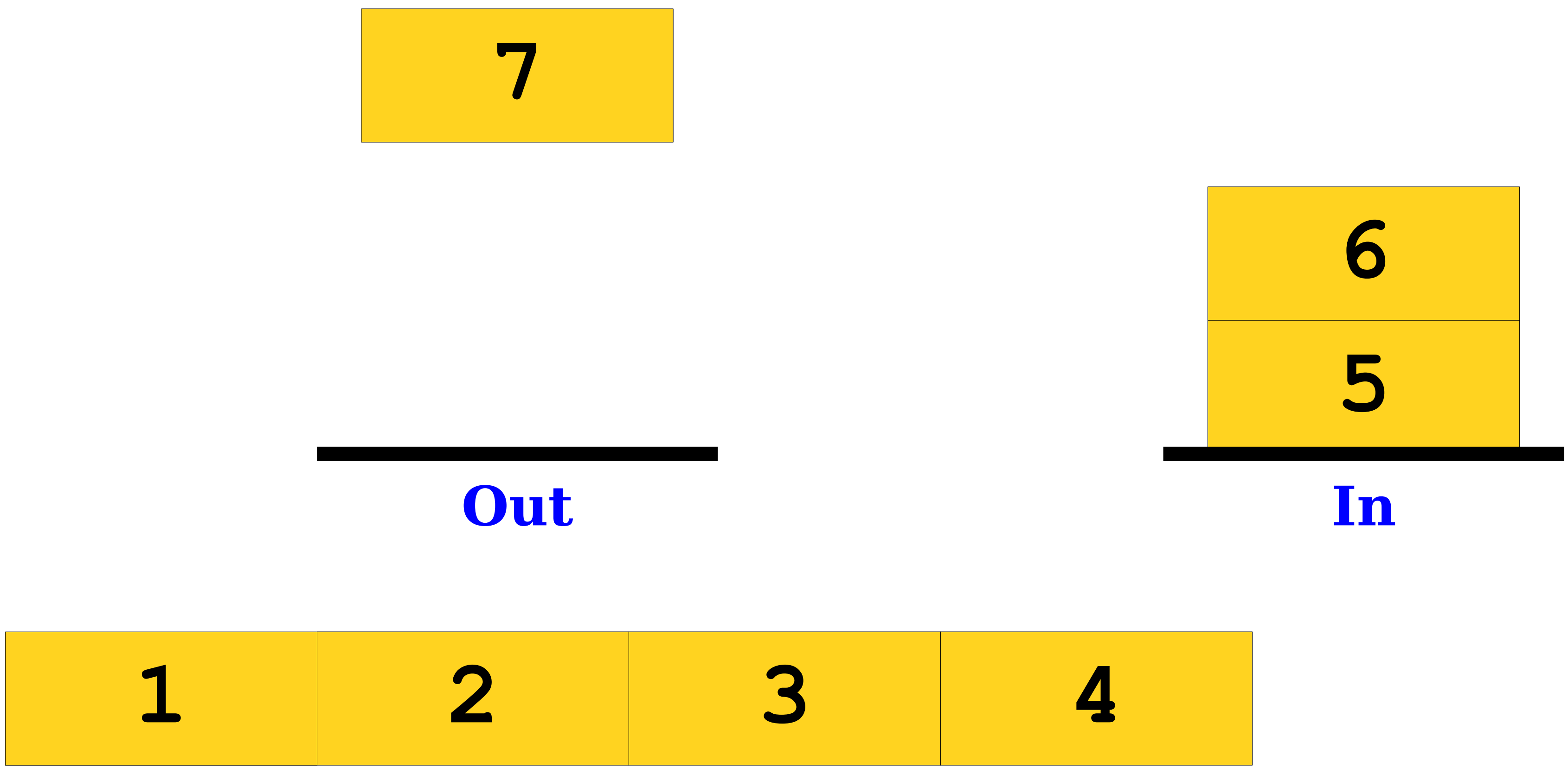

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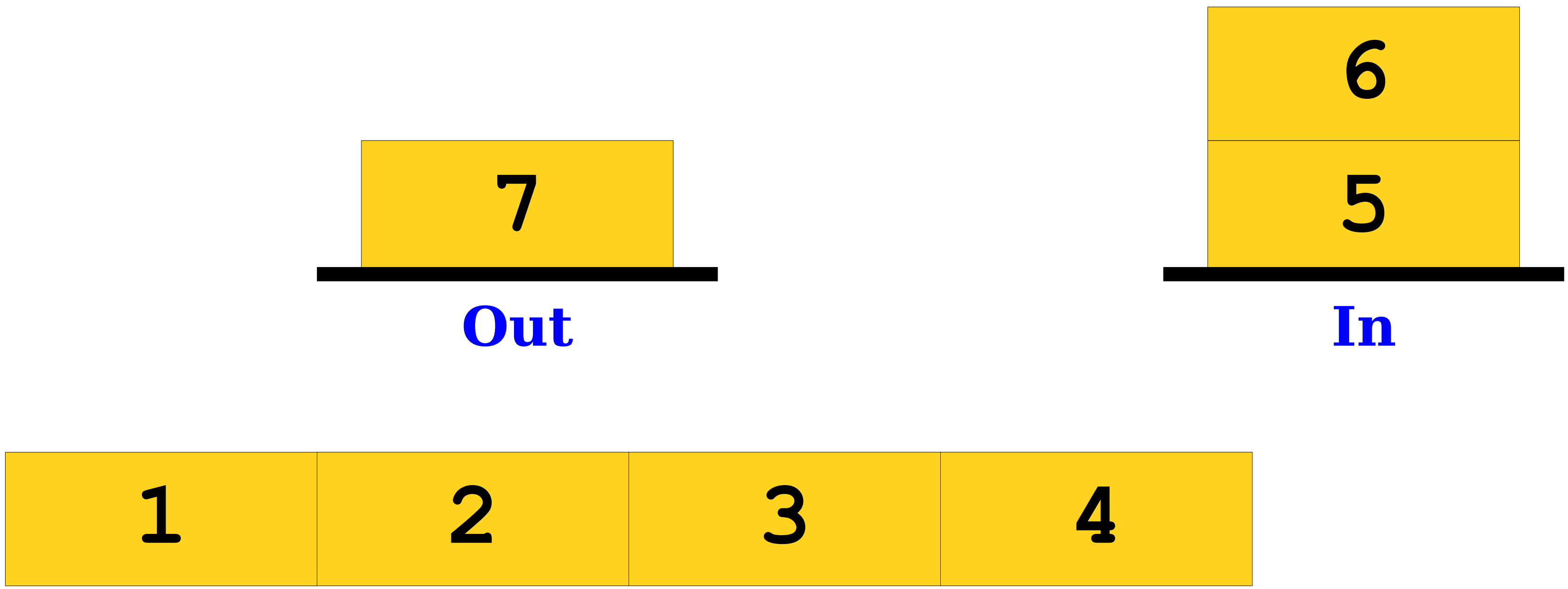

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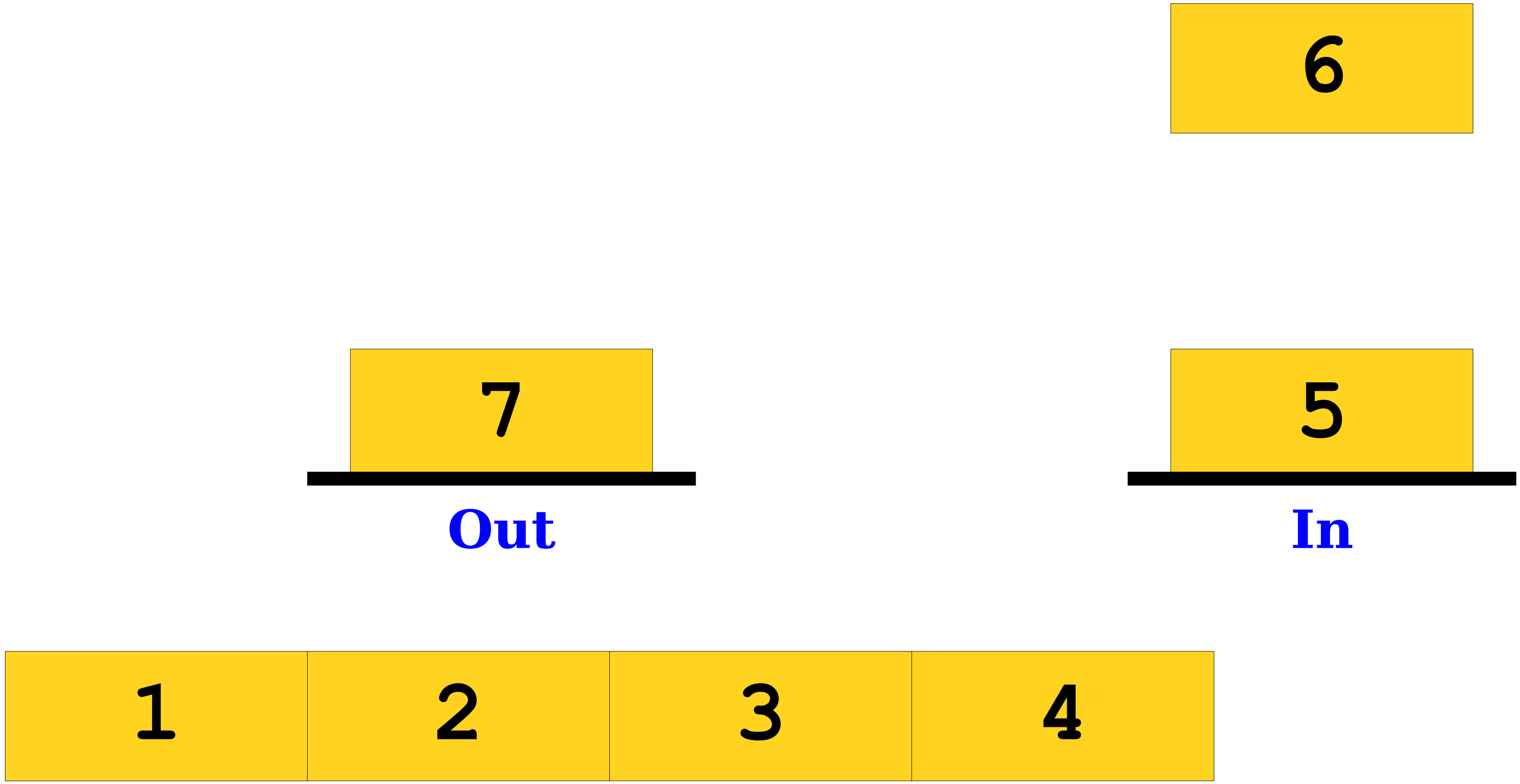

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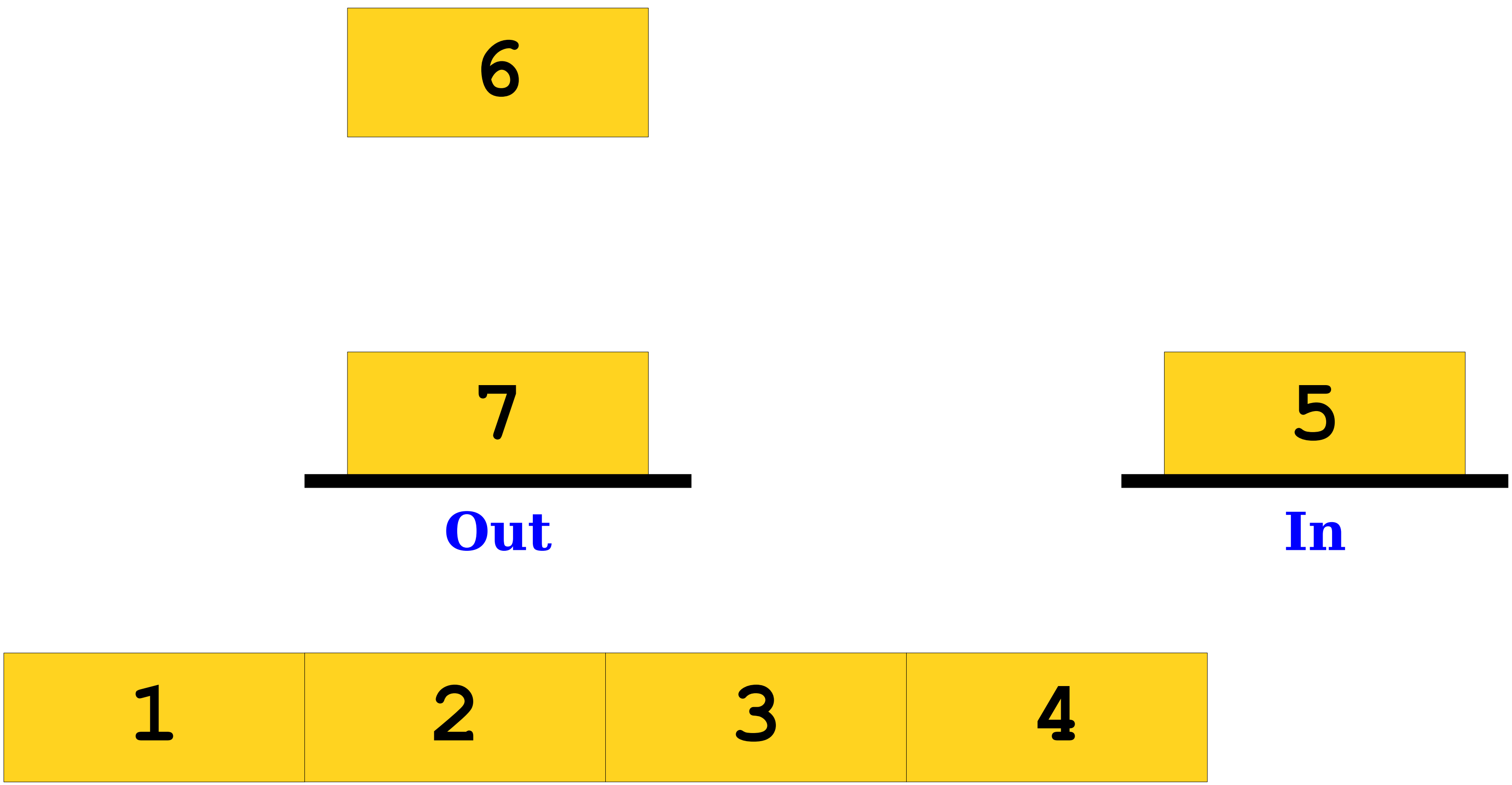

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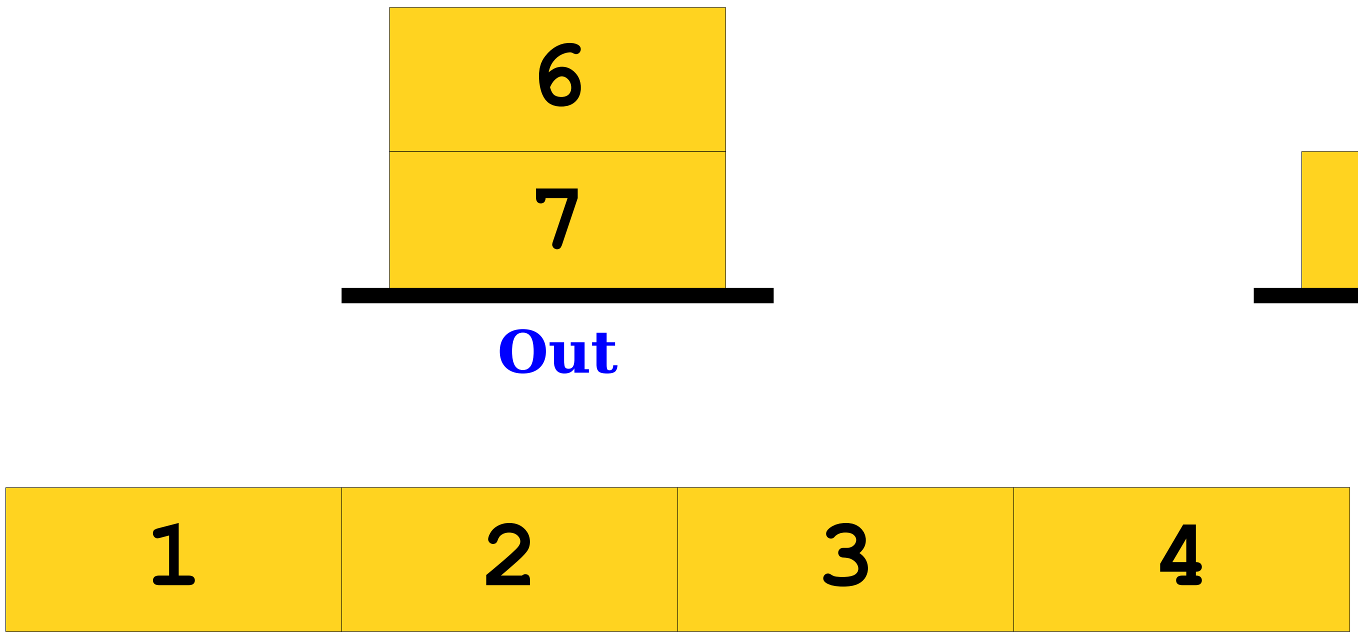

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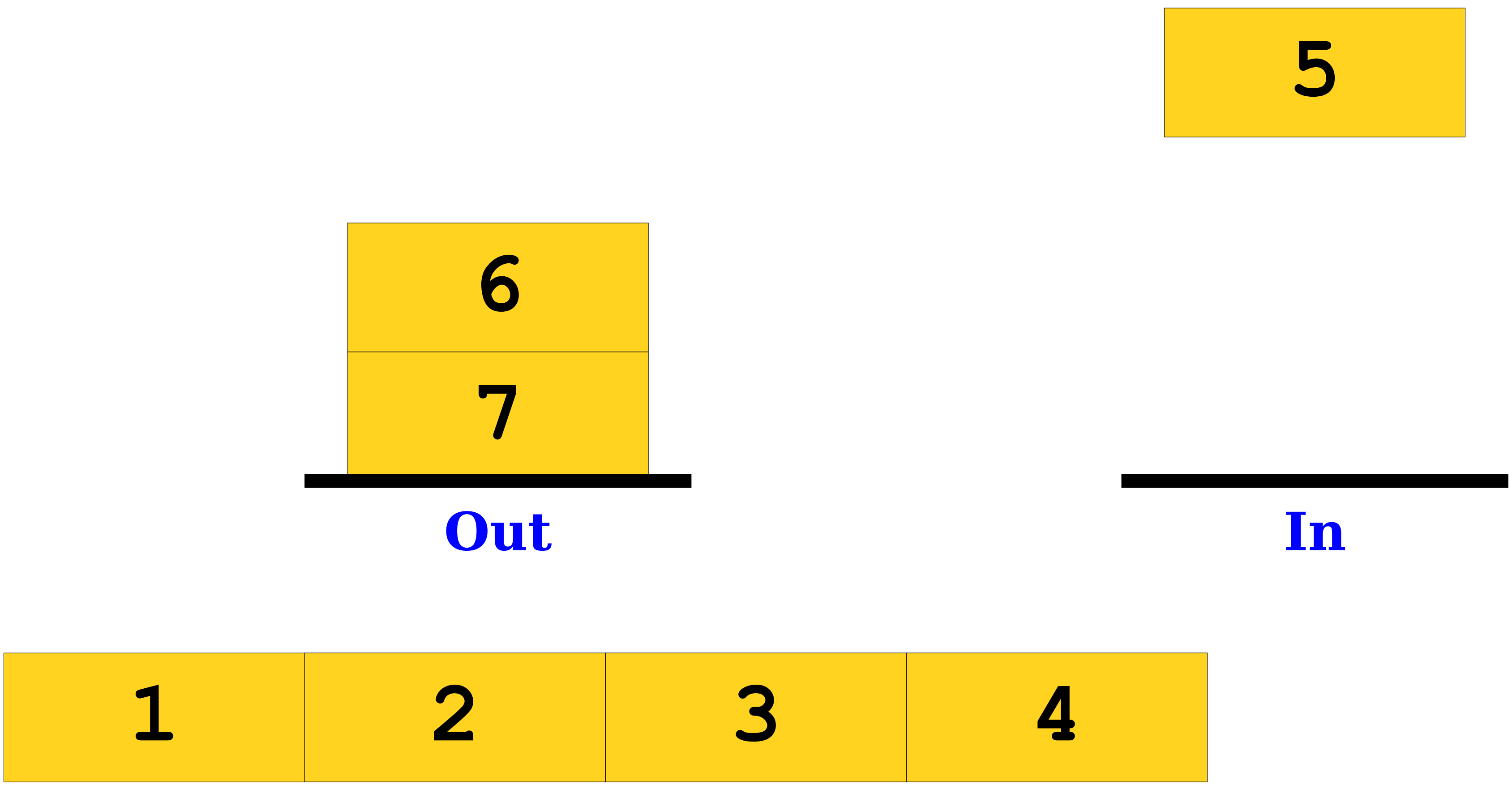

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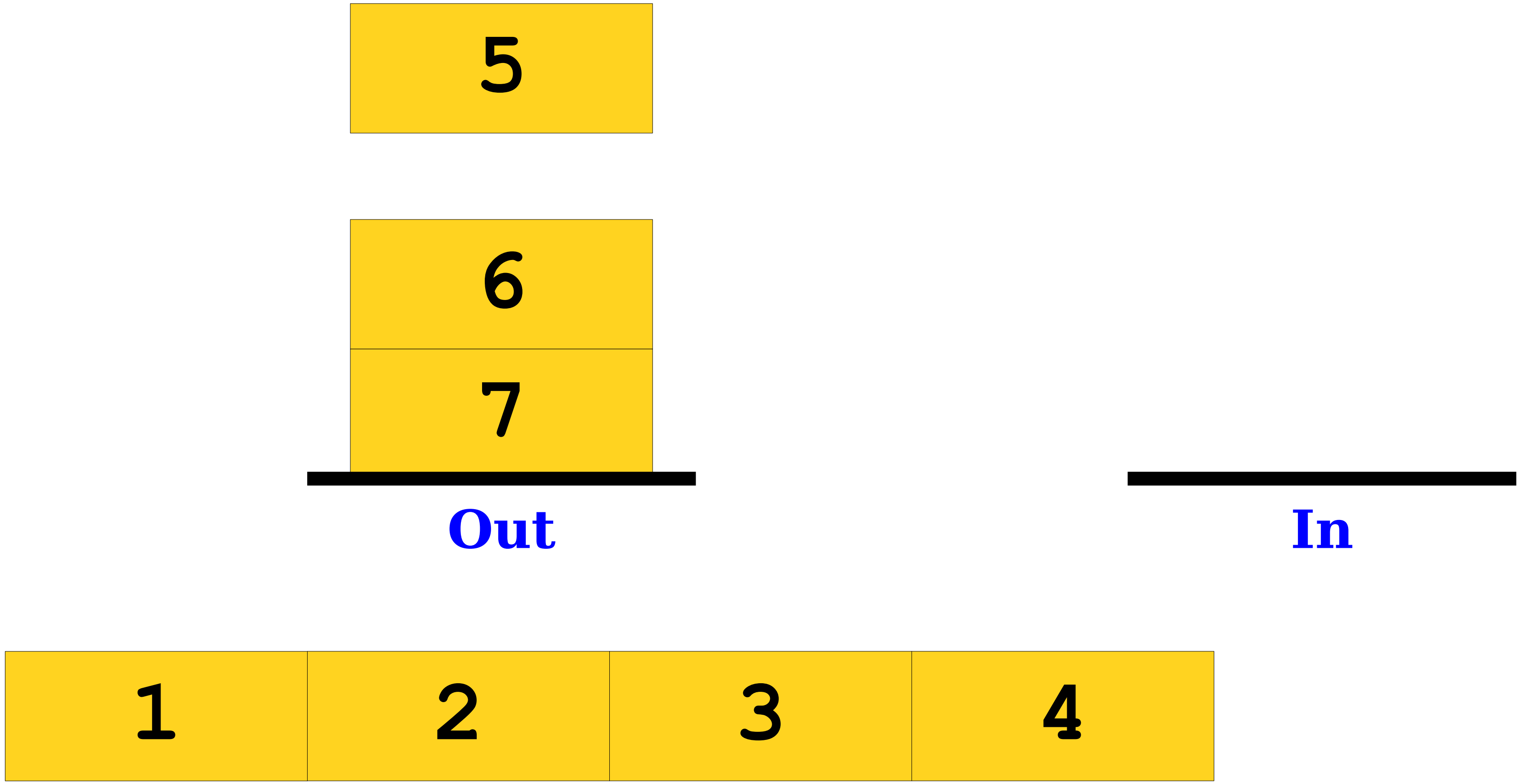

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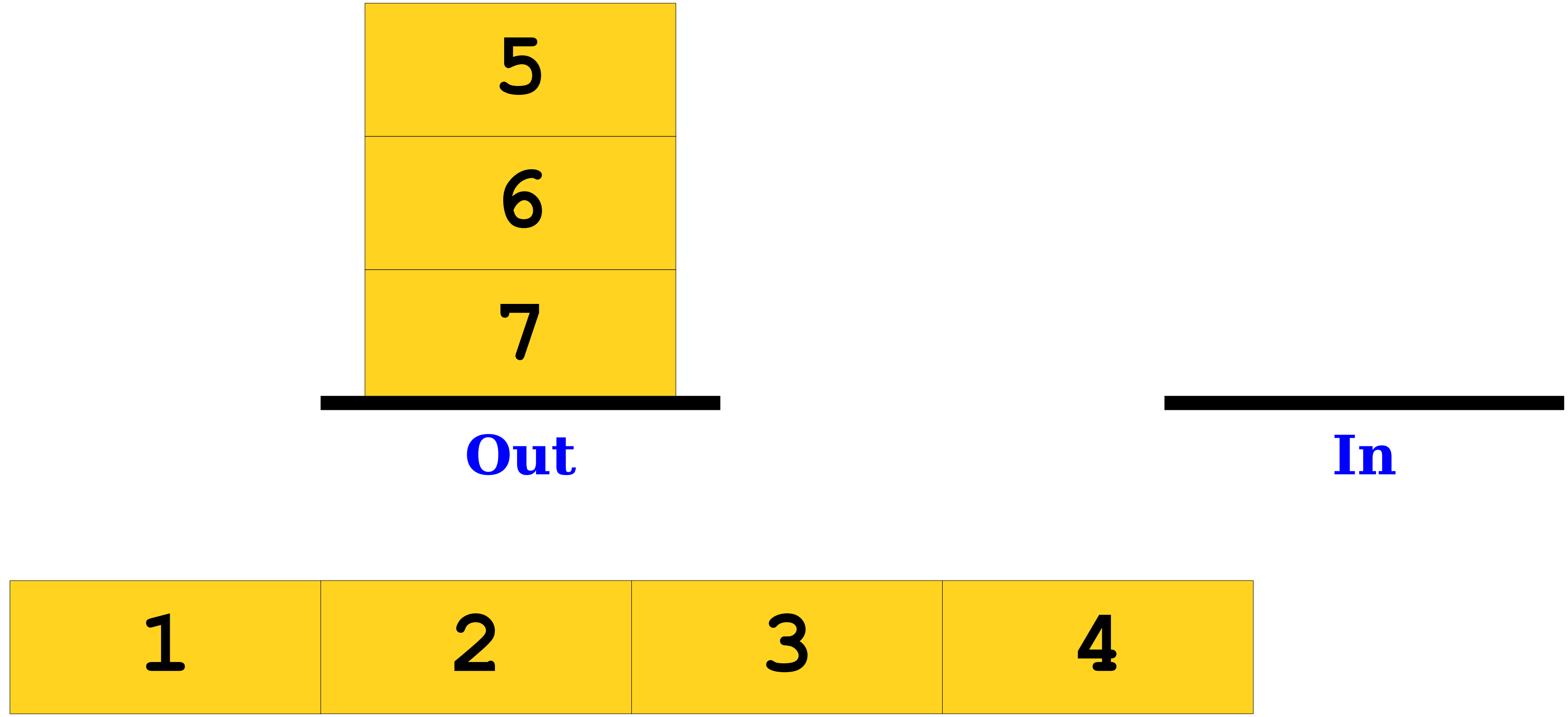

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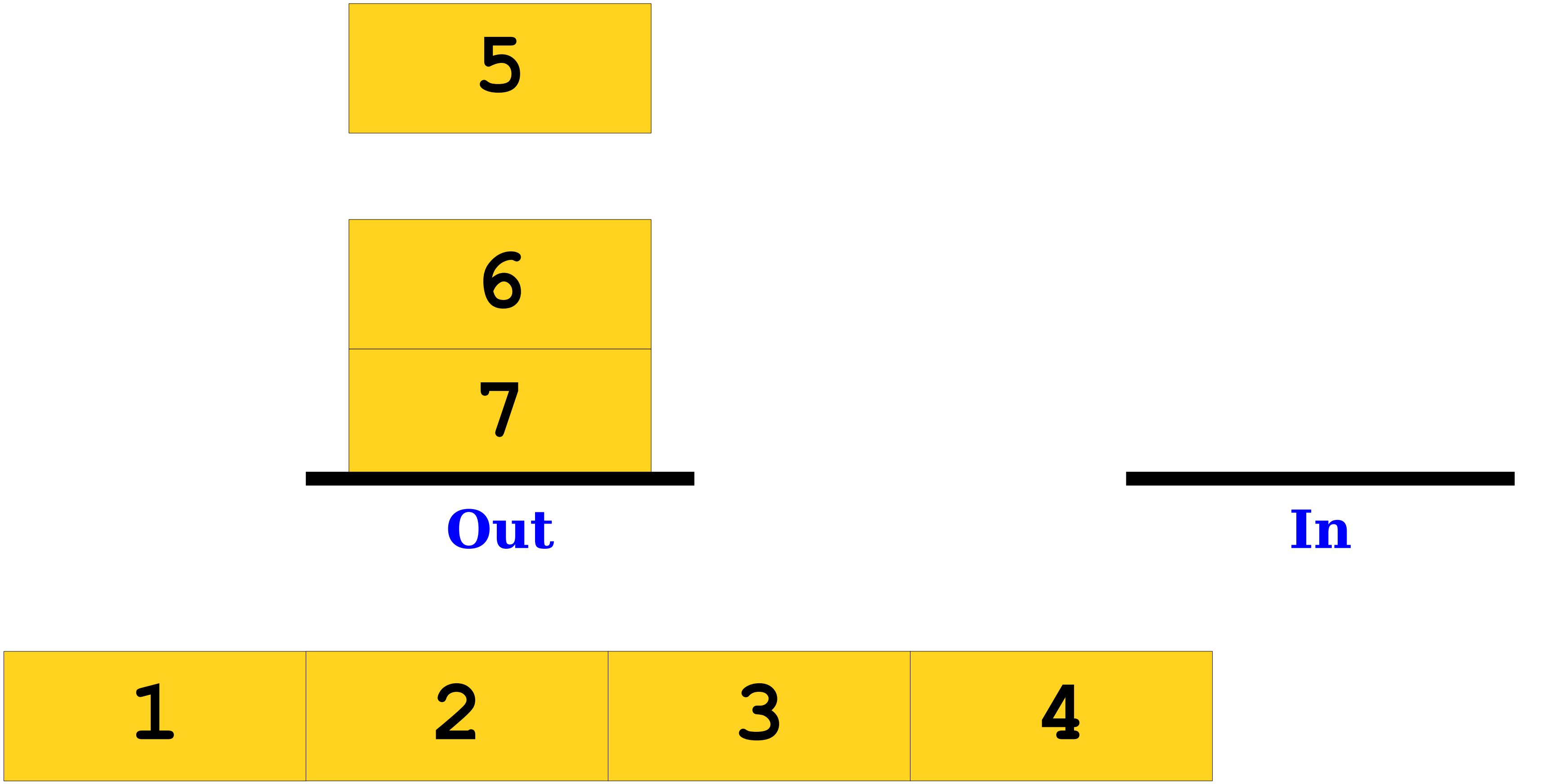

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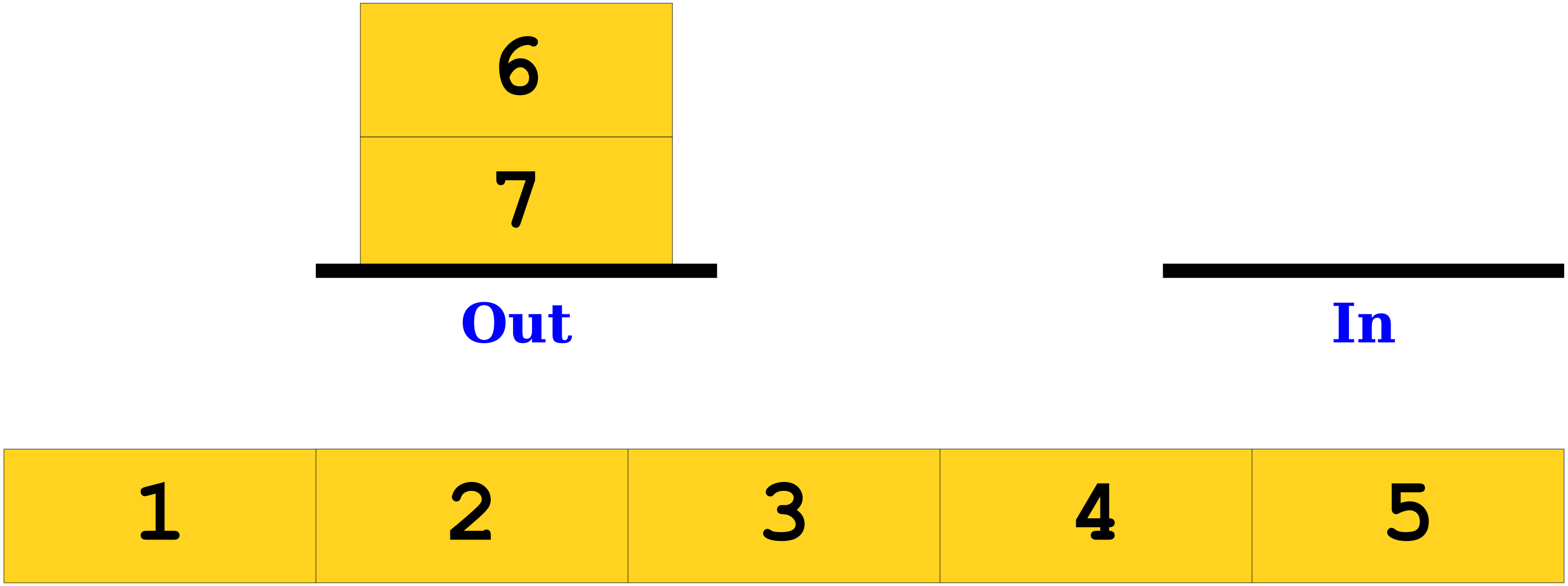


The Two-Stack Queue

- Maintain an ***In*** stack and an ***Out*** stack.
- To enqueue an element, push it onto the ***In*** stack.
- To dequeue an element:
  - If the ***Out*** stack is nonempty, pop it.
  - If the ***Out*** stack is empty, pop elements from the ***In*** stack, pushing them into the ***Out*** stack. Then dequeue as usual.

The Two-Stack Queue

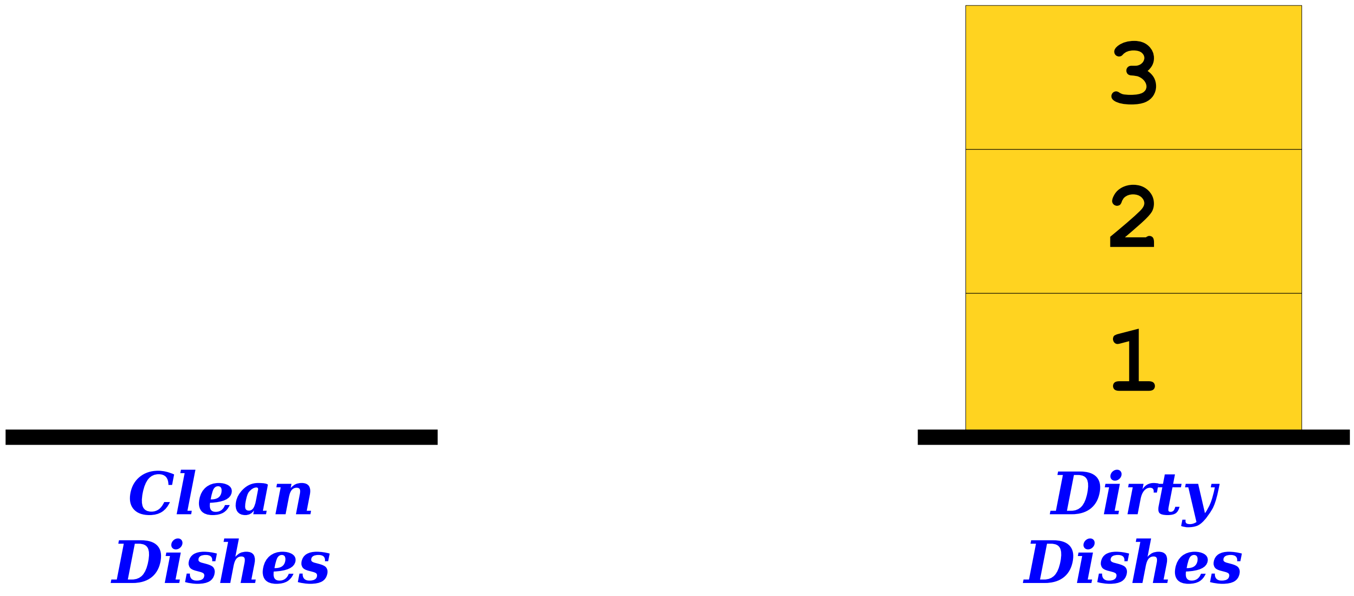
Clean Dishes

Dirty Dishes

The Two-Stack Queue

The Two-Stack Queue

The Two-Stack Queue



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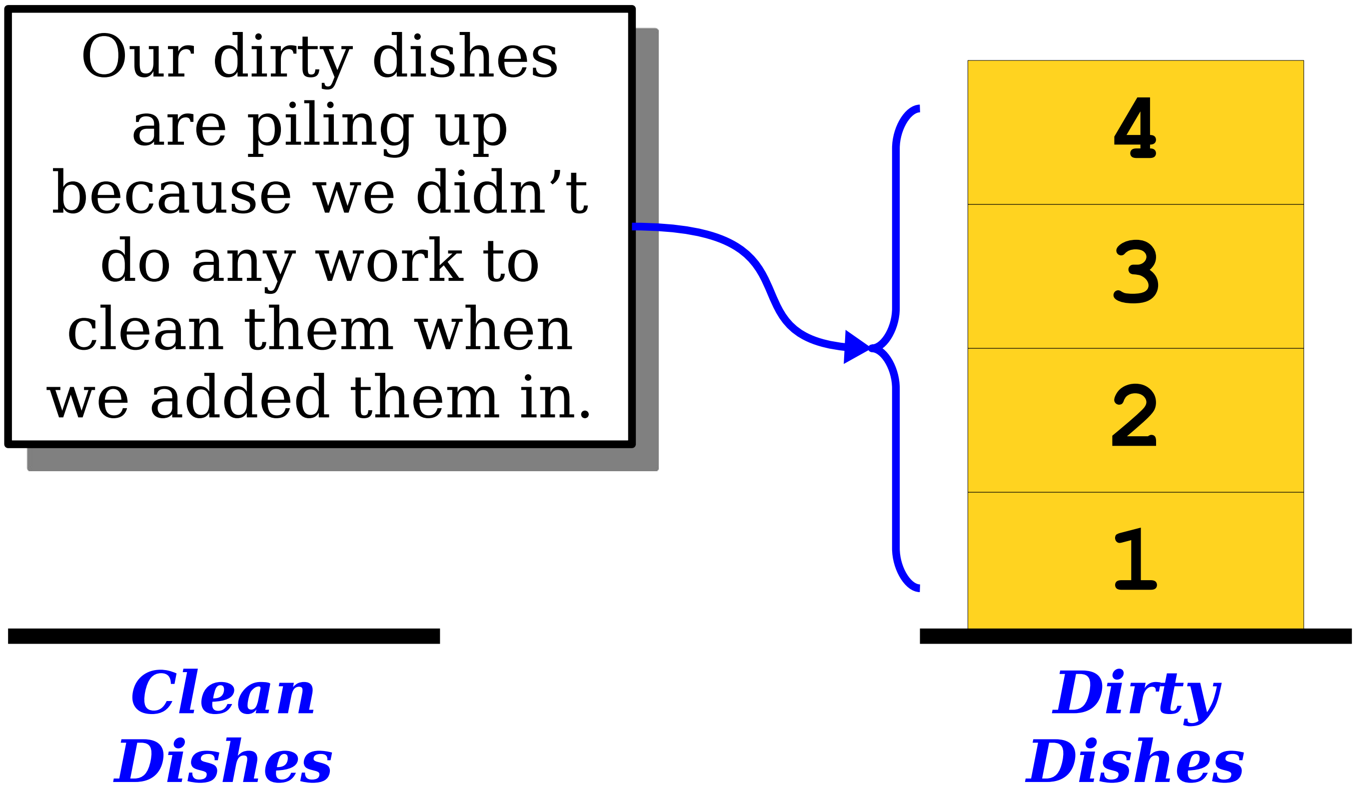

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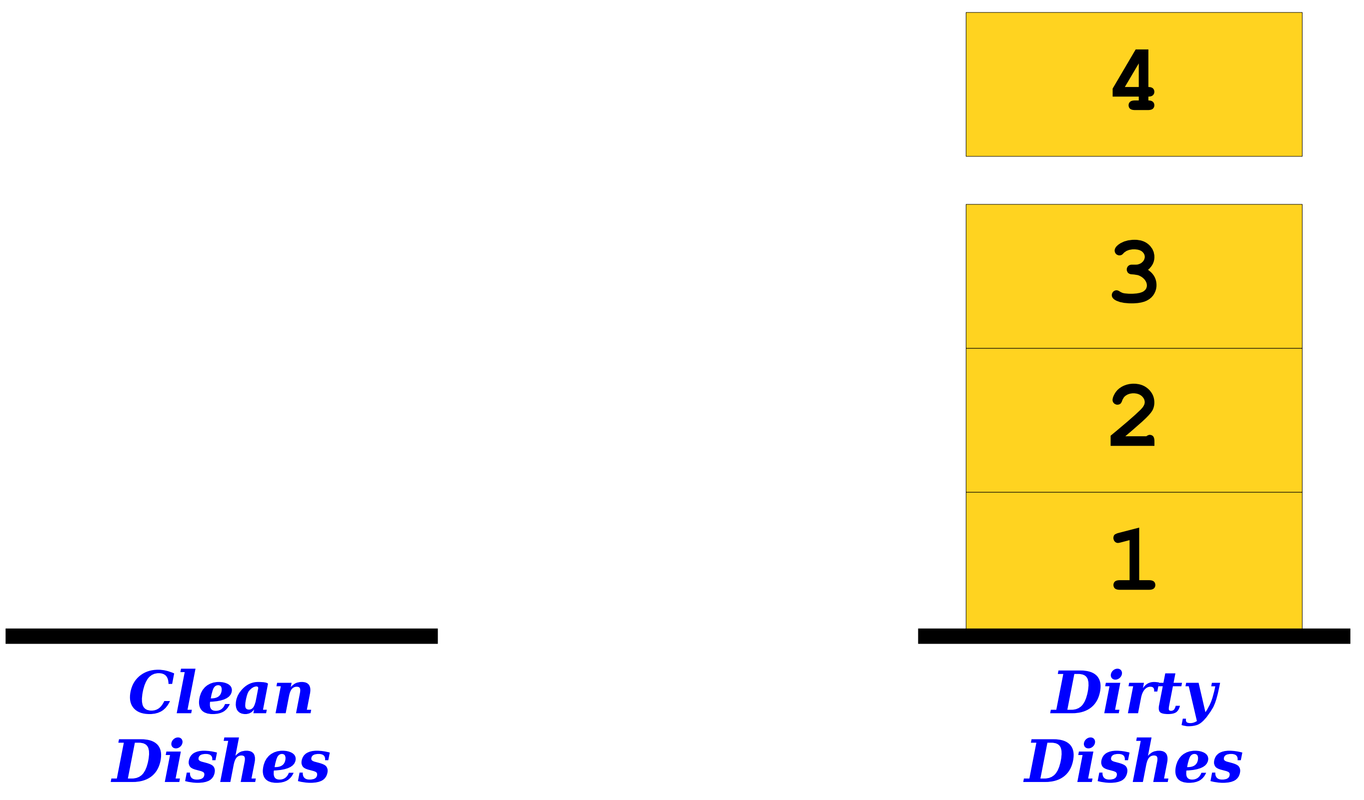

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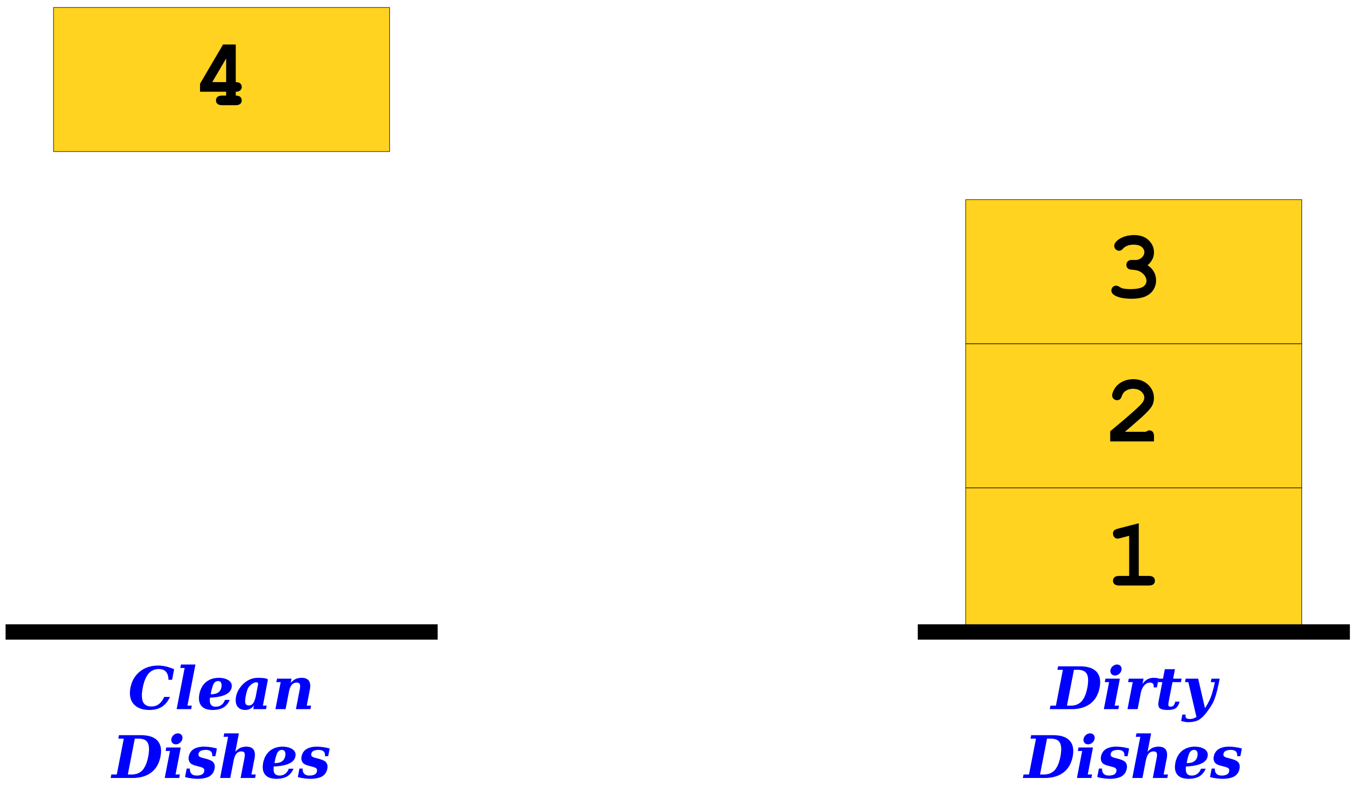

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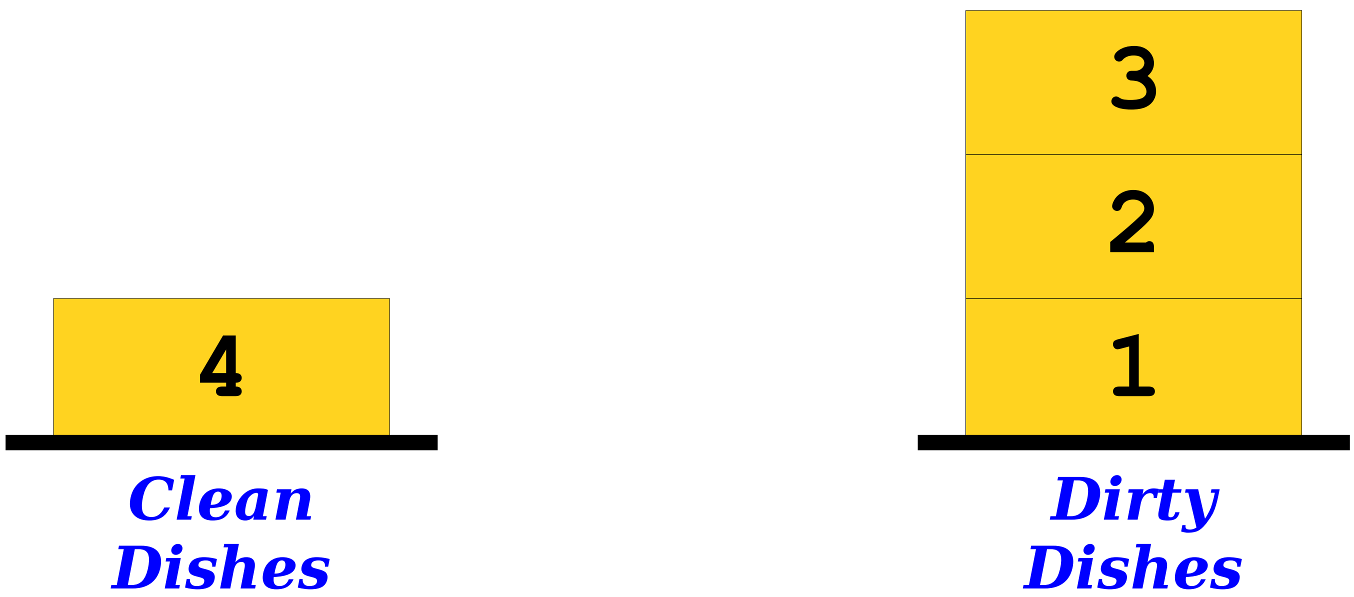

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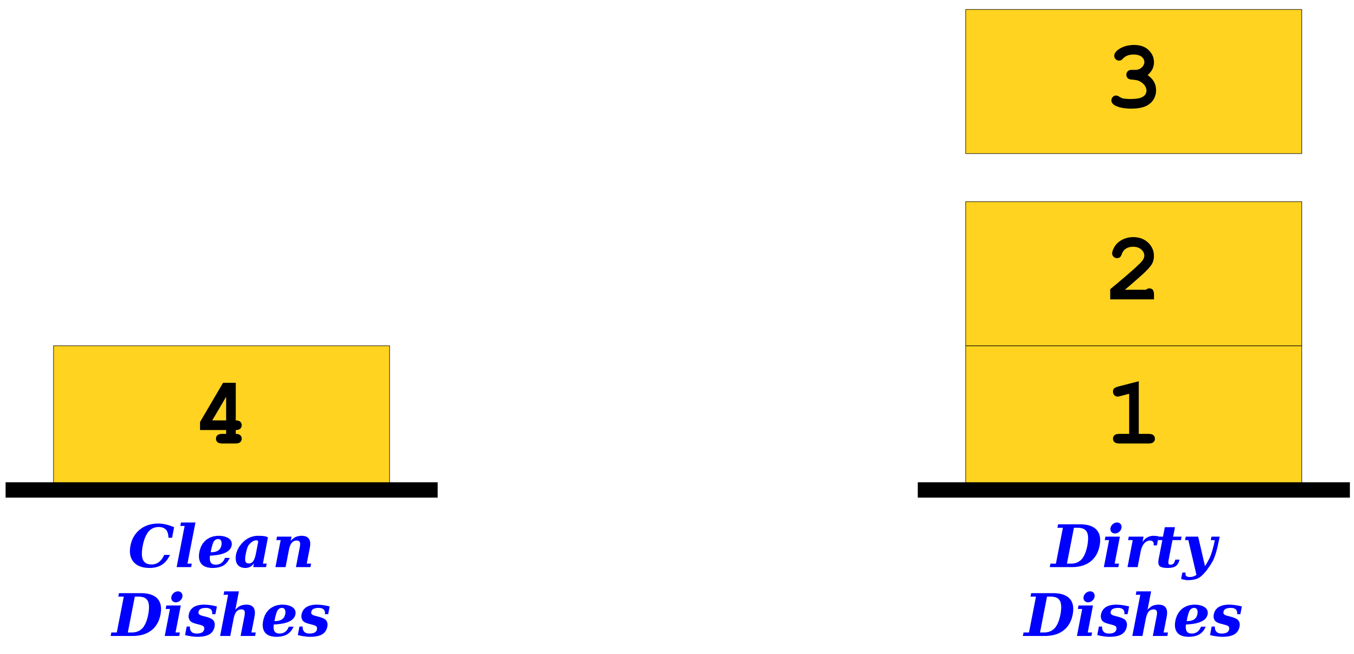

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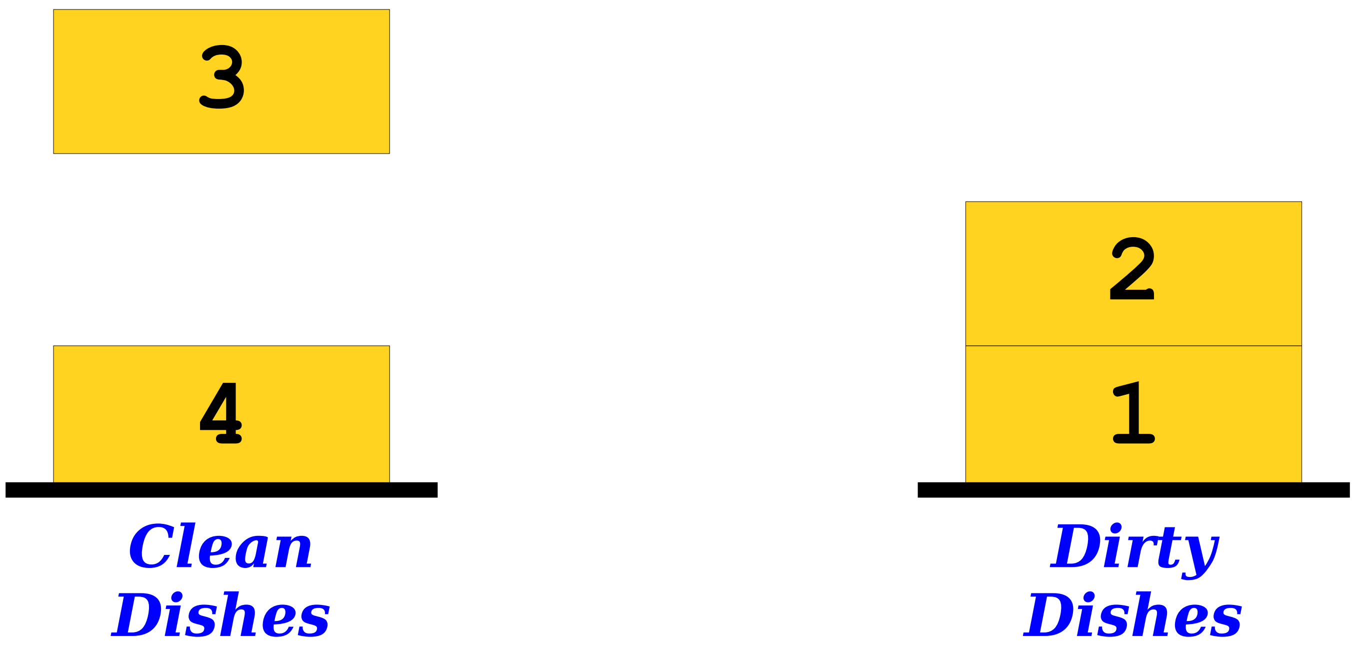

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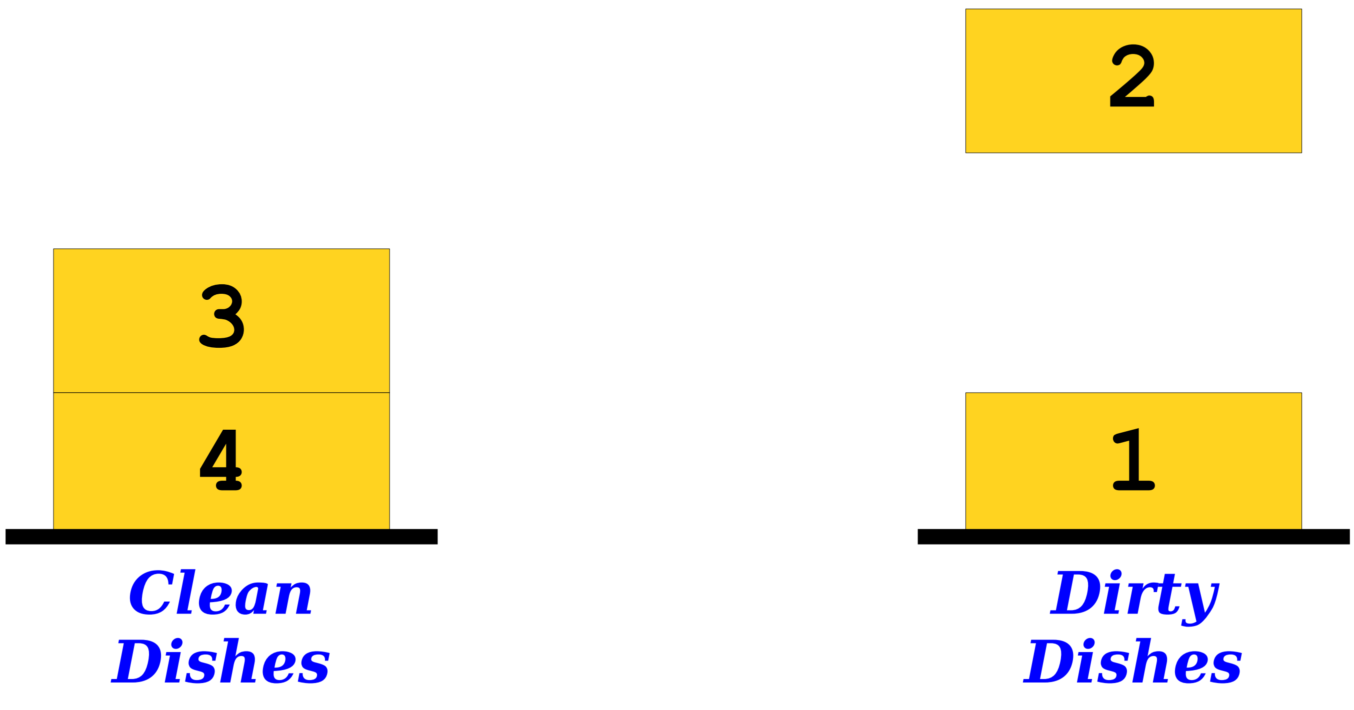

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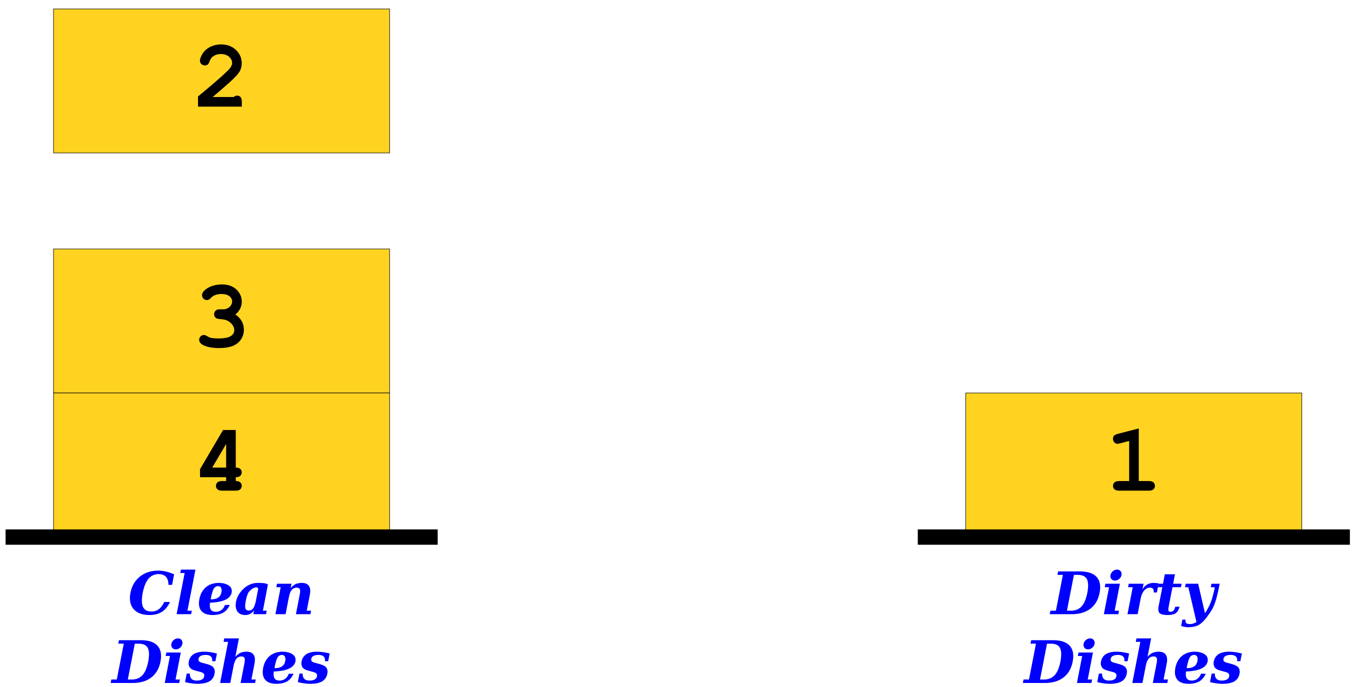

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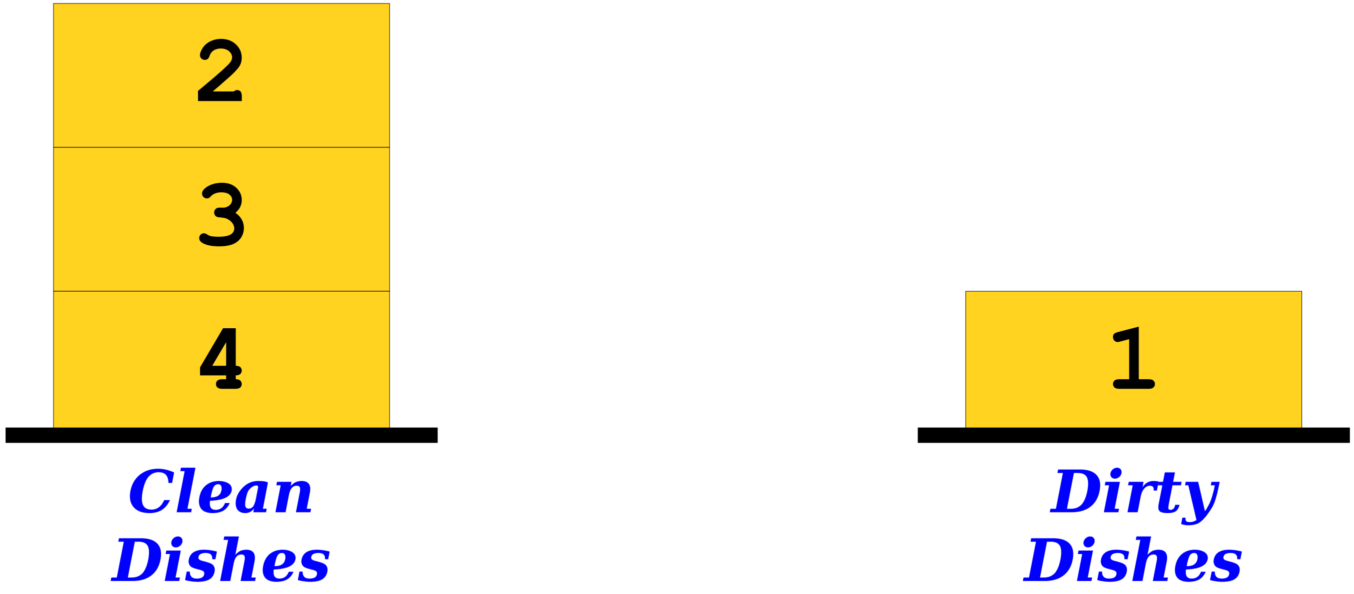

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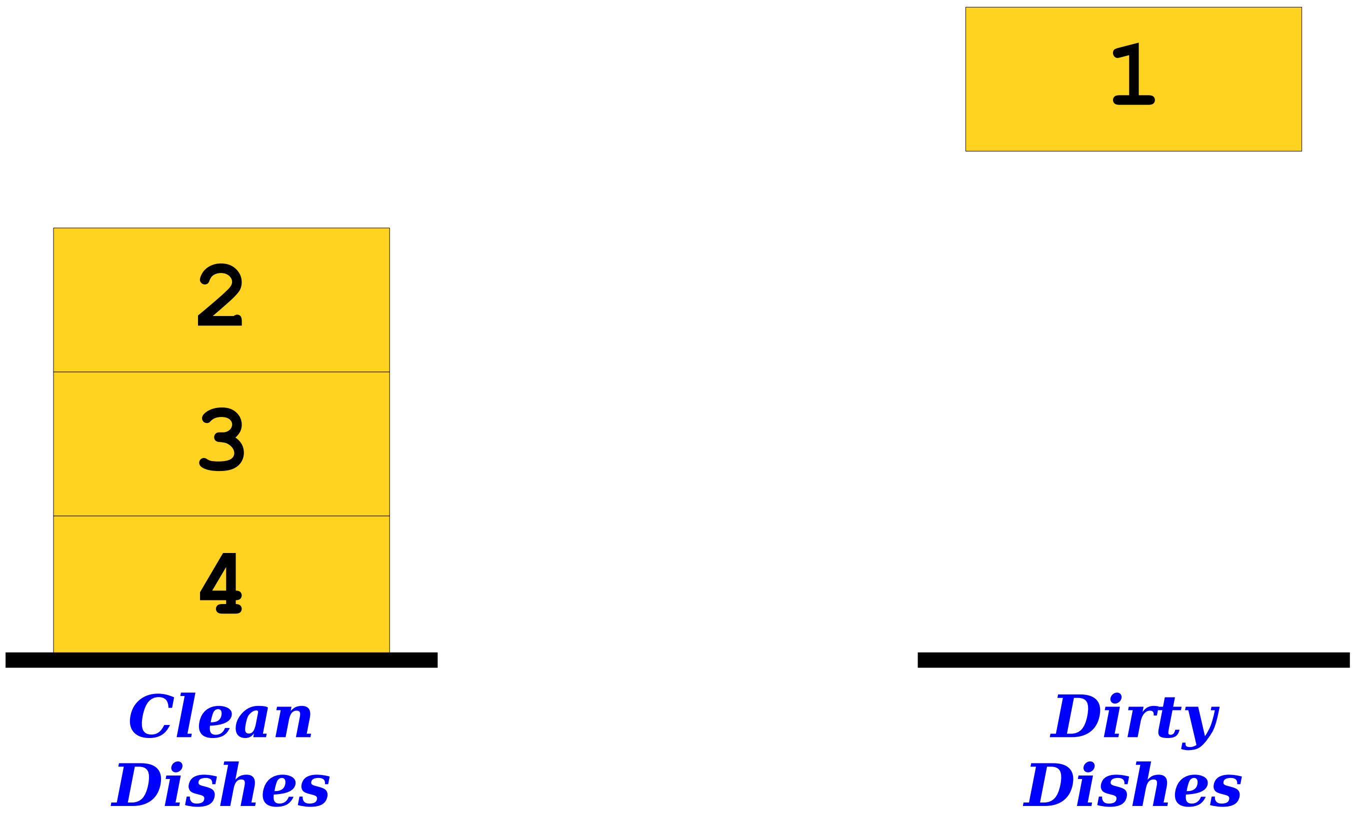

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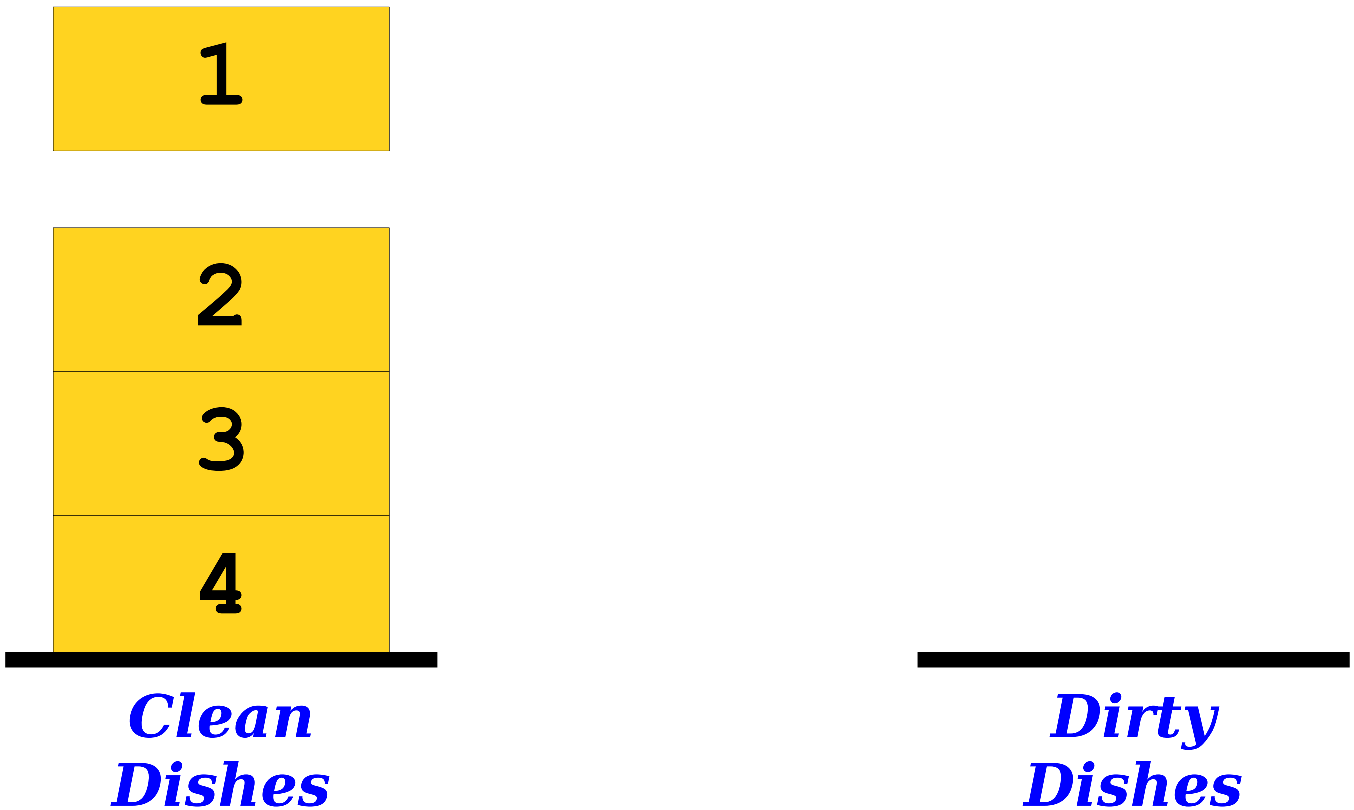

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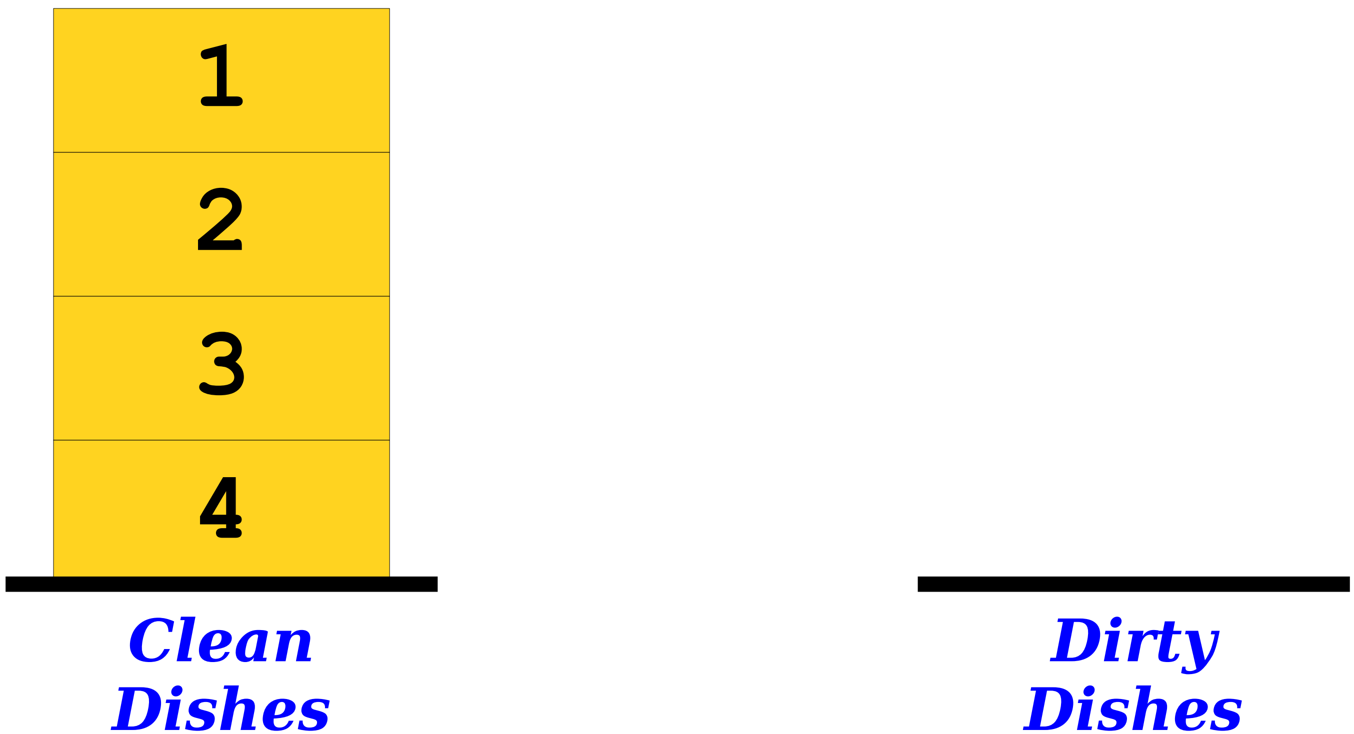

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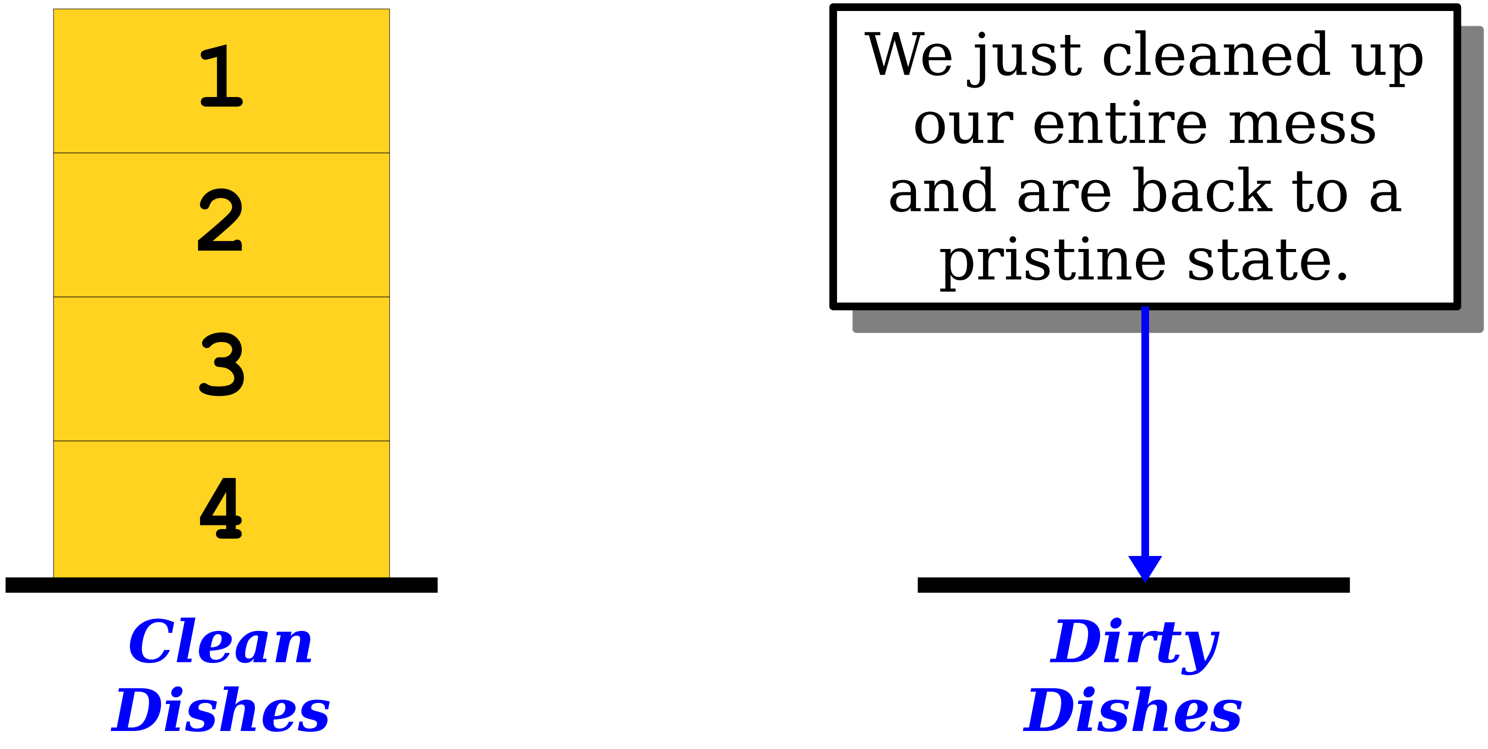

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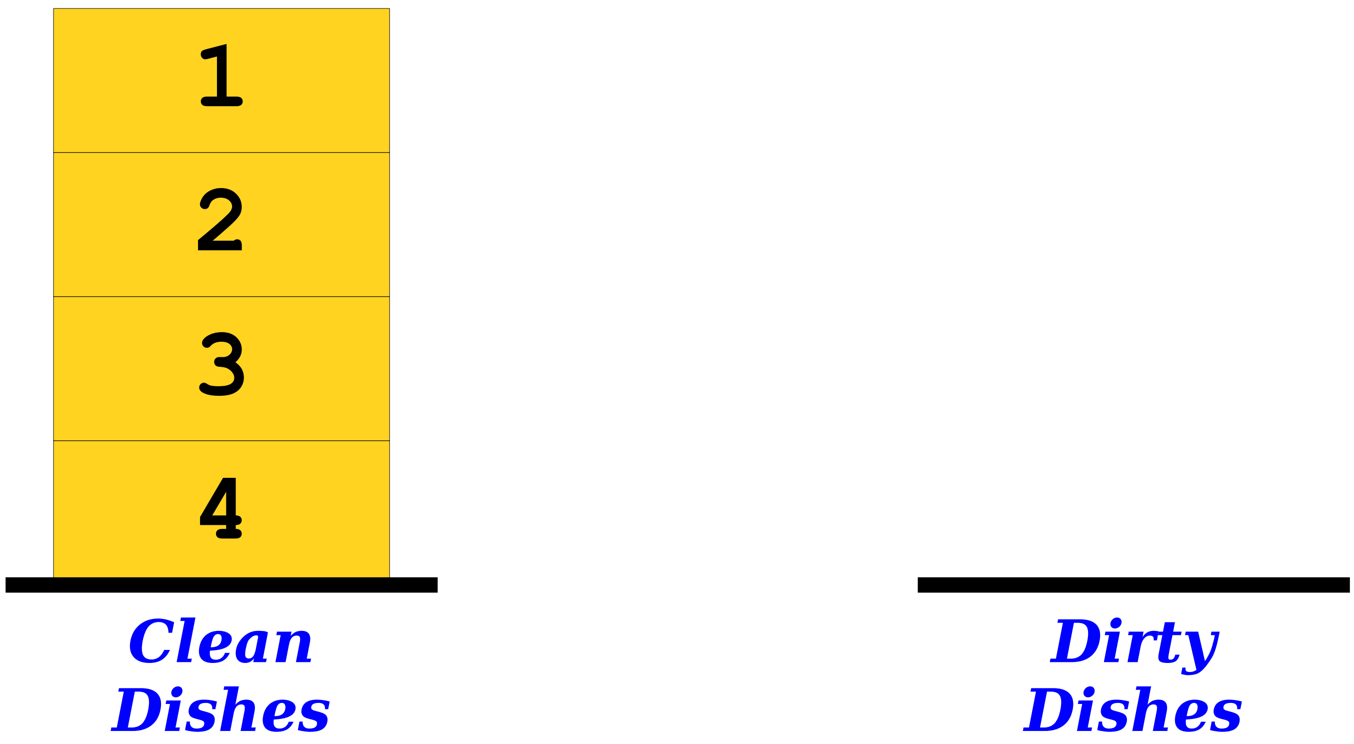

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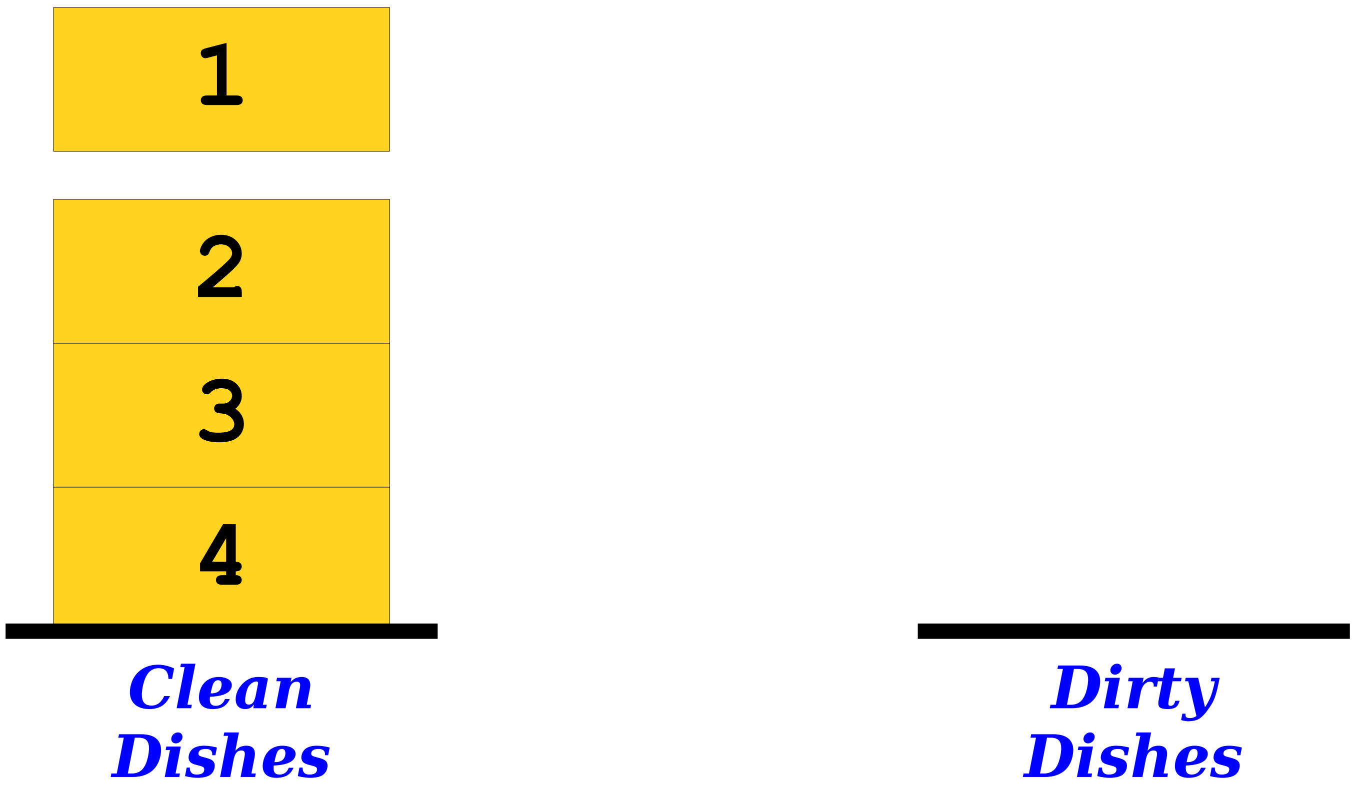

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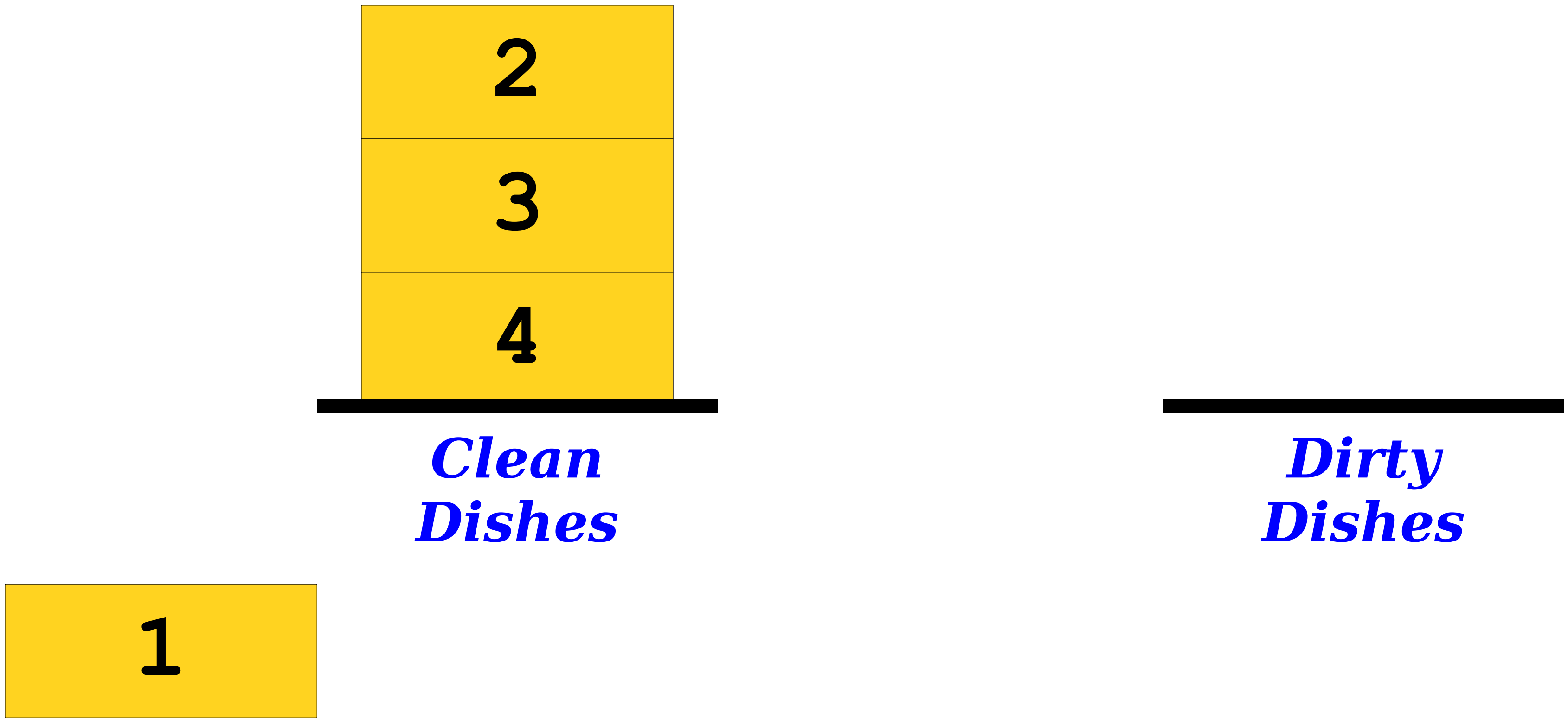

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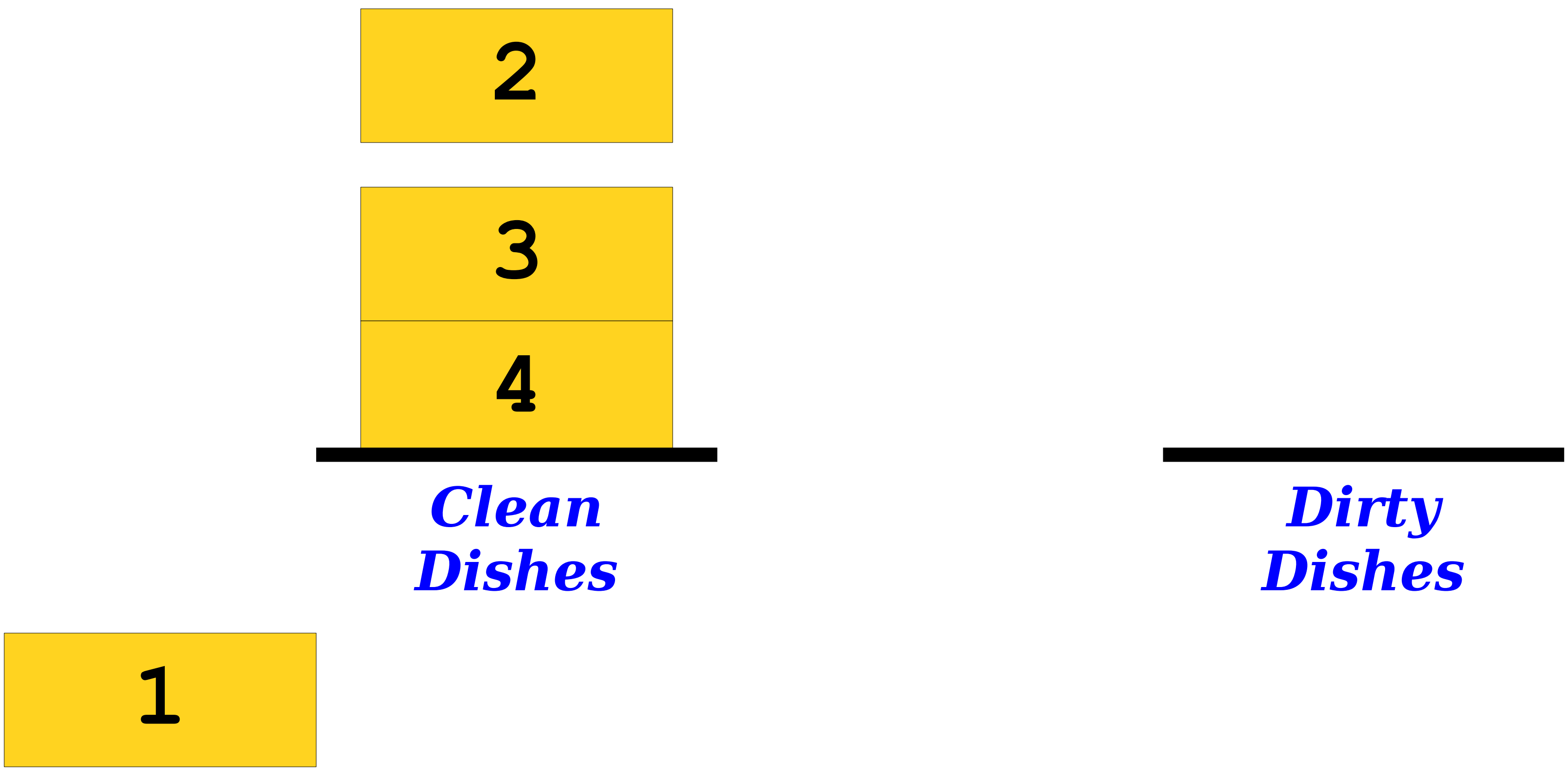

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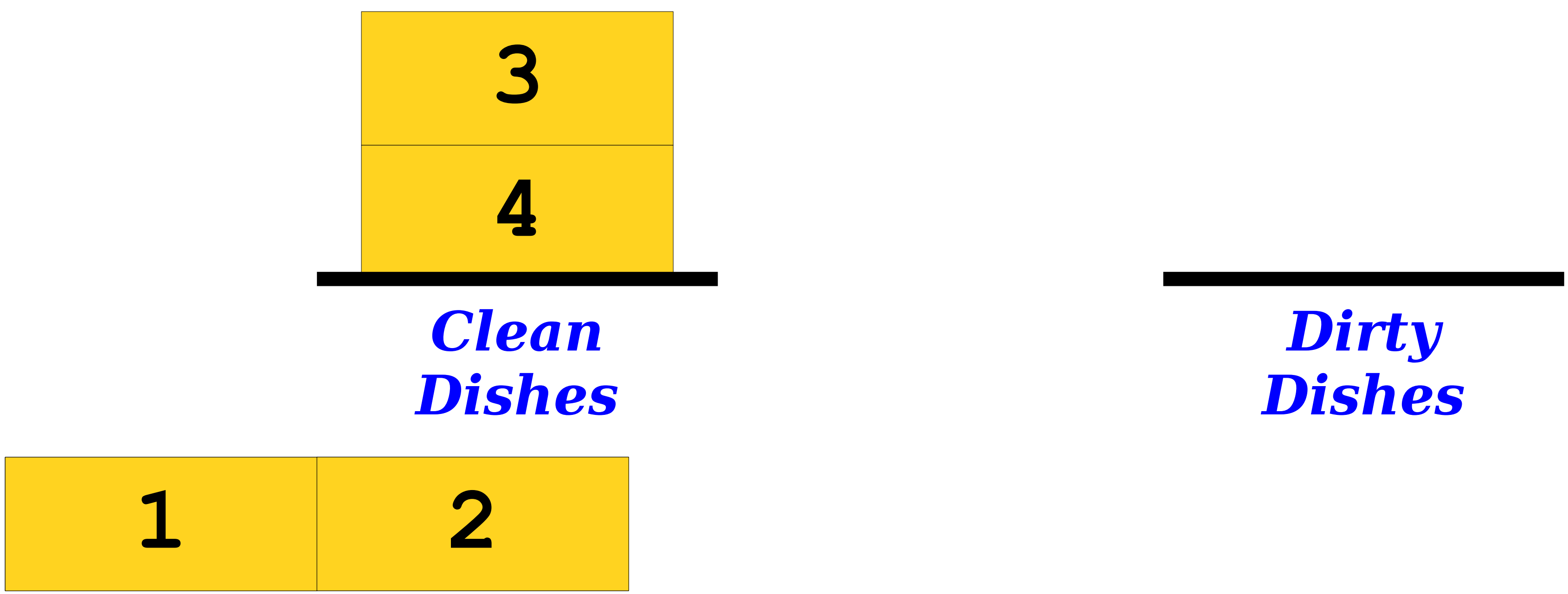

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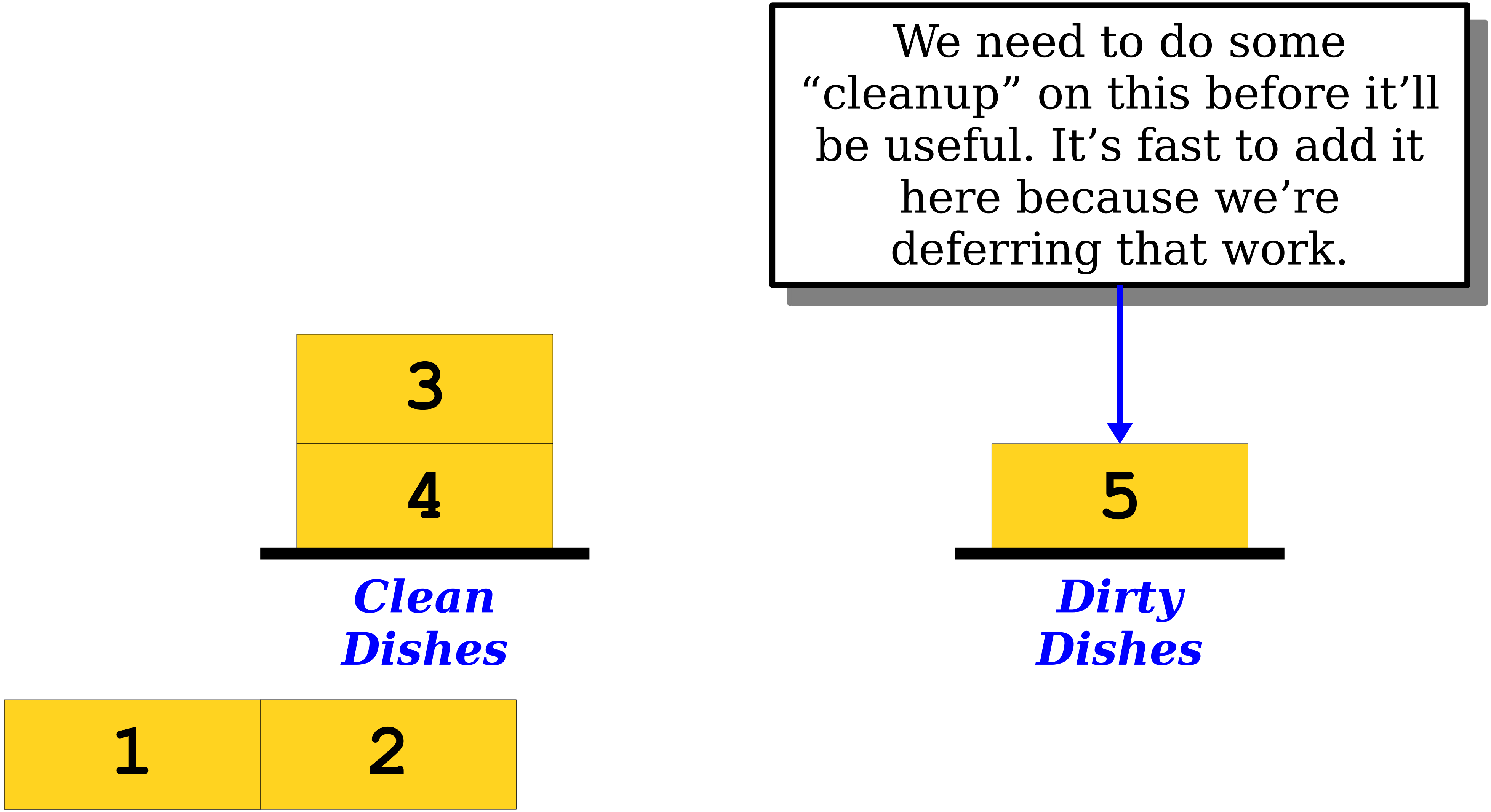

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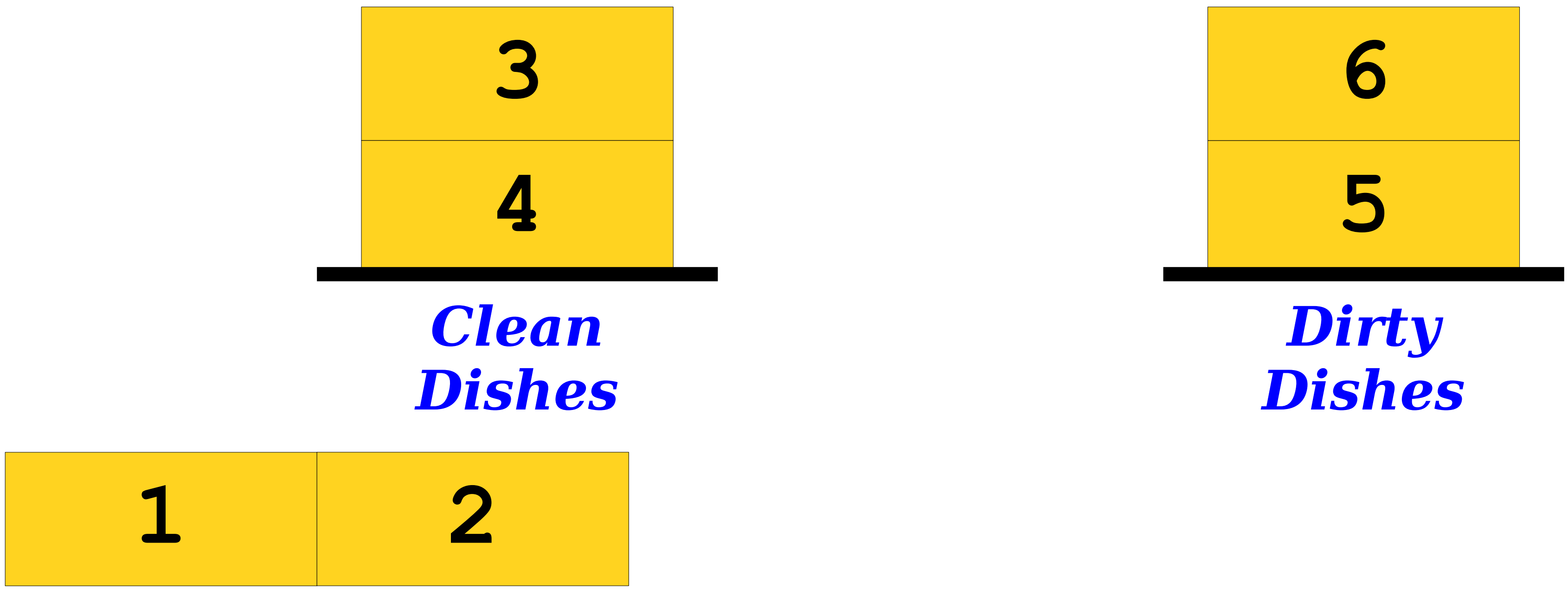

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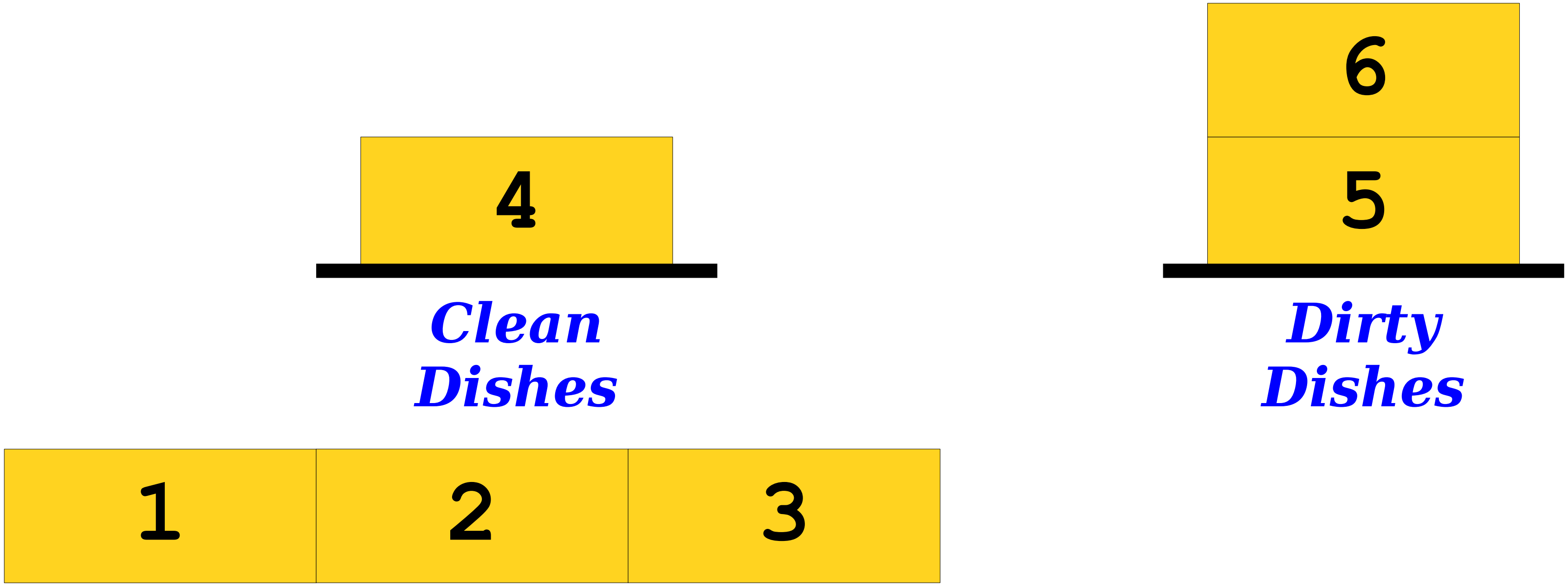

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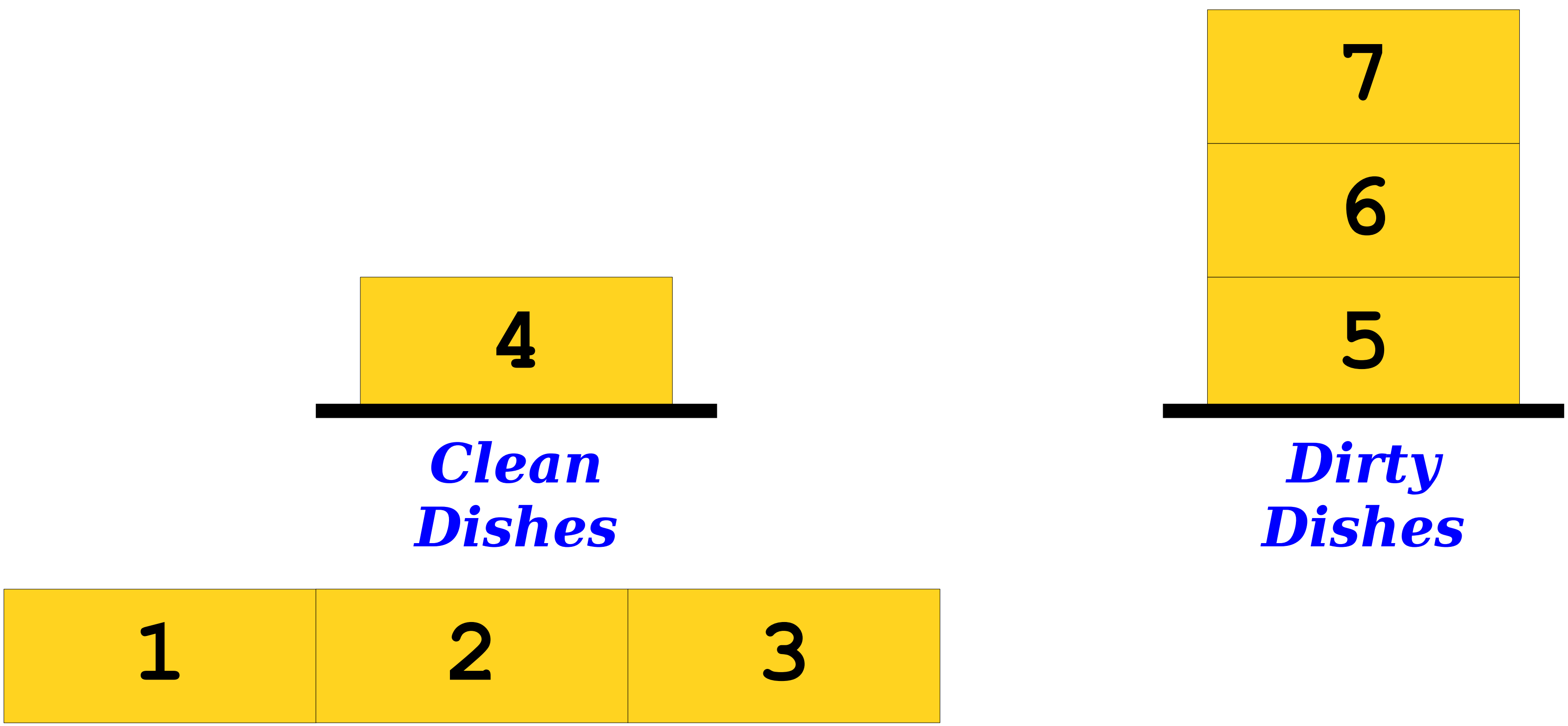

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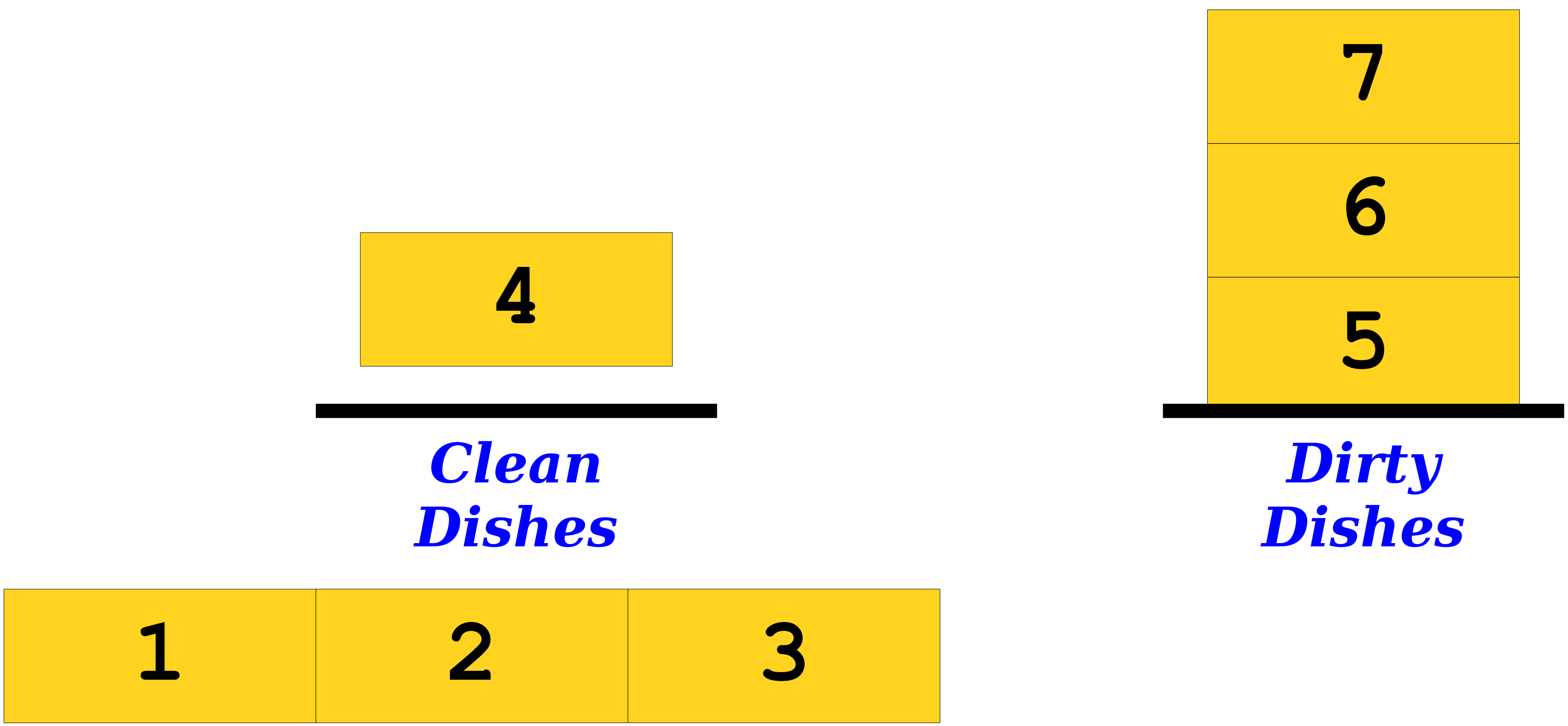

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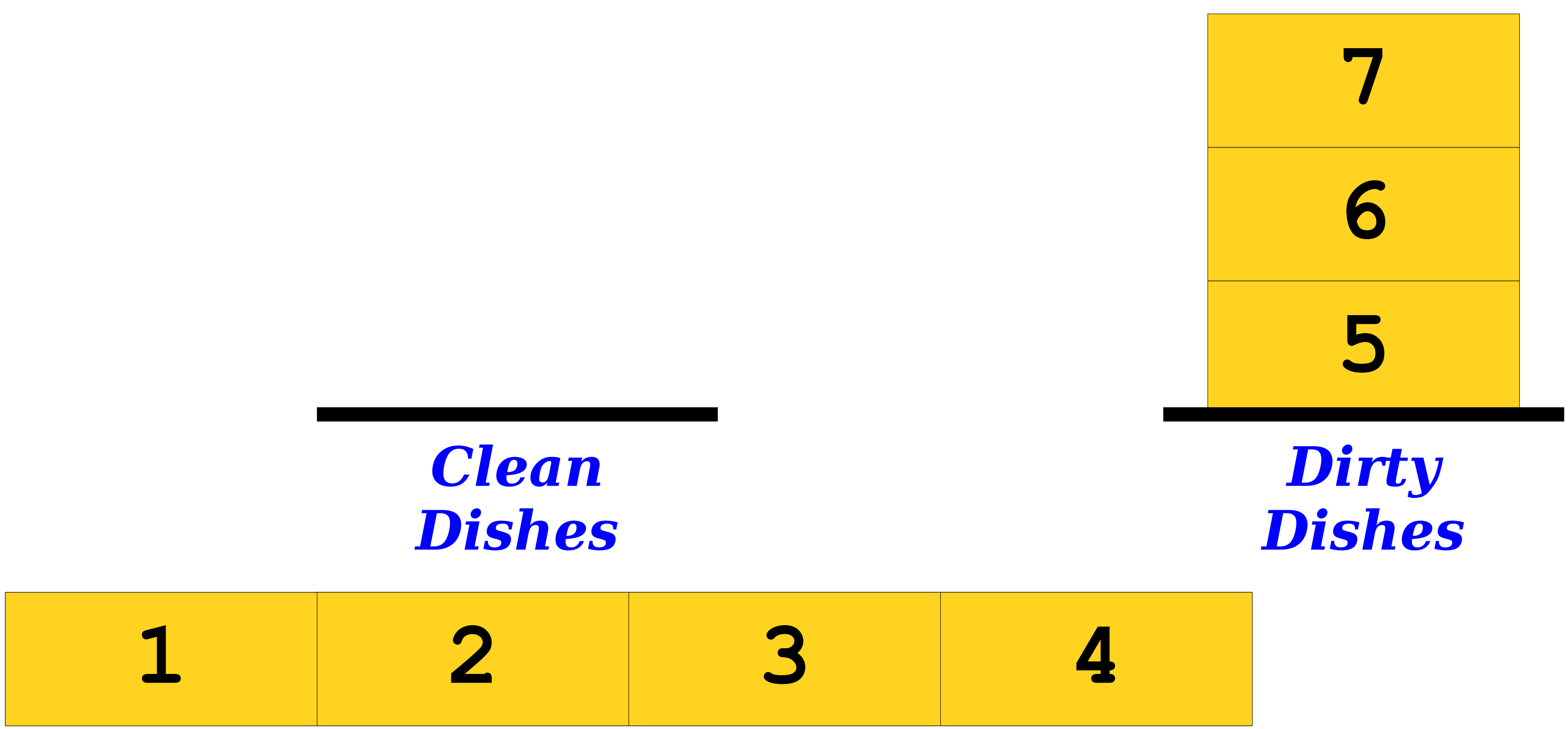

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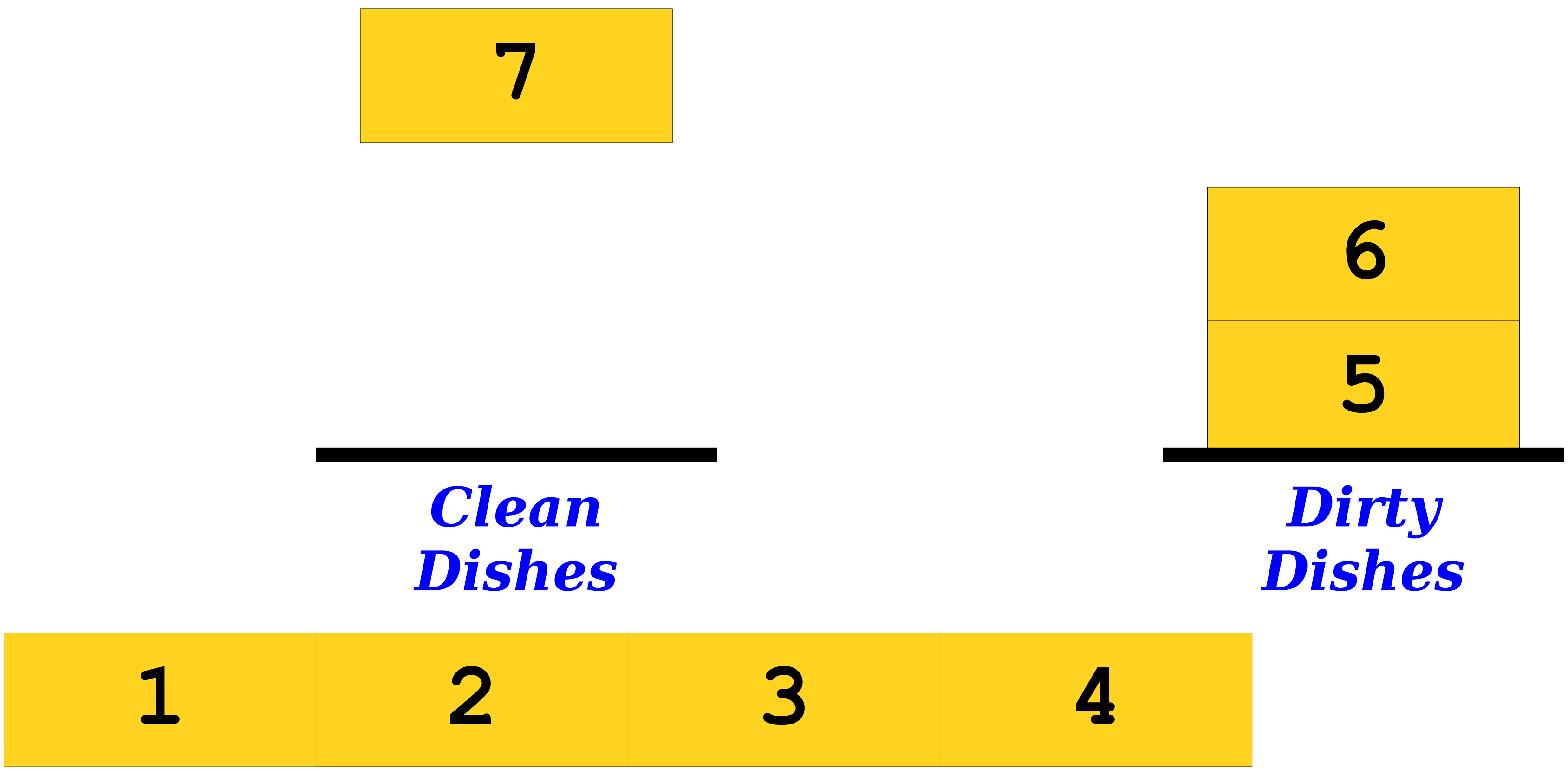

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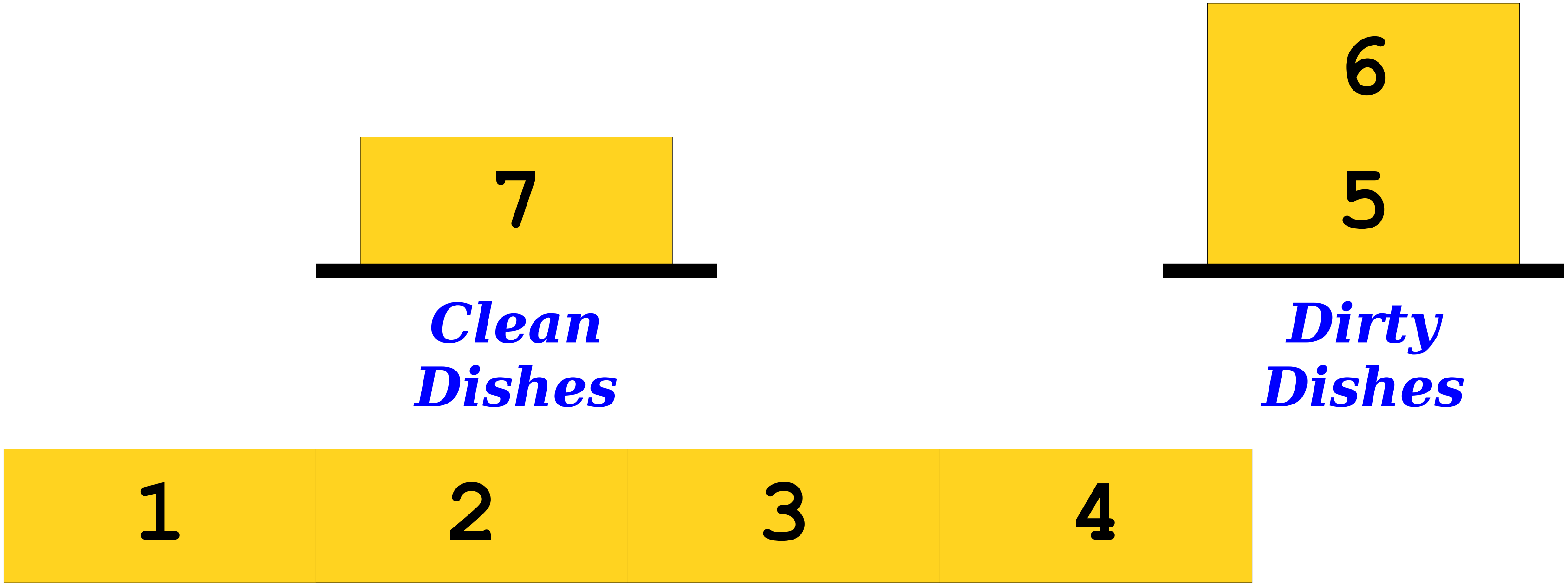

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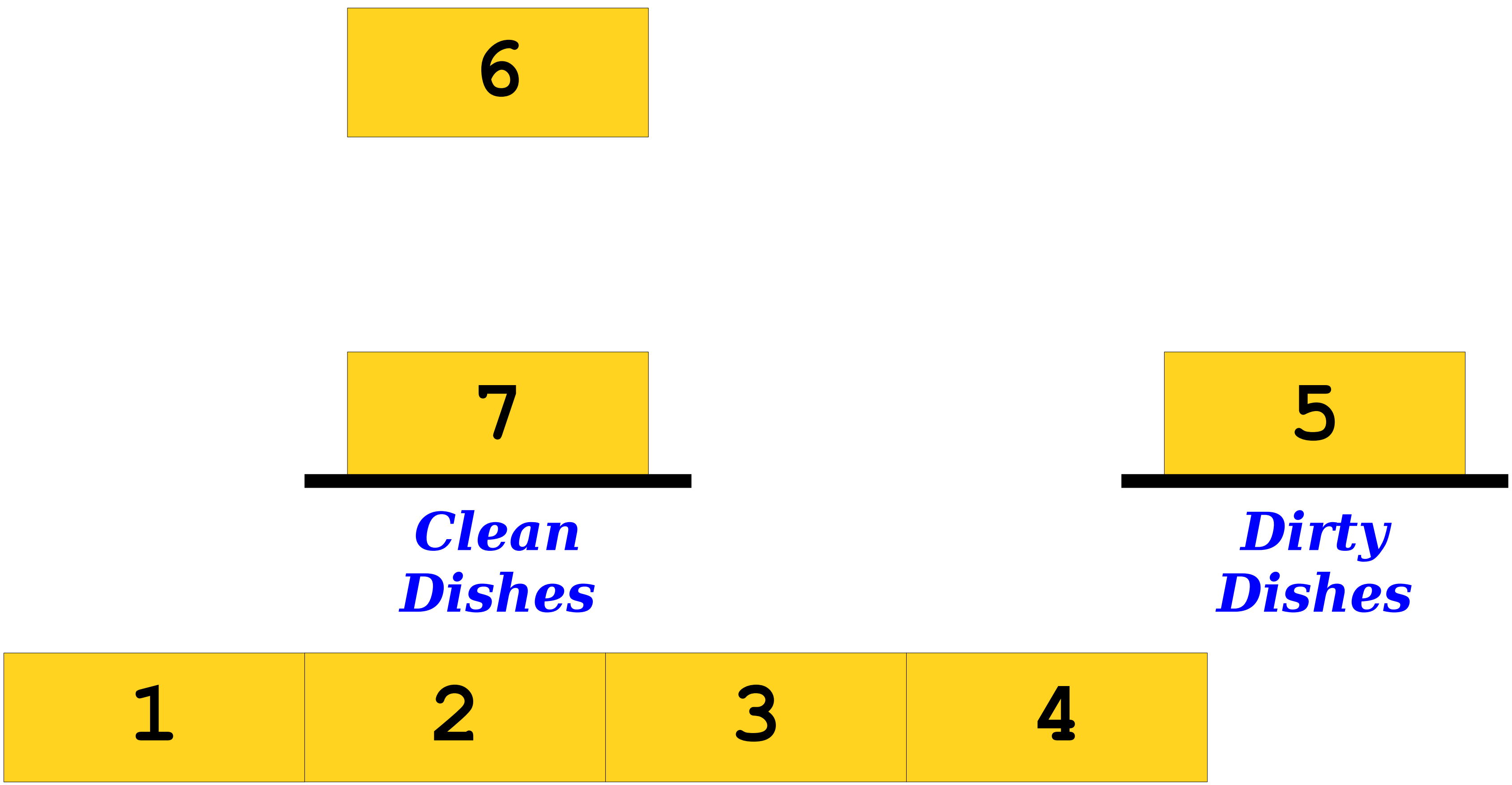

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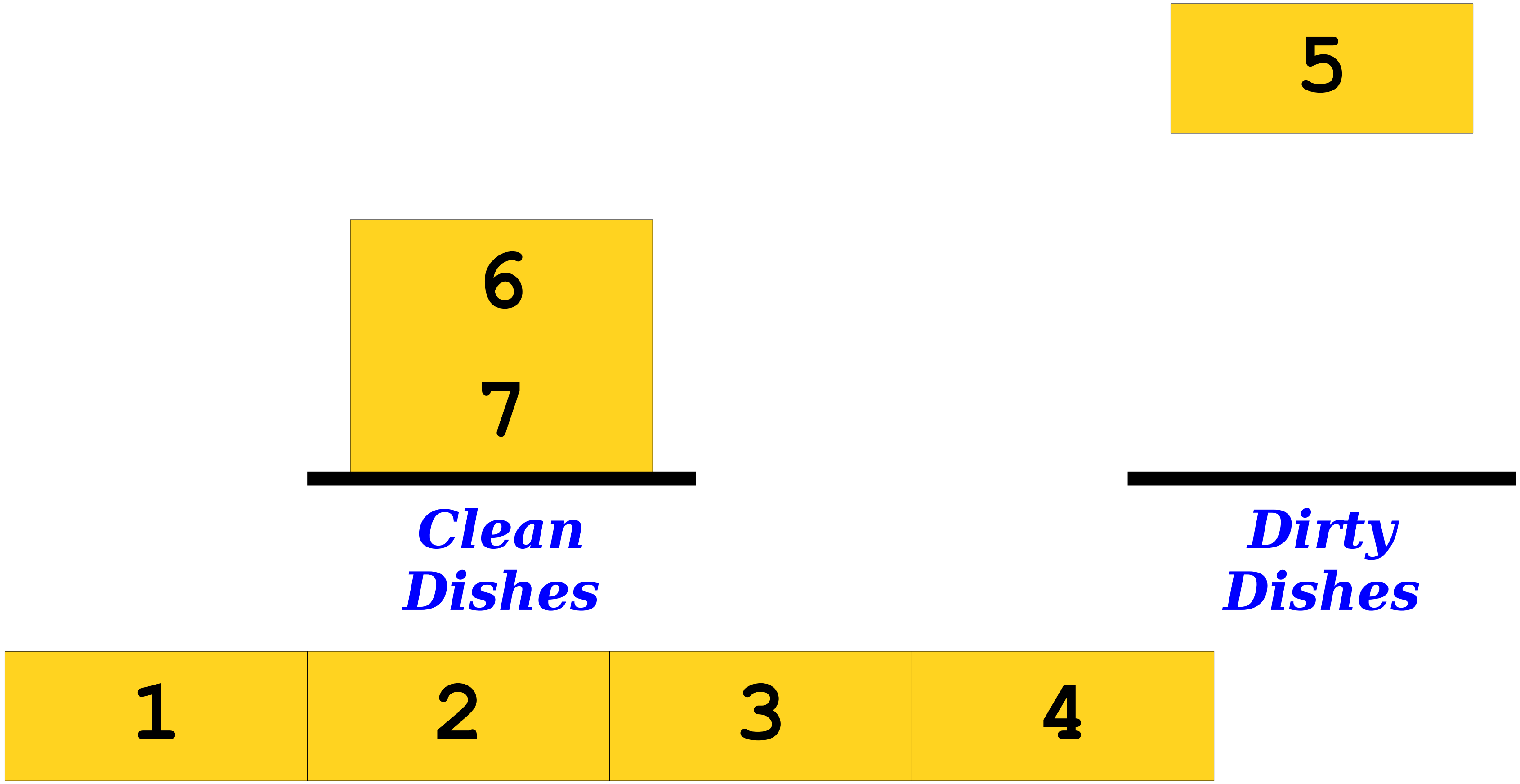

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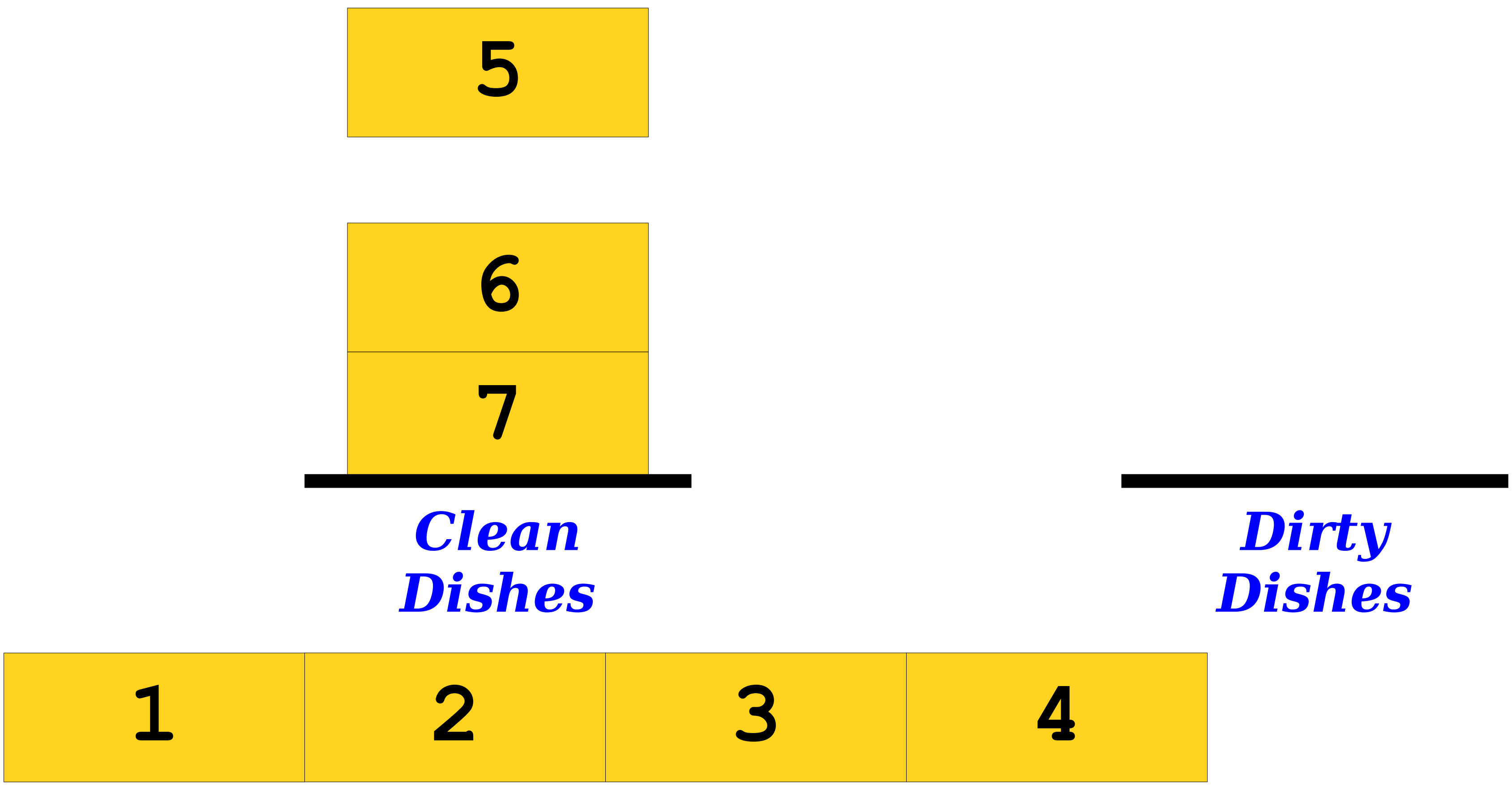

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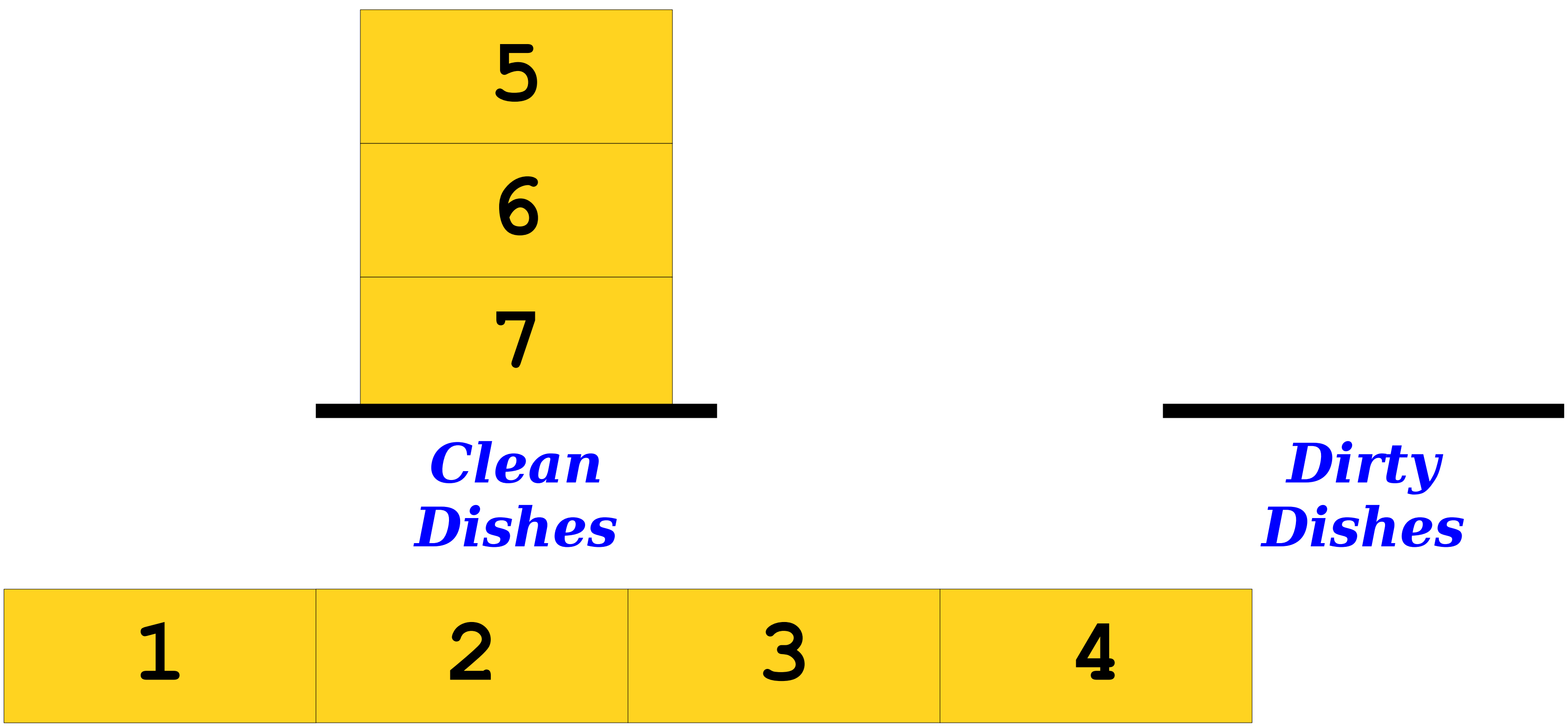

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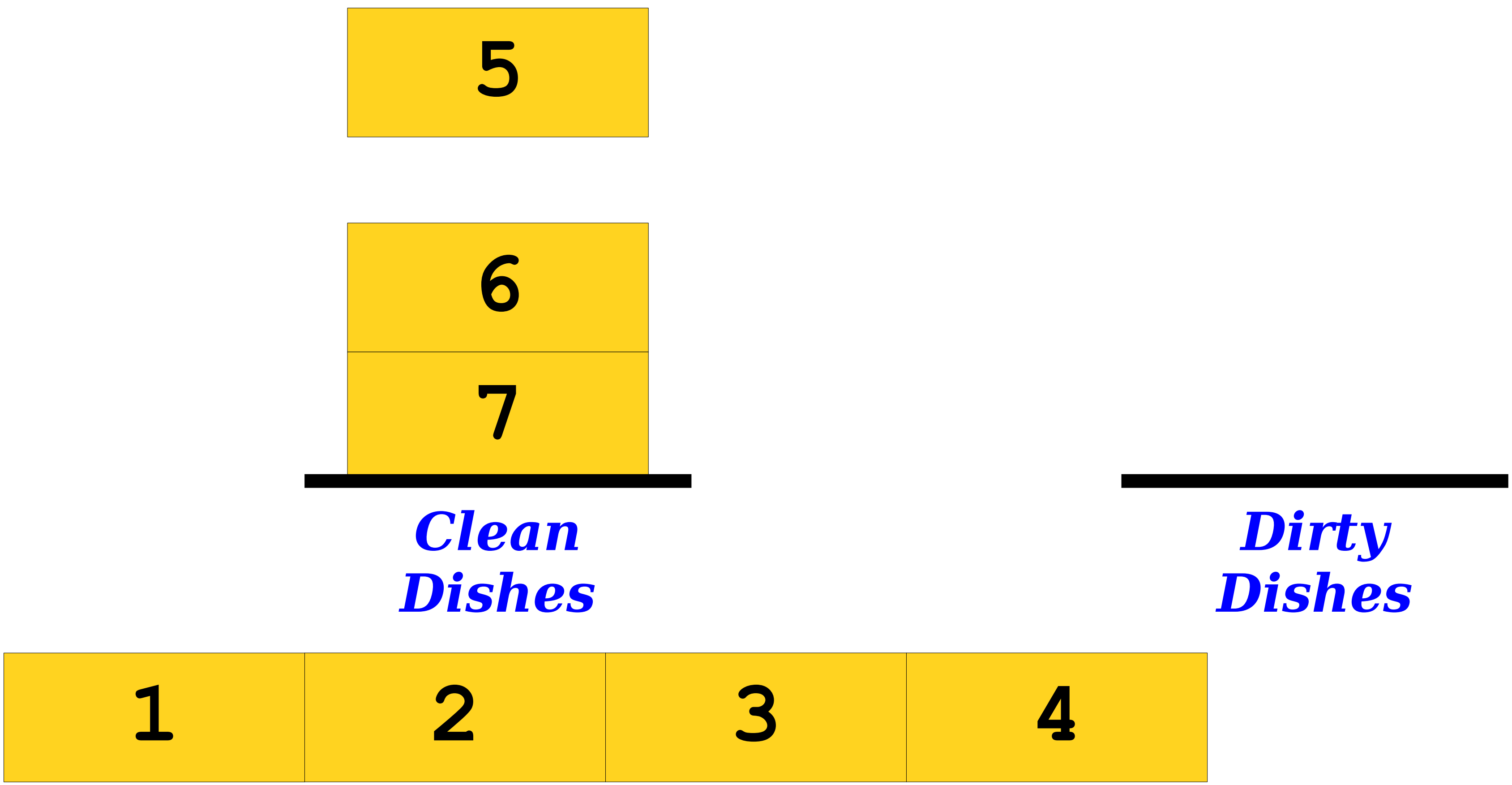

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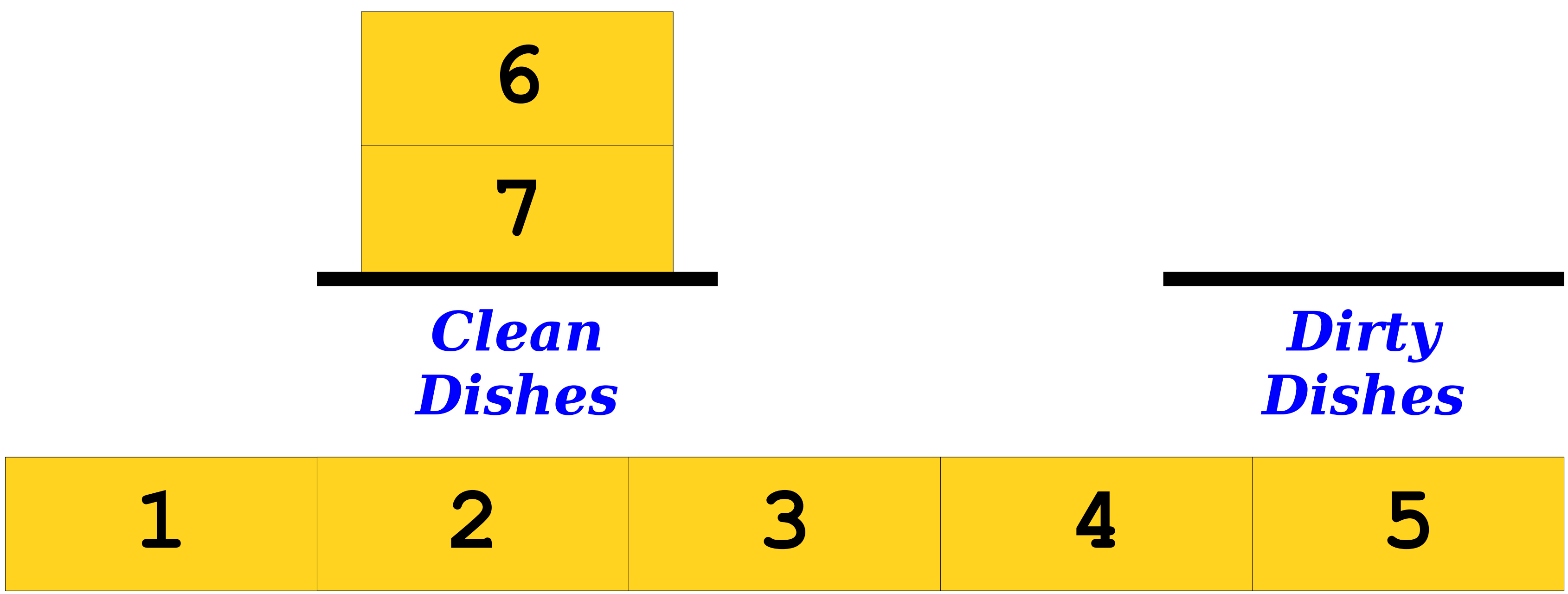
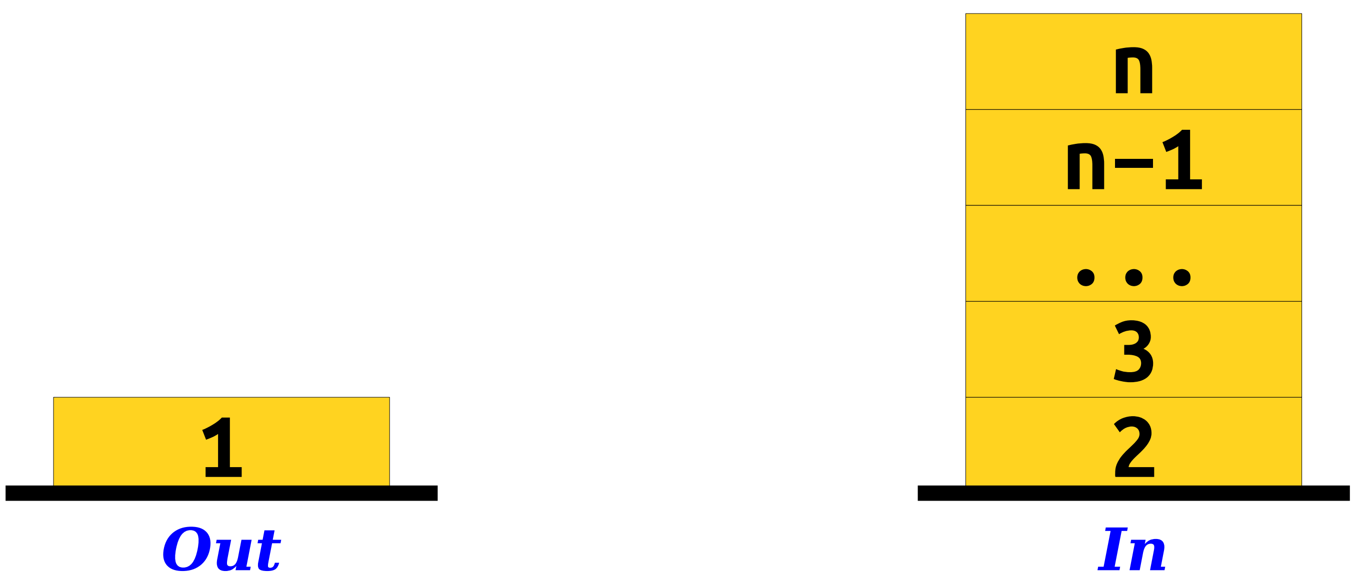


The Two-Stack Queue

- Each enqueue takes time O(1).
  - Just push an item onto the ***In*** stack.
- Dequeues can vary in their runtime.
  - Could be O(1) if the ***Out*** stack isn't empty.
  - Could be Θ(n) if the ***Out*** stack is empty.



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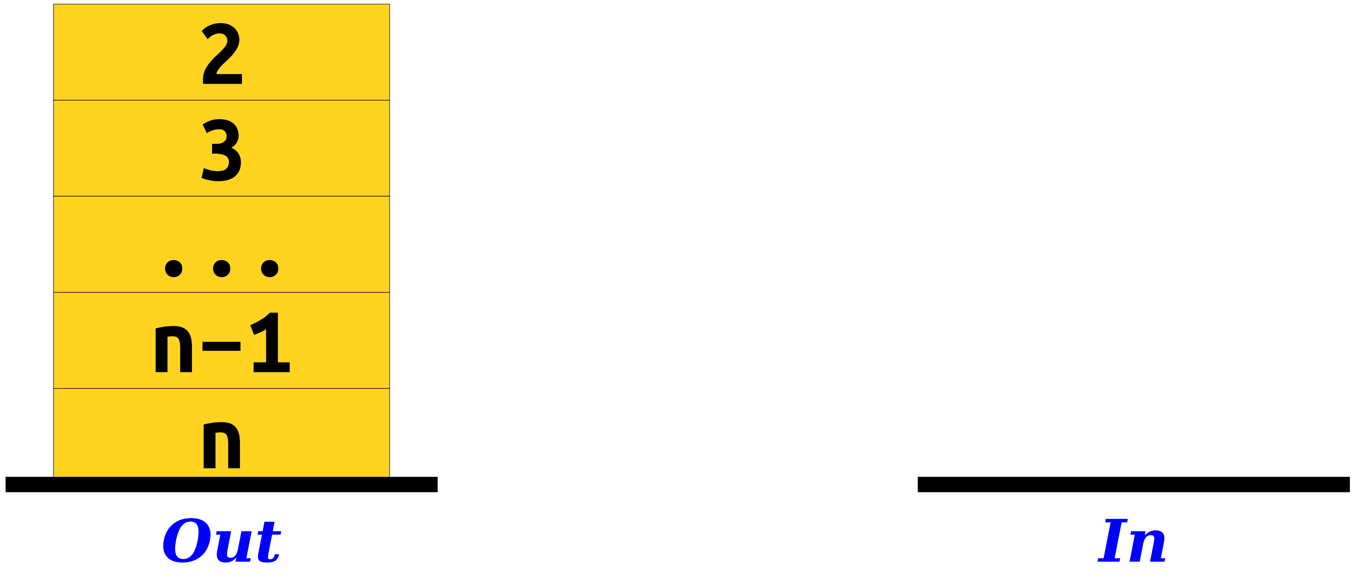

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The Two-Stack Queue

- ***Intuition:*** We only do expensive dequeues after a long run of cheap enqueues.
- Think "dishwasher:" we very slowly introduce a lot of dirty dishes that get cleaned up all at once.
- Provided we clean up all the dirty dishes at once, and provided that dirty dishes accumulate slowly, this is a fast strategy!

The Two-Stack Queue

- ***Key Fact:*** Any series of *n* operations on an (initially empty) two-stack queue will take time O(*n*).
- ***Why?***

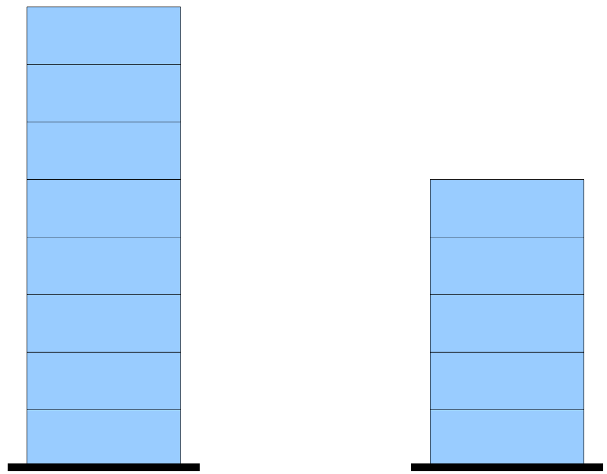
Answer at
https://pollev.com/cs166spr23

The Two-Stack Queue

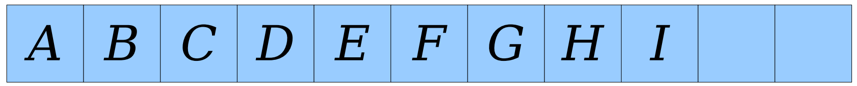
- ***Key Fact:*** Any series of *n* operations on an (initially empty) two-stack queue will take time O(*n*).
- ***Why?***
- Each item is pushed into at most two stacks and popped from at most two stacks.
- Adding up the work done per element across all *n* operations, we can do at most O(*n*) work.

The Two-Stack Queue

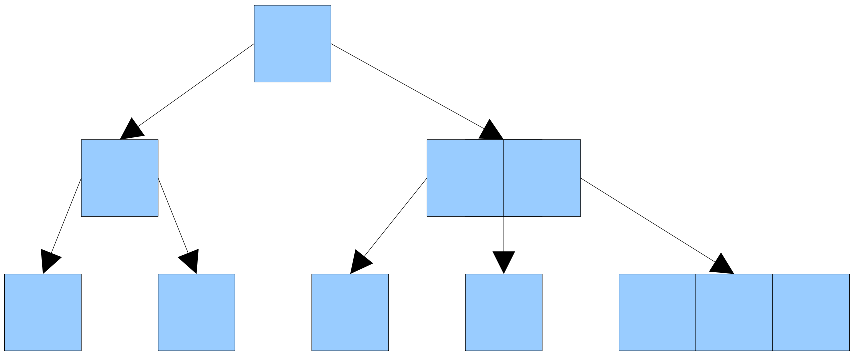
- It's correct but misleading to say the cost of a dequeue is O(n).
  - This is comparatively rare.
- It's wrong, but useful, to pretend the cost of a dequeue is O(1).
  - Some operations take more time than this.
  - However, if we pretend each operation takes time O(1), then the sum of all the costs never underestimates the total.
- ***Question:*** What's an honest, accurate way to describe the runtime of the two-stack queue?



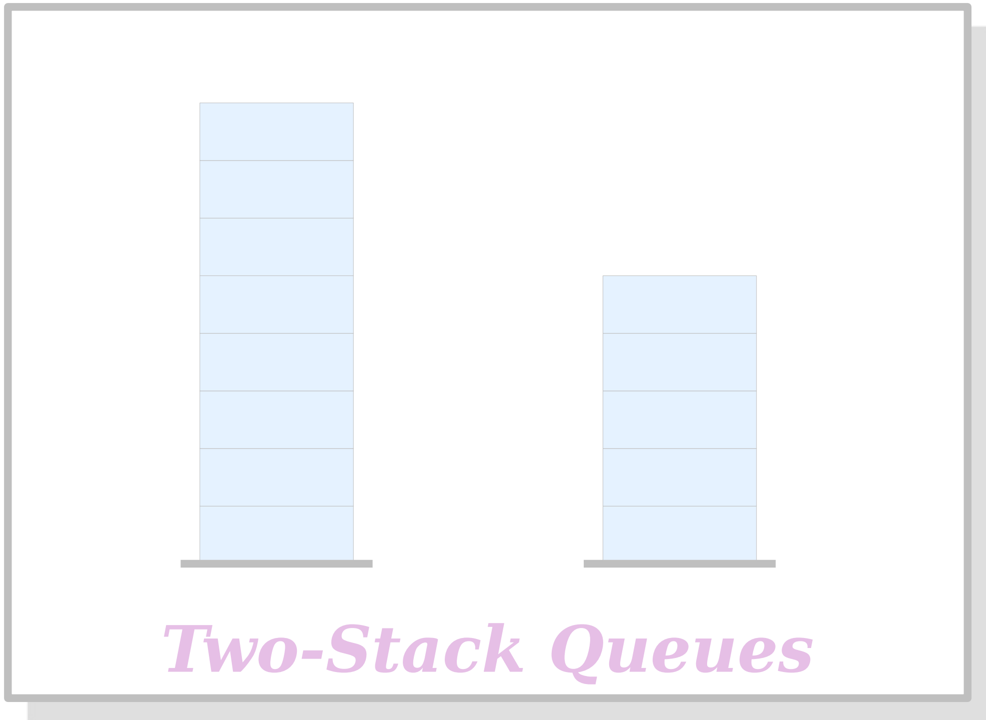
Two-Stack Queues



Dynamic Arrays



Building B-Trees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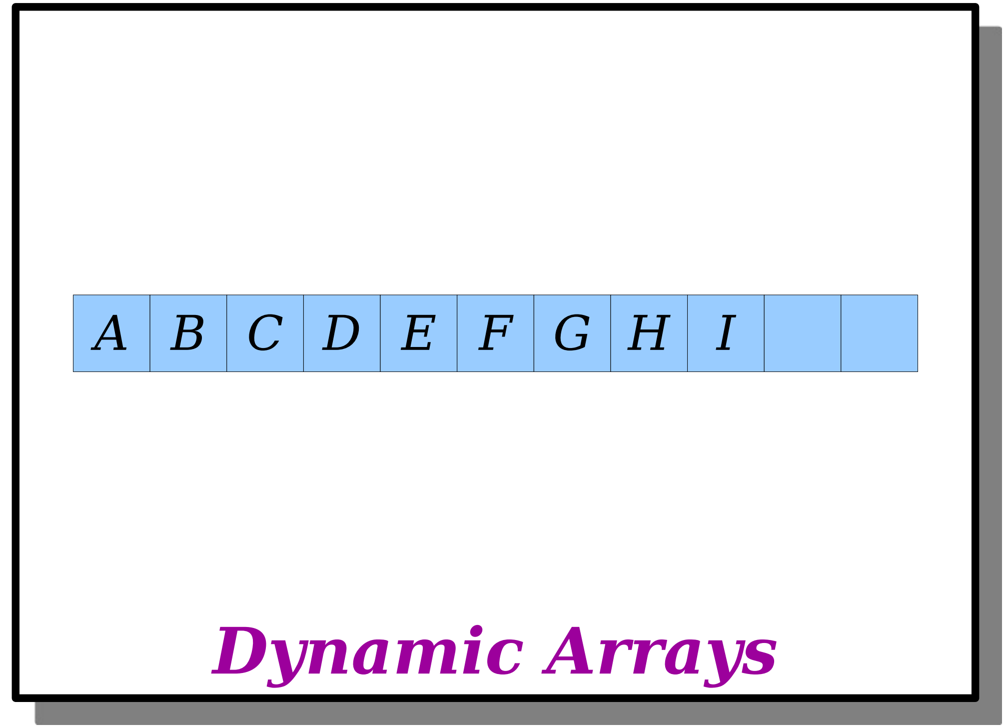
A
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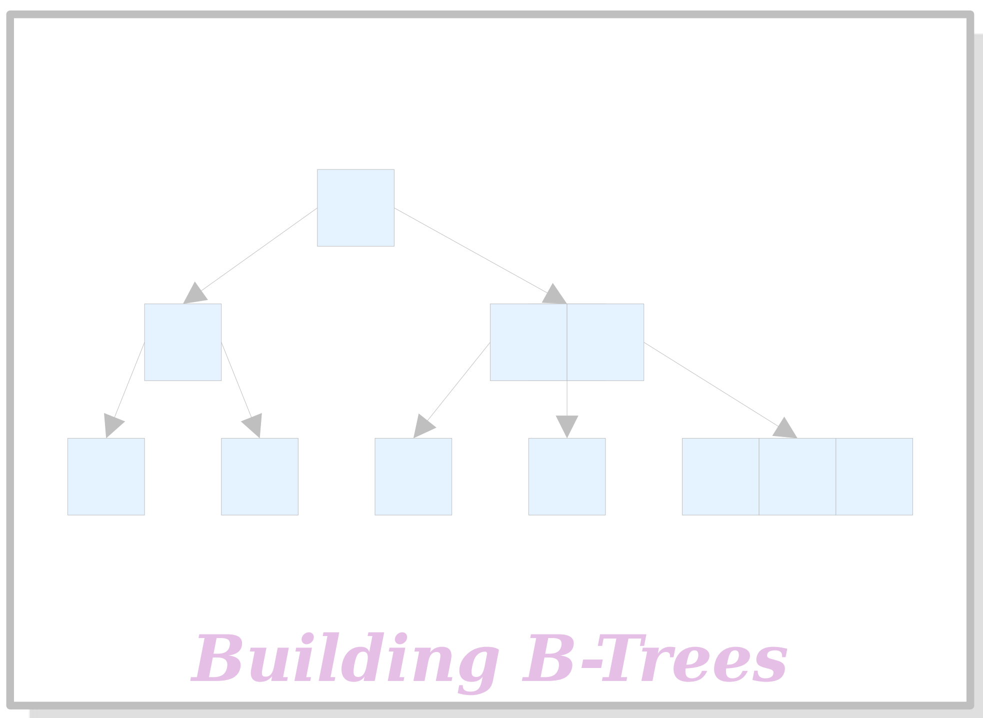
Dynamic Arrays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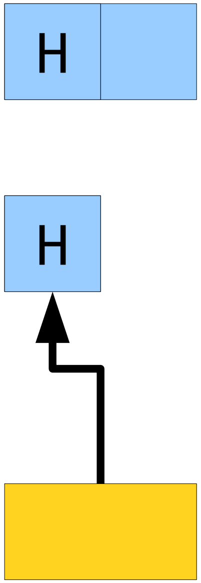
Dynamic Arrays

- A ***dynamic array*** is the most common way to implement a list of values.
- Maintain an array slightly bigger than the one you need. When you run out of space, double the array size and copy the elements over.



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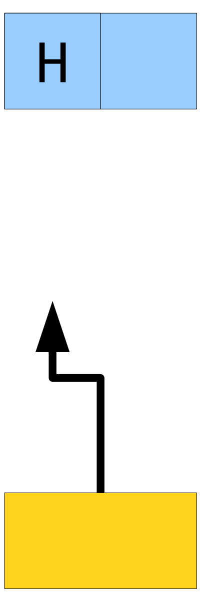

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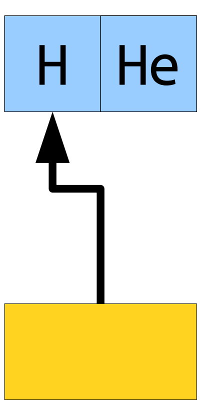

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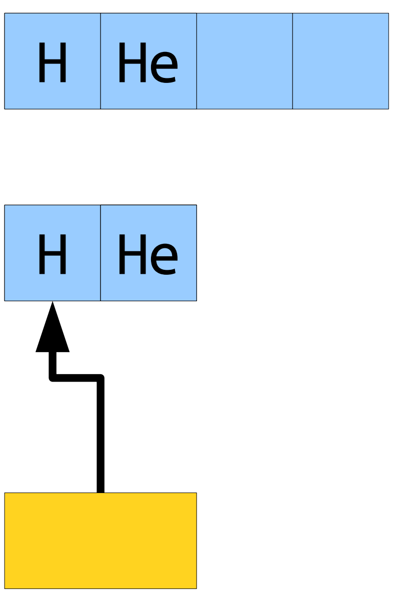

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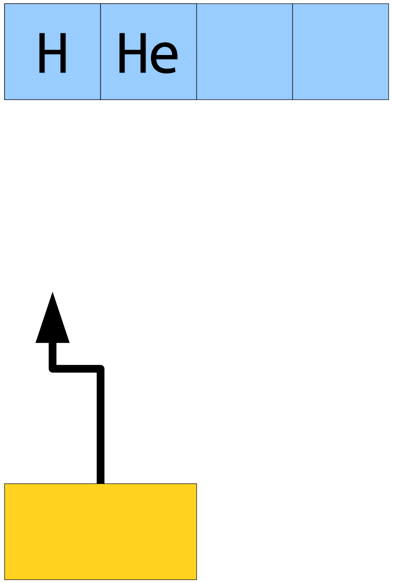

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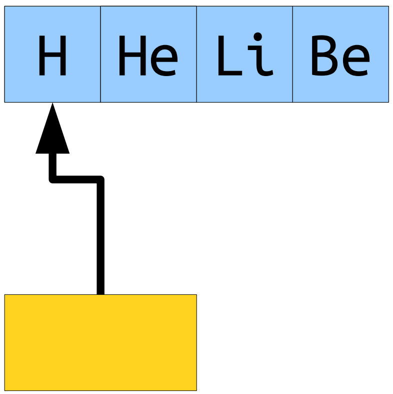

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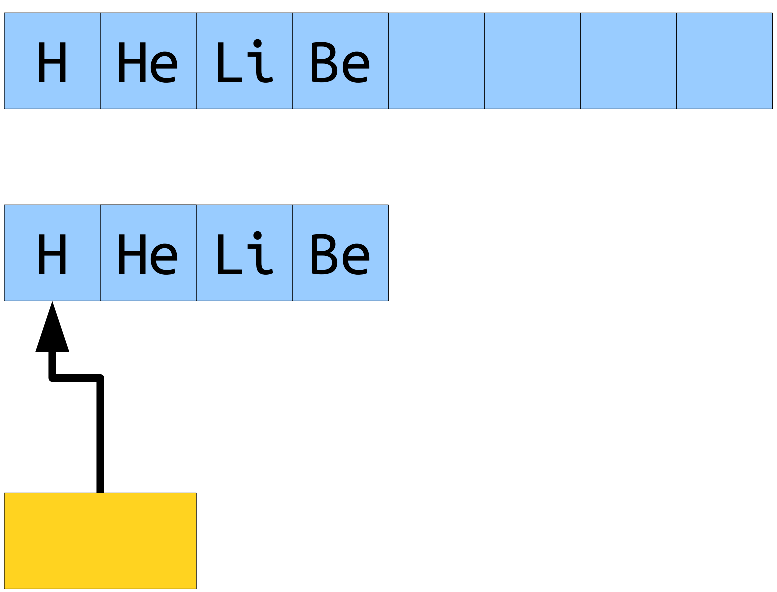

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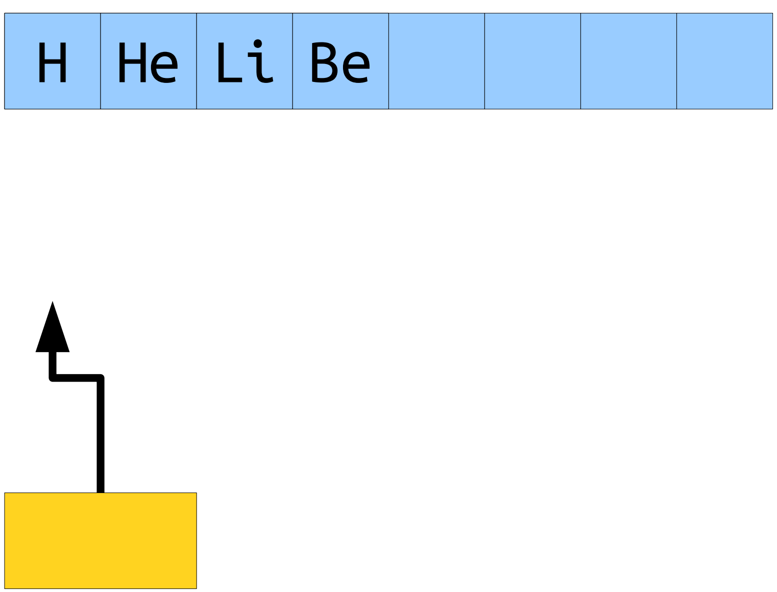

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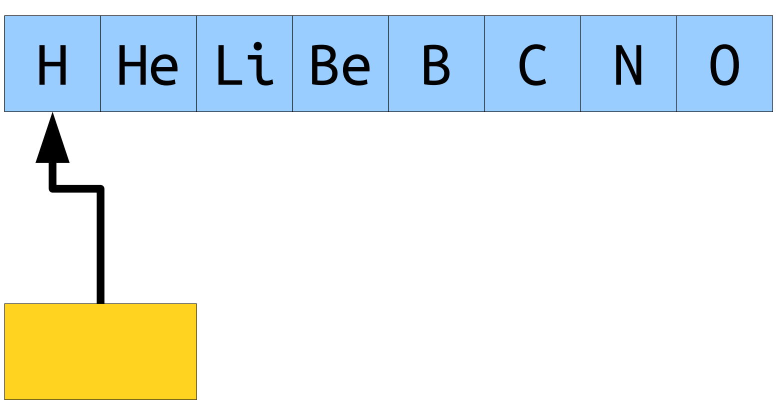

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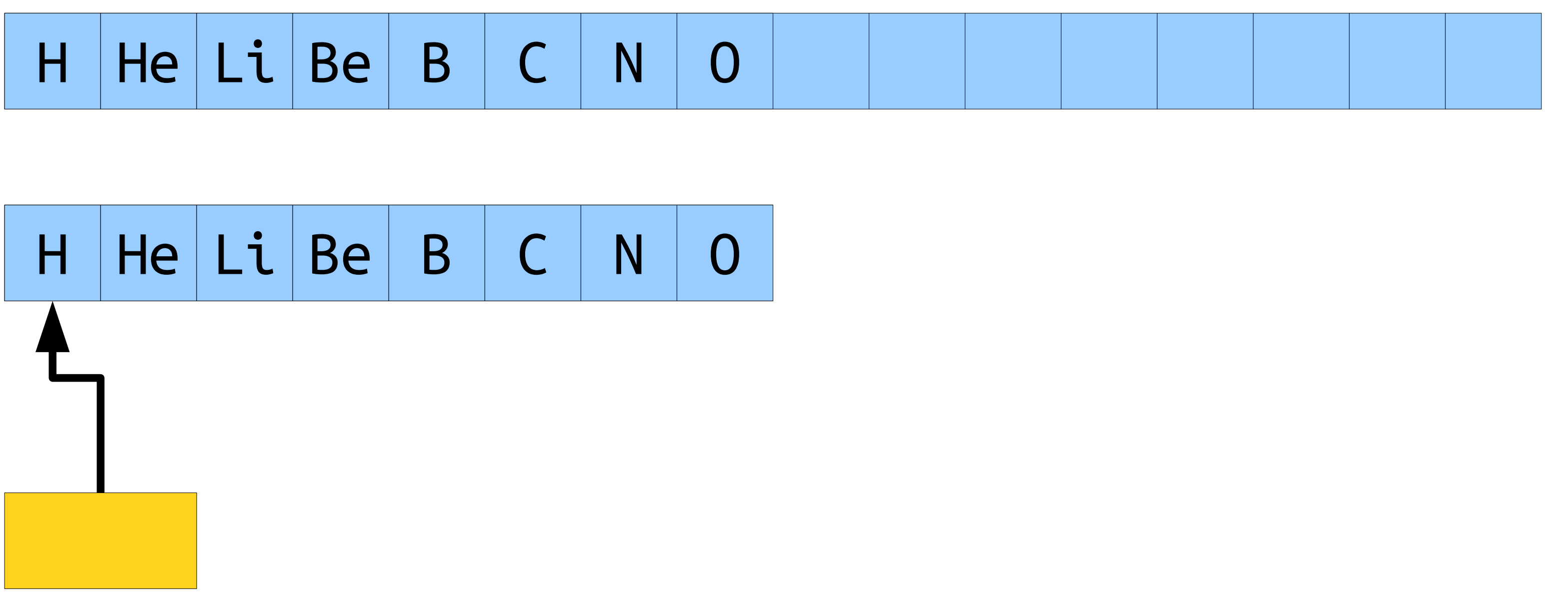

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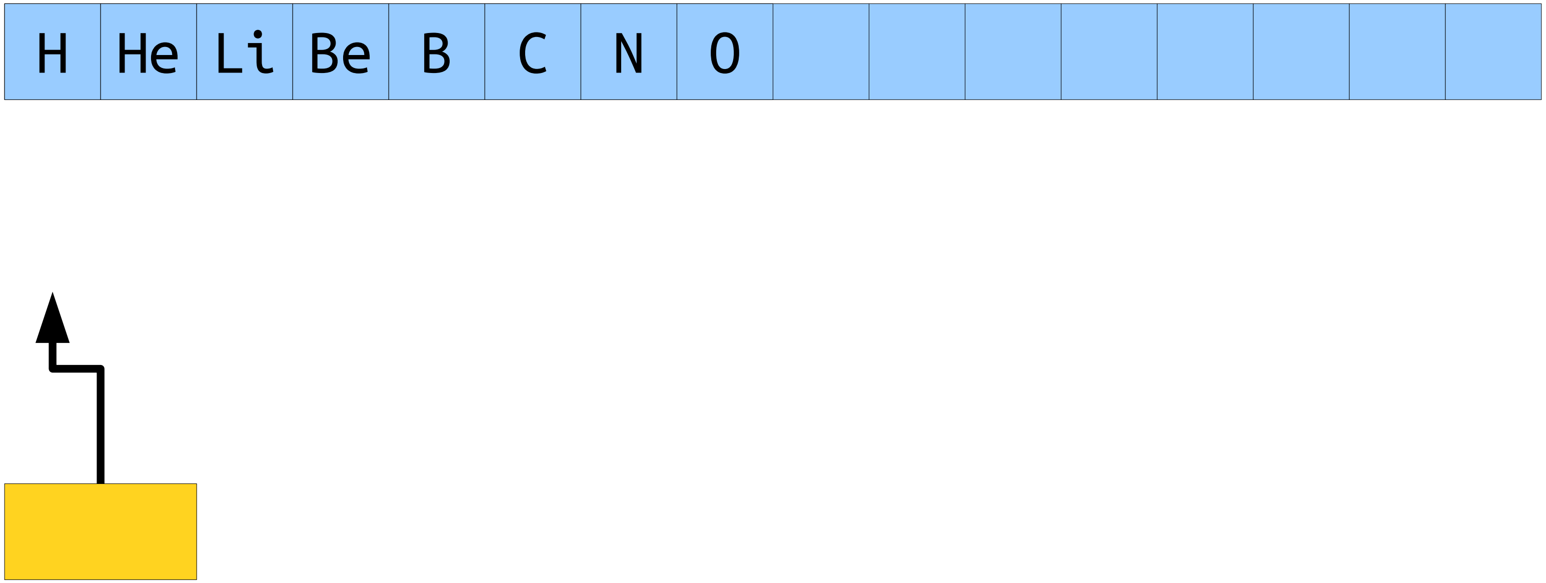

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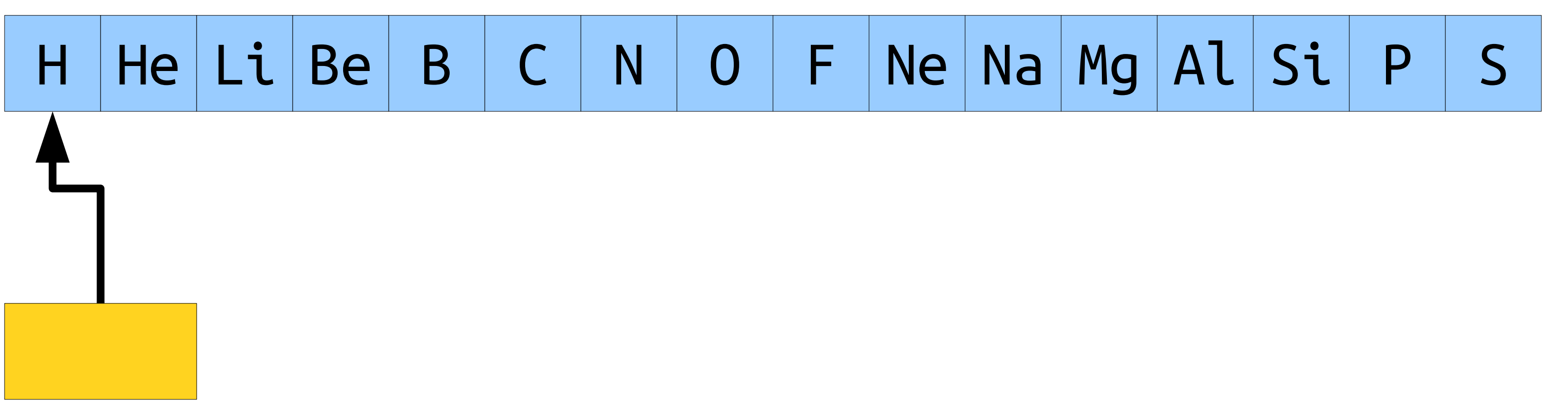
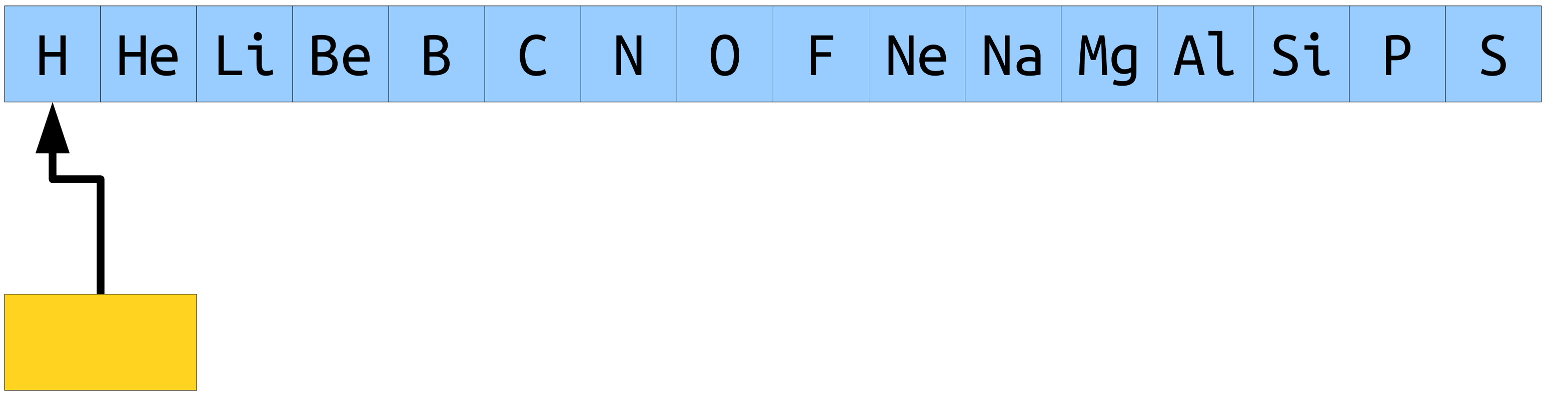


Dynamic Arrays

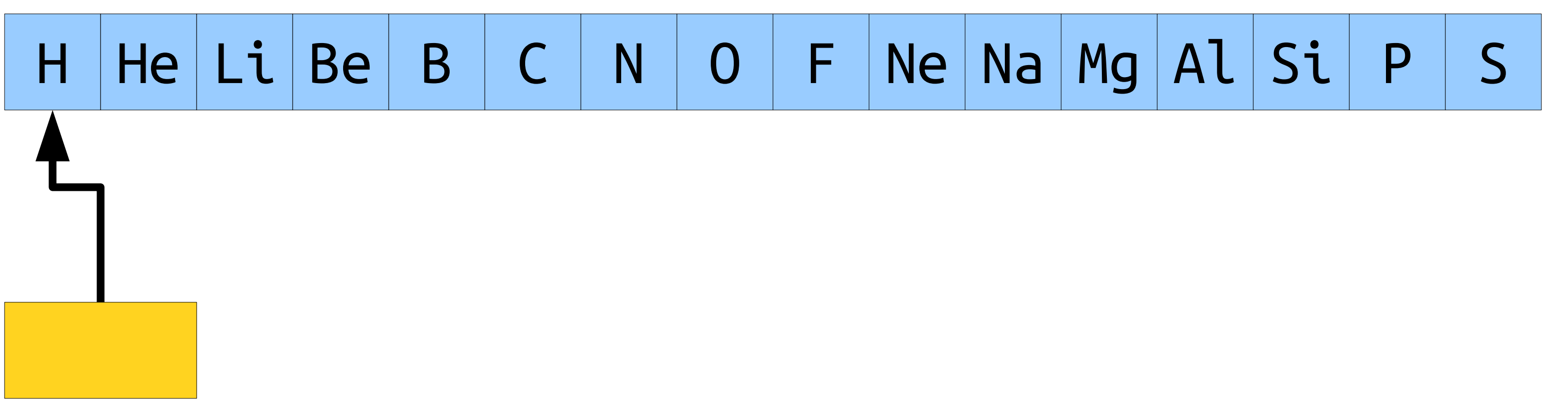
- Most appends to a dynamic array take time O(1).
- Infrequently, we do Θ(*n*) work to copy all *n* elements from the old array to a new one.
- Think "dishwasher:"
  - We slowly accumulate "messes" (filled slots).
  - We periodically do a large "cleanup" (copying the array).
- ***Claim:*** The cost of doing *n* appends to an initially empty dynamic array is always O(*n*).



Dynamic Arrays

- ***Claim:*** Appending n elements always takes time $O(n)$.
- The array doubles at sizes 2^0, 2^1, 2^2, …, etc.
- The very last doubling is at the largest power of two less than n. This is at most $2^{\lfloor \log_2 n \rfloor}$. (Do you see why?)
- Total work done across all doubling is at most

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{\lfloor \log_2 n \rfloor} &= 2^{\lfloor \log_2 n \rfloor + 1} - 1 \\ &\leq 2^{\log_2 n + 1} \\ &= 2n. \end{aligned}$$



Dynamic Arrays

- It's correct but misleading to say the cost of an append is O(n).
  - This is comparatively rare.
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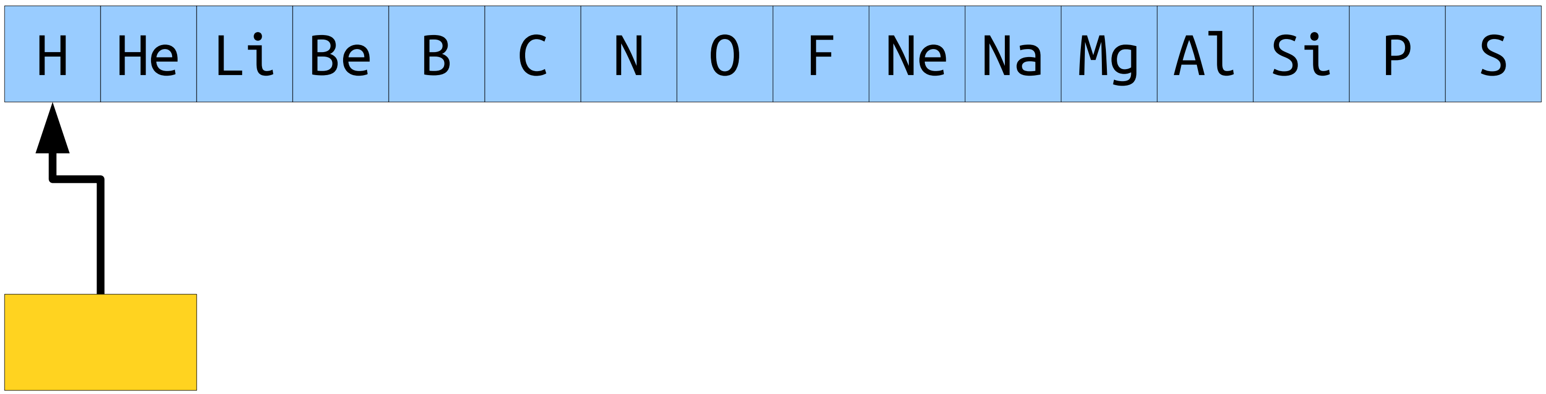
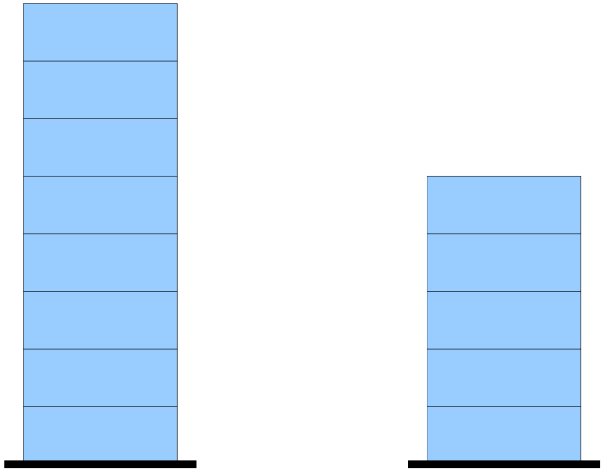
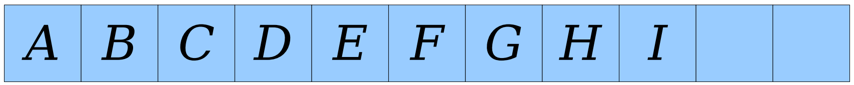
  - However, pretending each operation takes O(1) time never underestimates the true total runtime.
- ***Question:*** What's an honest, accurate way to describe the runtime of the dynamic array?





Two-Stack Queues



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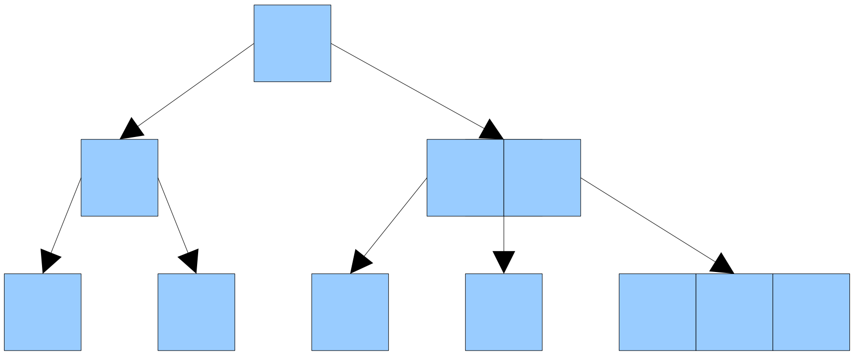
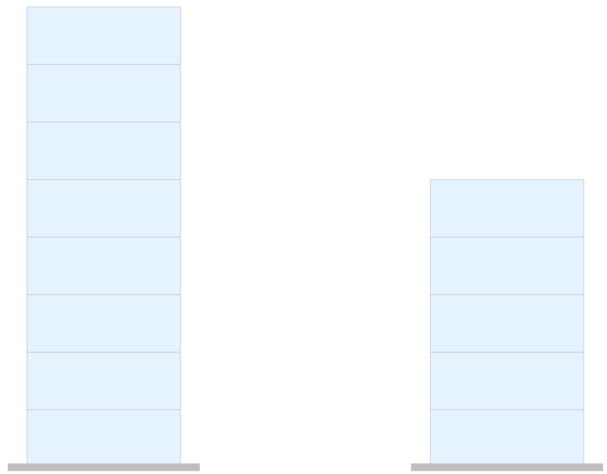


Building B-Trees



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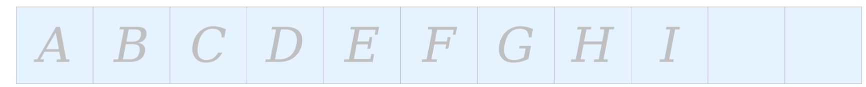

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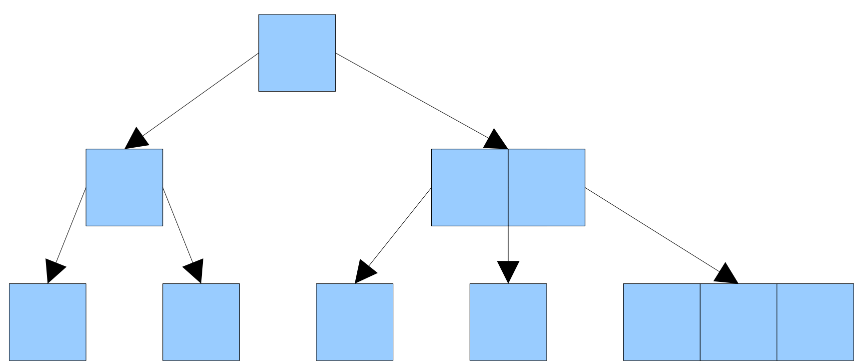
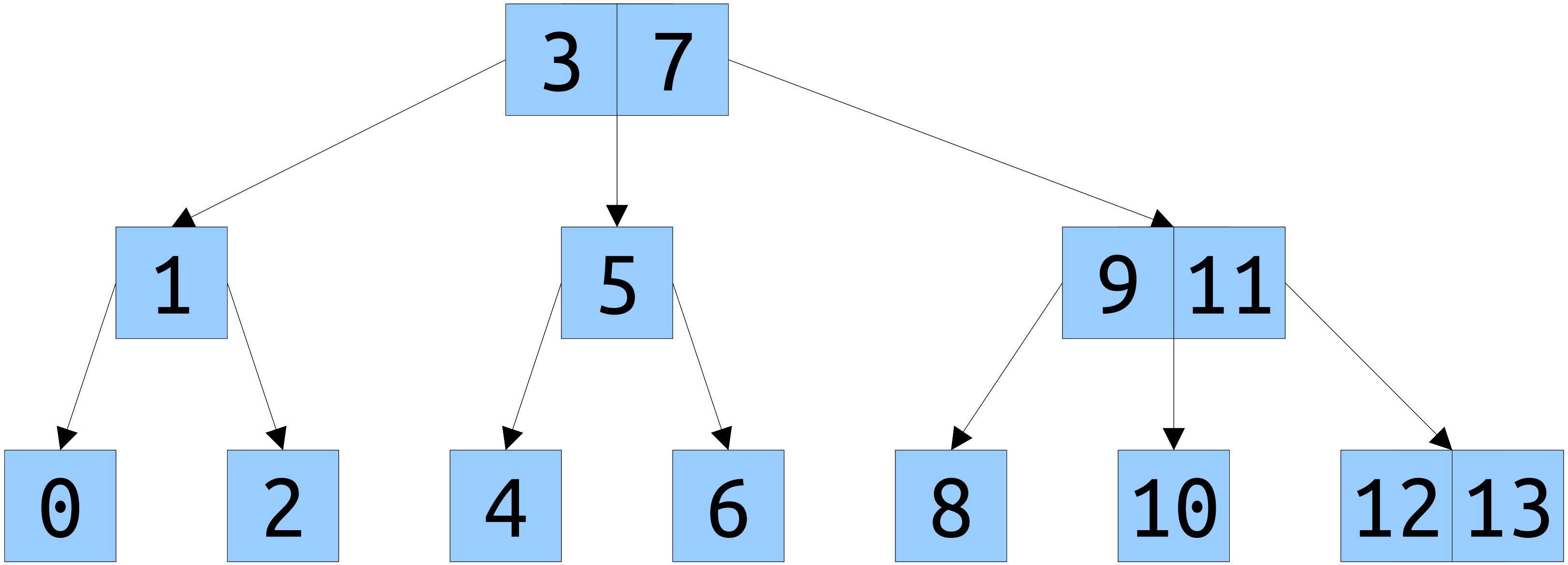


Building B-Trees

Building B-Trees

- You're given a sorted list of *n* values and a value of *b*.
- What's the most efficient way to construct a B-tree of order *b* holding these *n* values?
- ***One Option:*** Think really hard, calculate the shape of a B-tree of order *b* with *n* elements in it, then place the items into that B-tree in sorted order.
- Is there an easier option?



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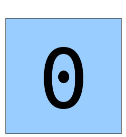

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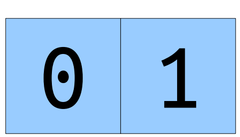

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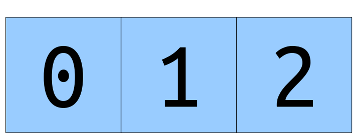

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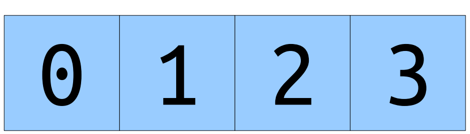

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0	1	2	3

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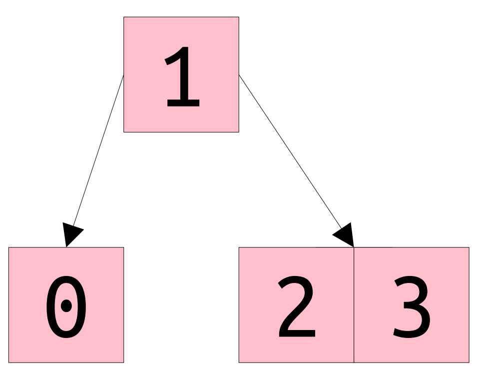

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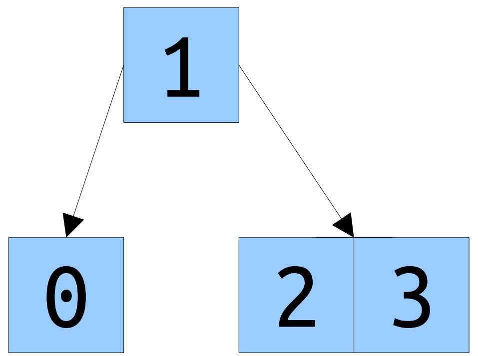

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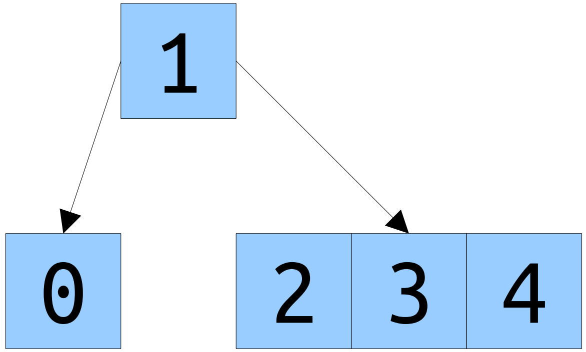

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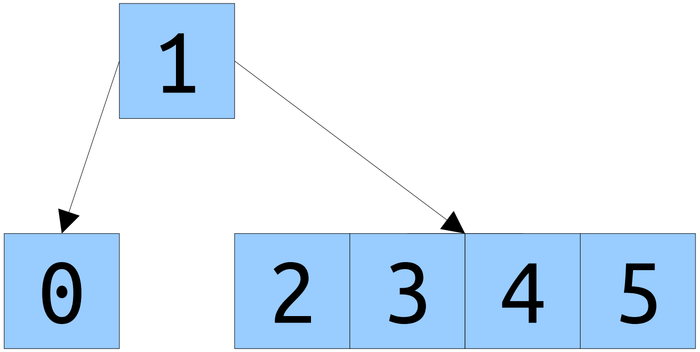

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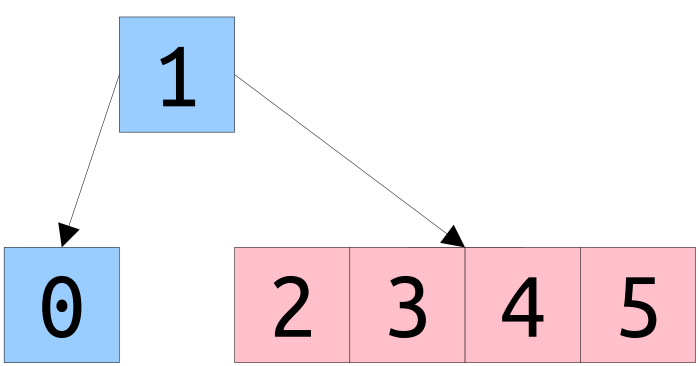

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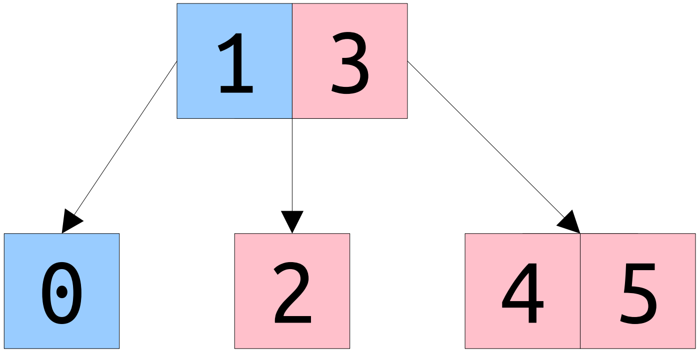

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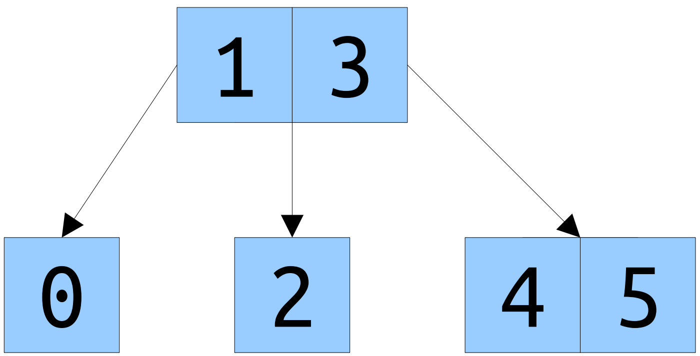

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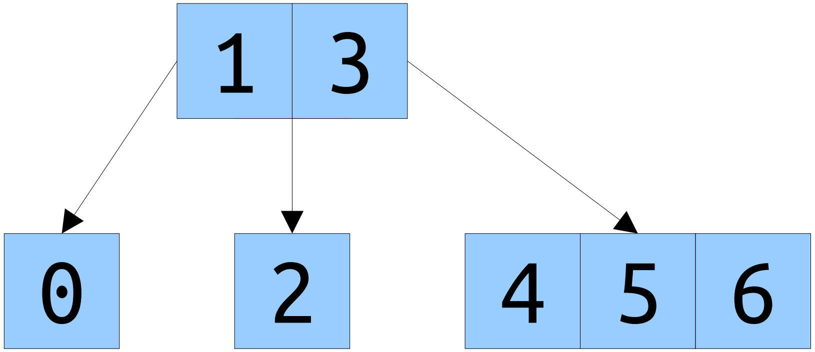

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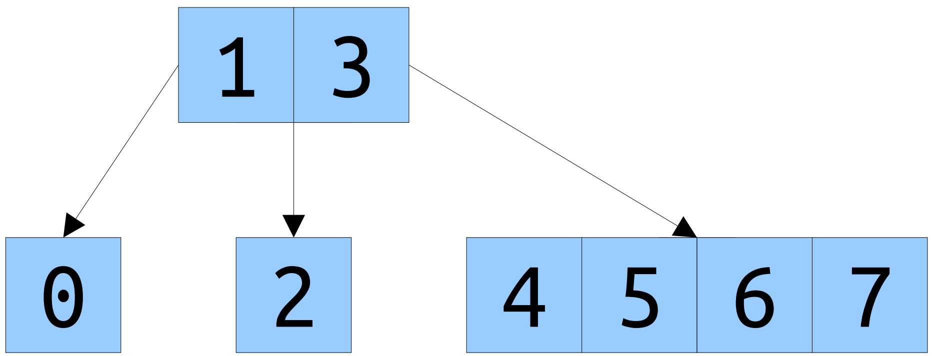

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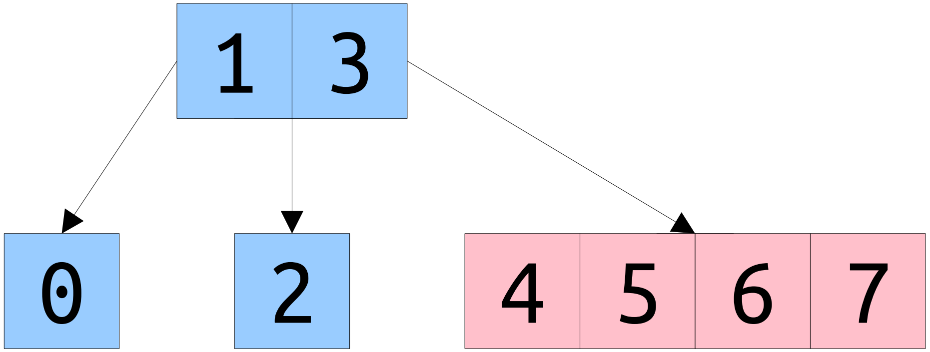

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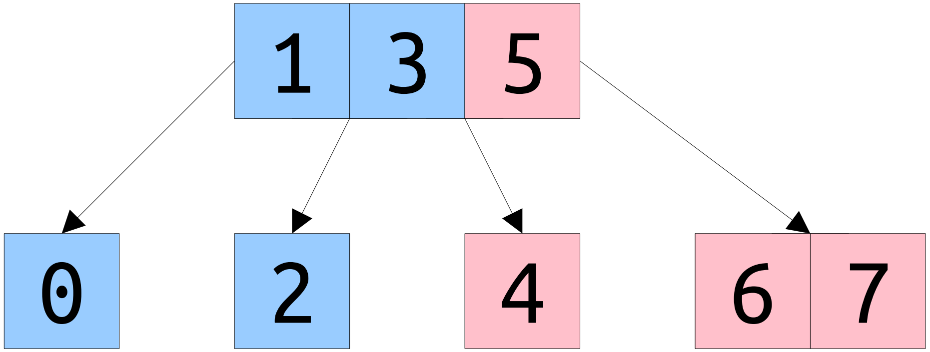

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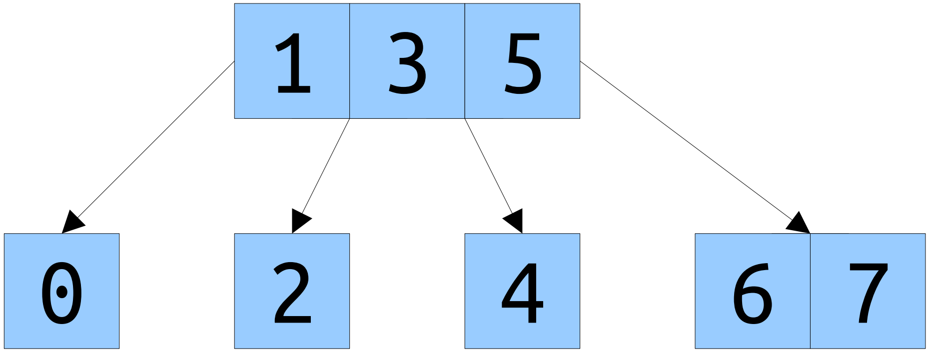
- ***Idea 1:*** Insert the items into an empty B-tree in sorted order.



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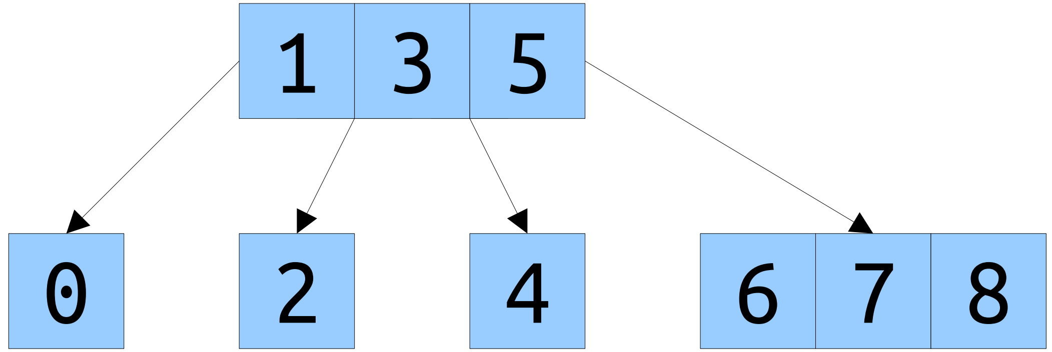

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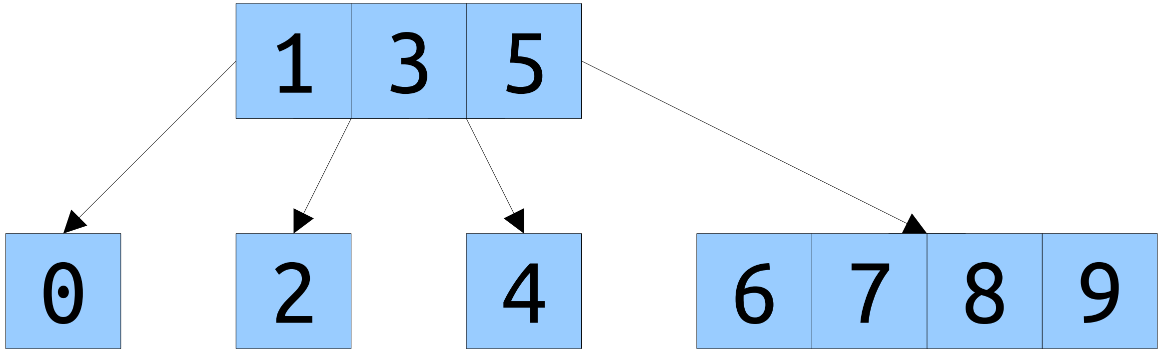

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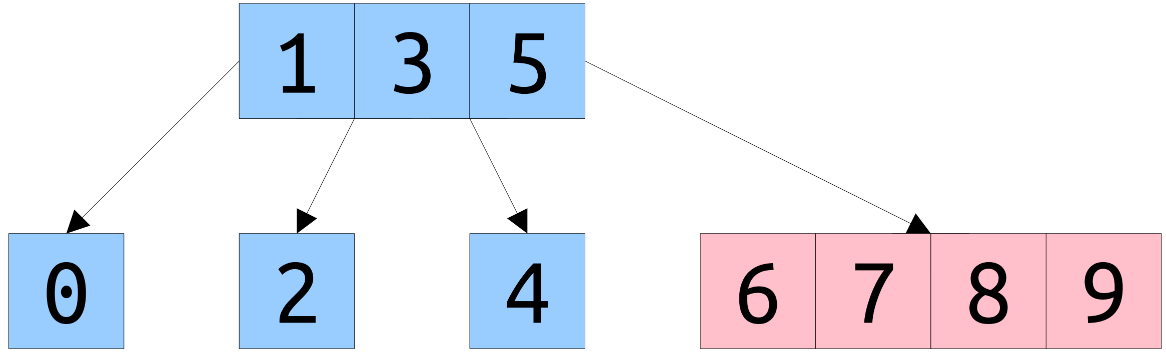

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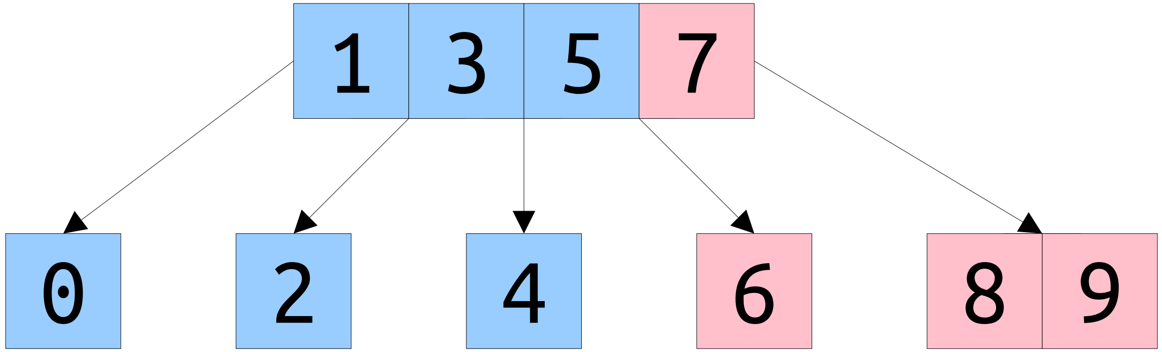

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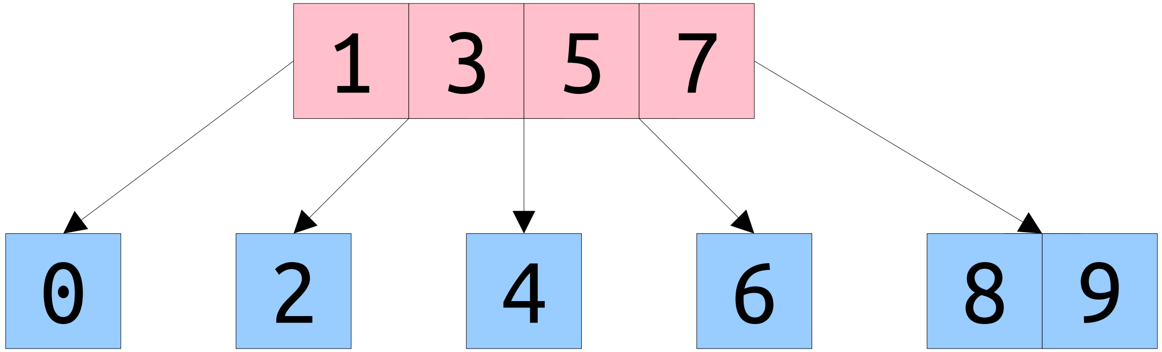

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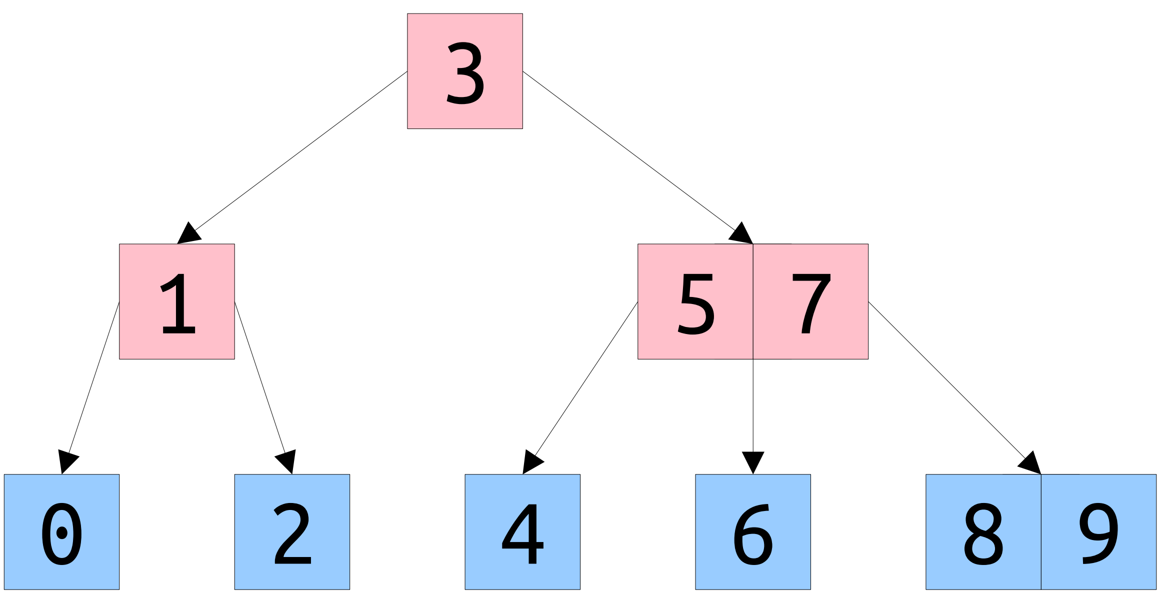

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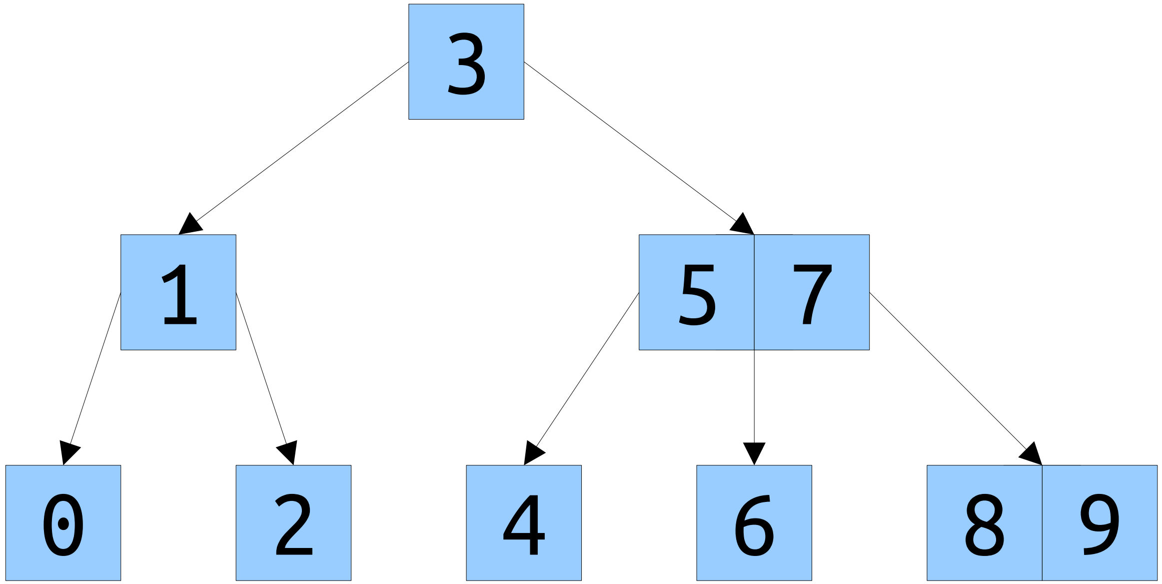
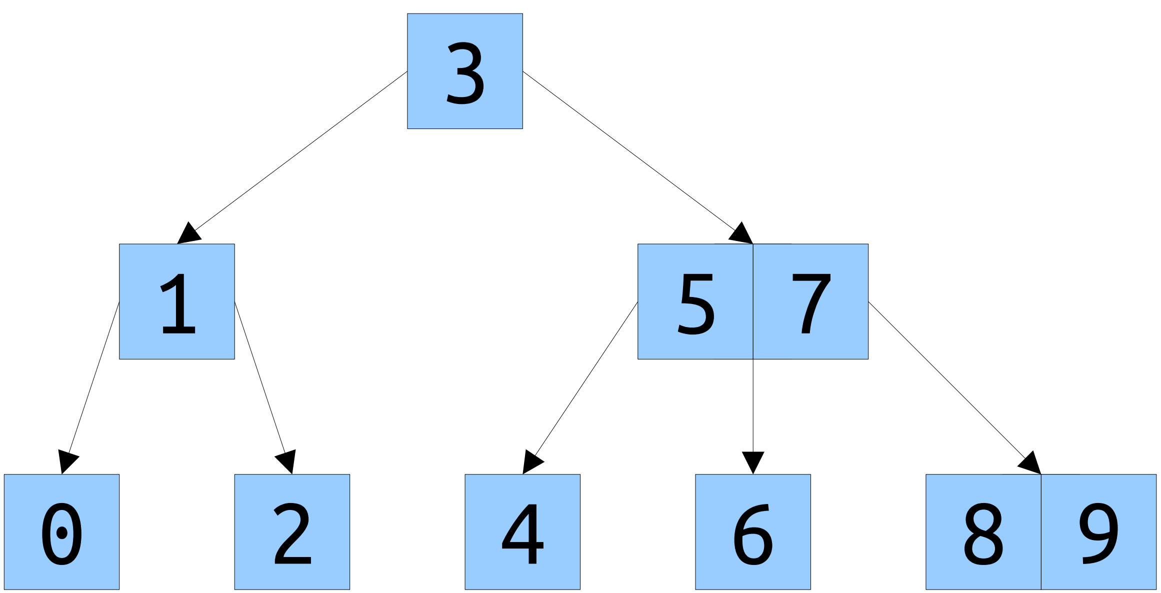


Building B-Trees

- ***Idea 1:*** Insert the items into an empty B-tree in sorted order.
- Cost: $\Omega(n \log_b n)$, due to the top-down search.
- ***Can we do better?***



Building B-Trees

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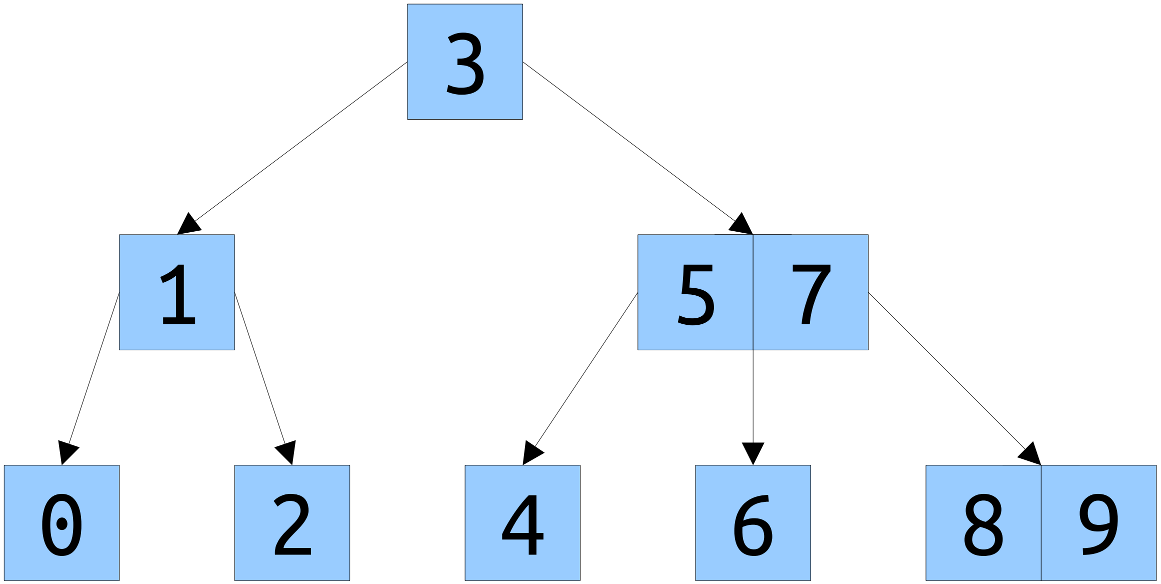

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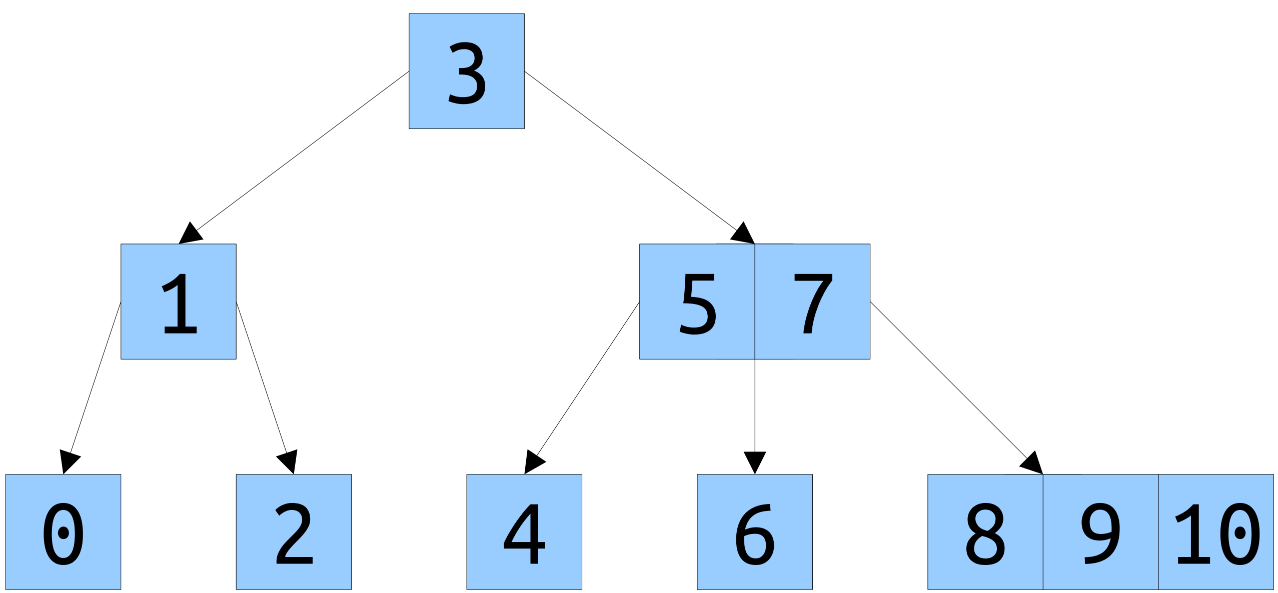

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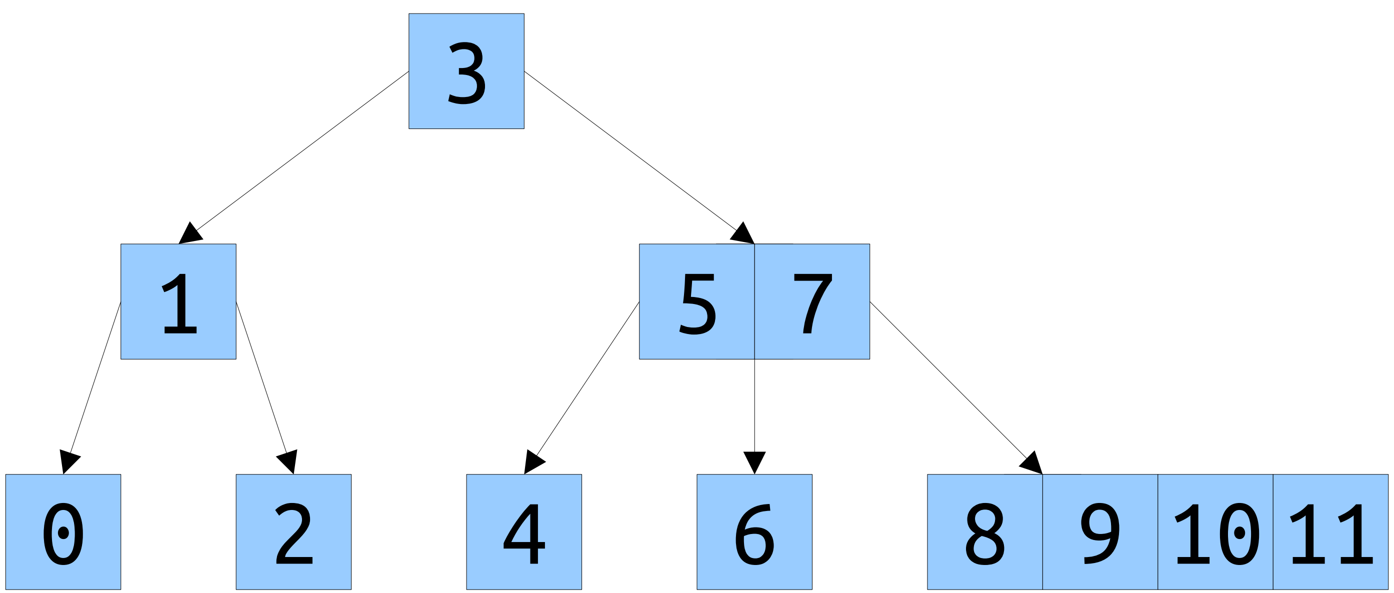

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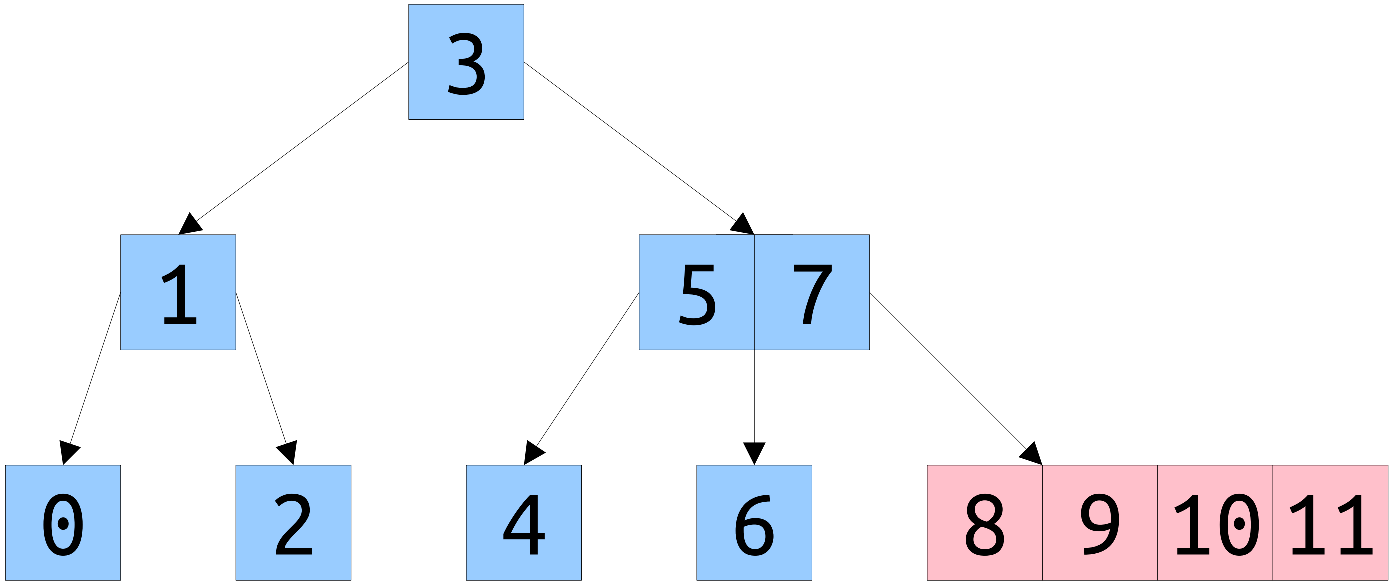

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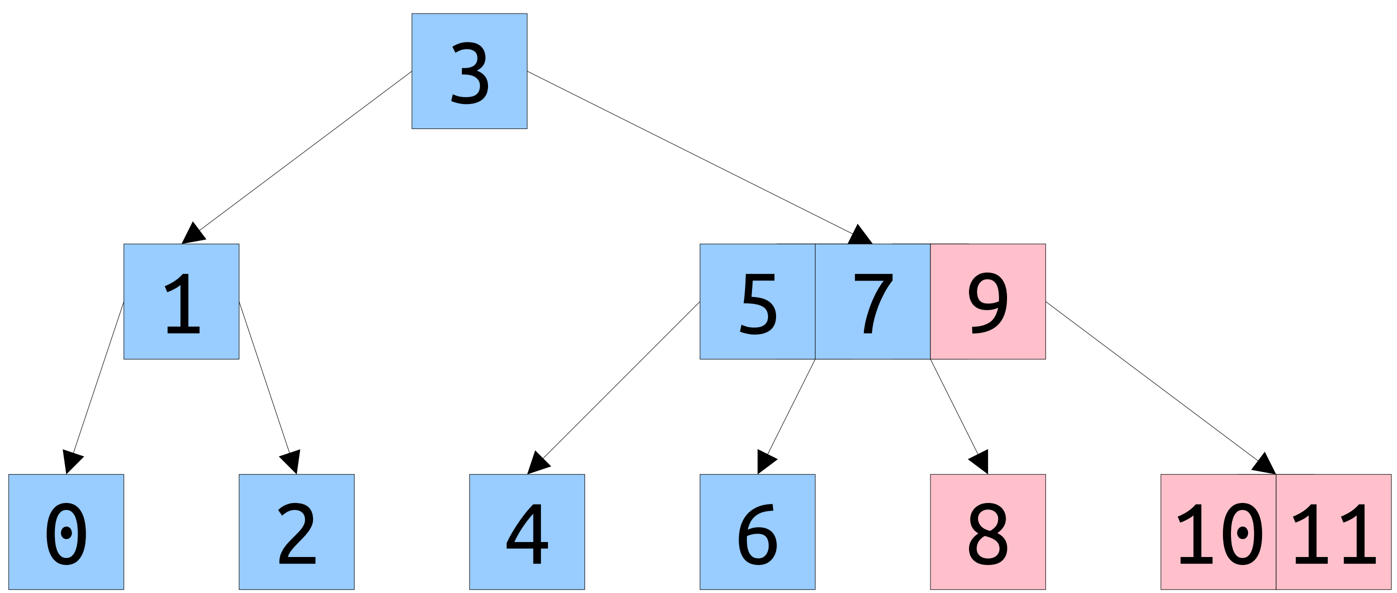

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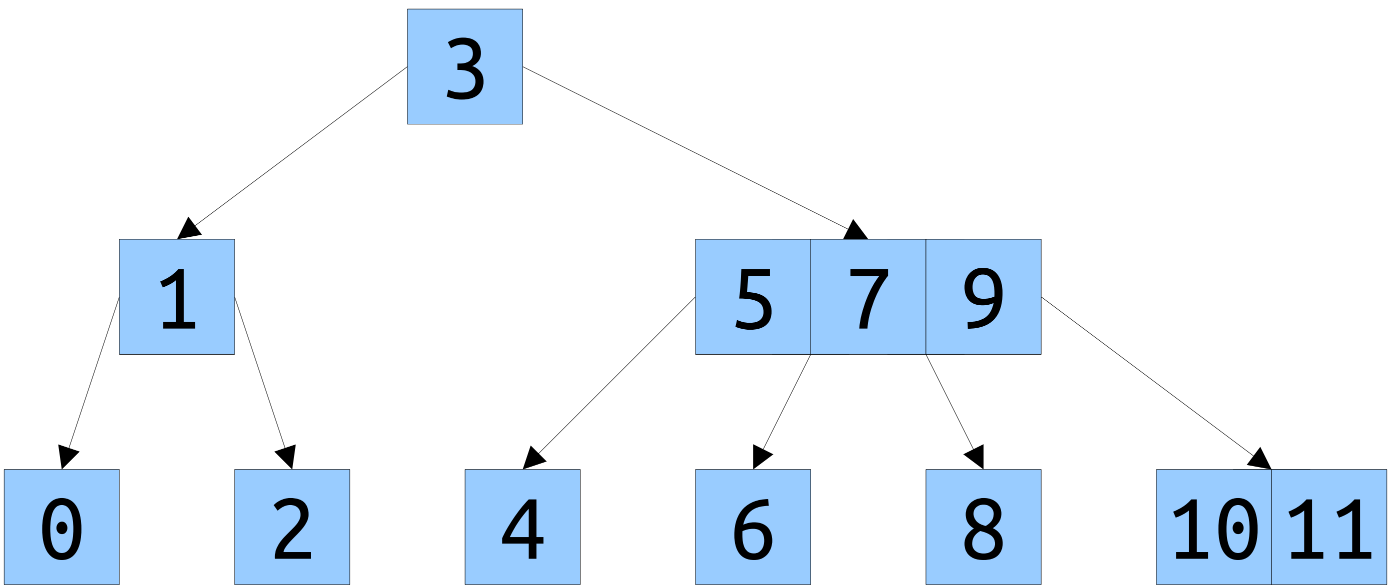

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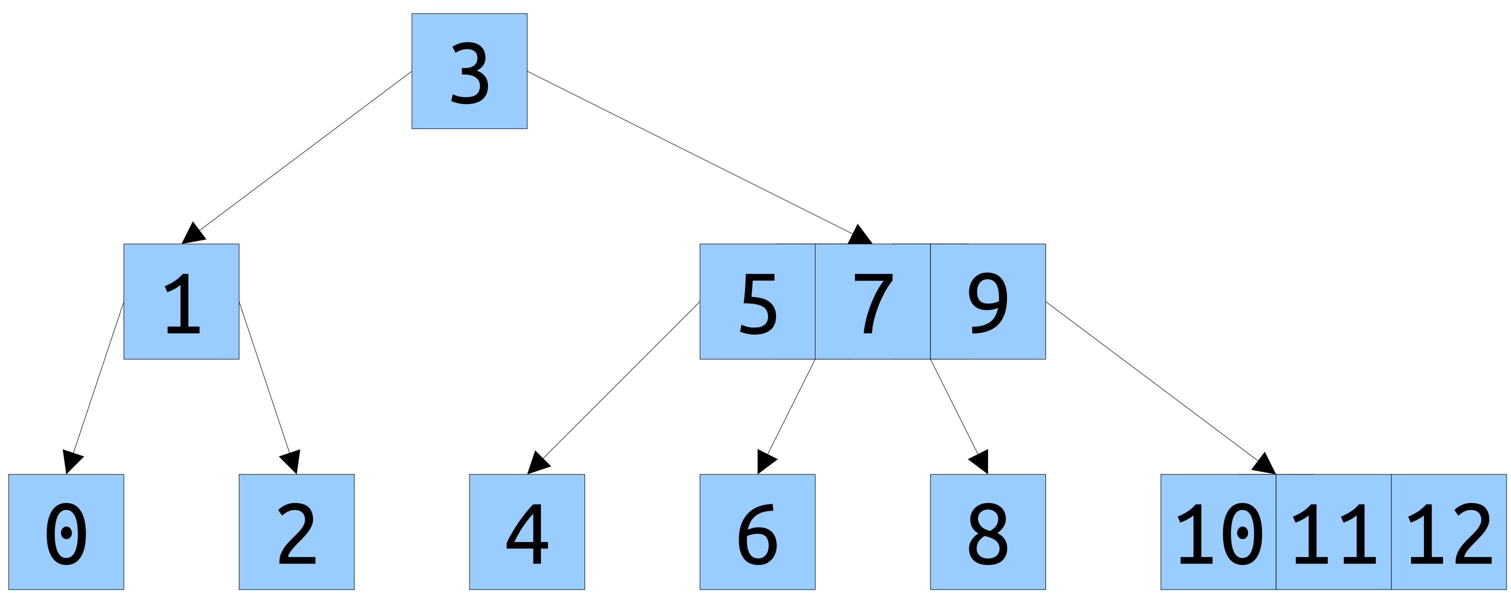

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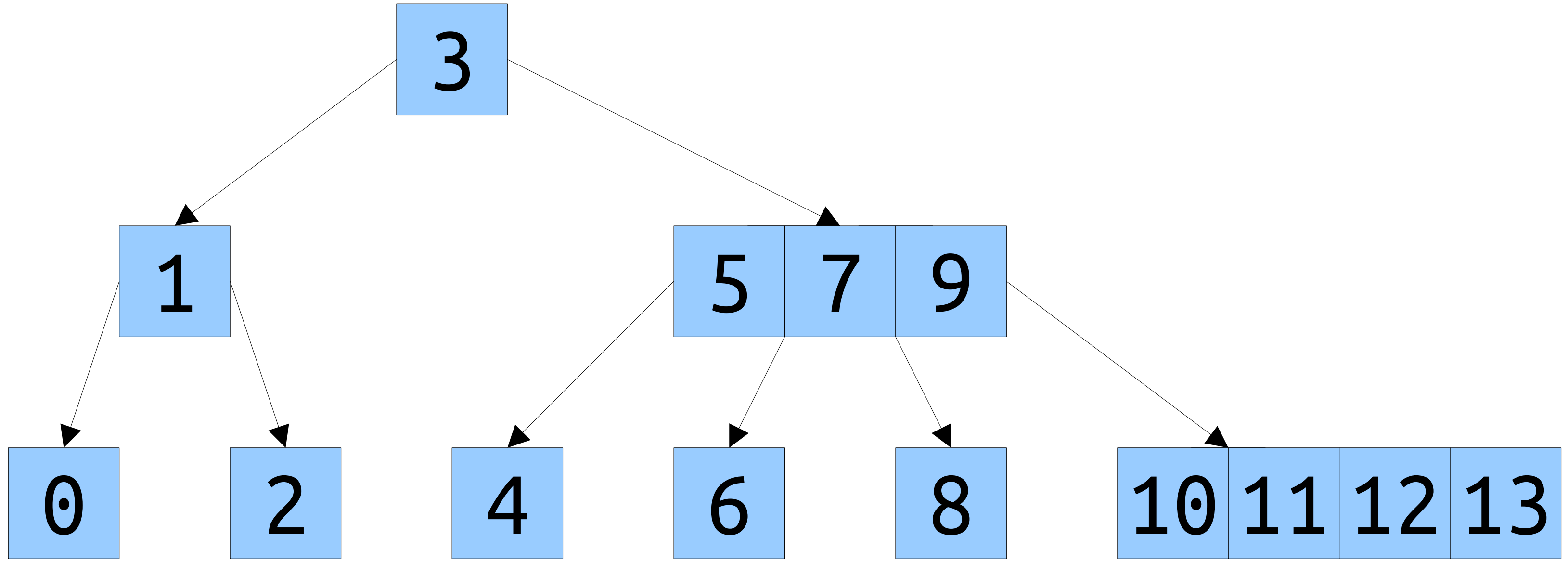

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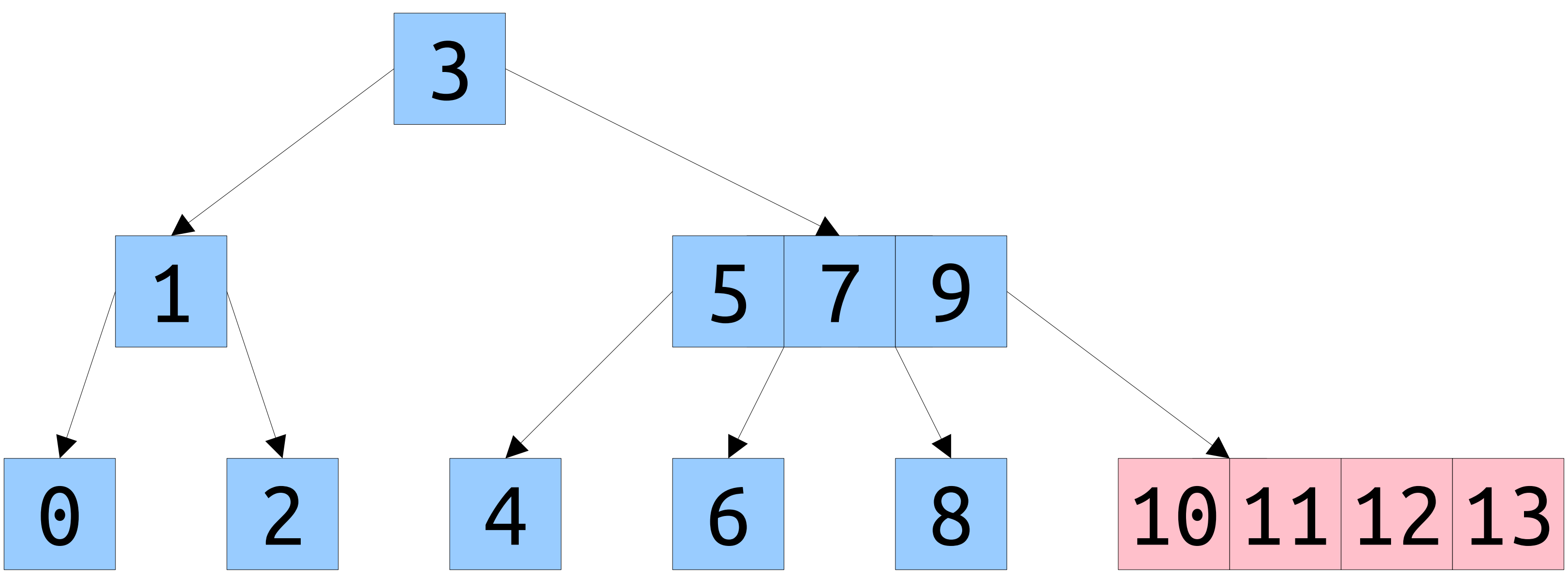

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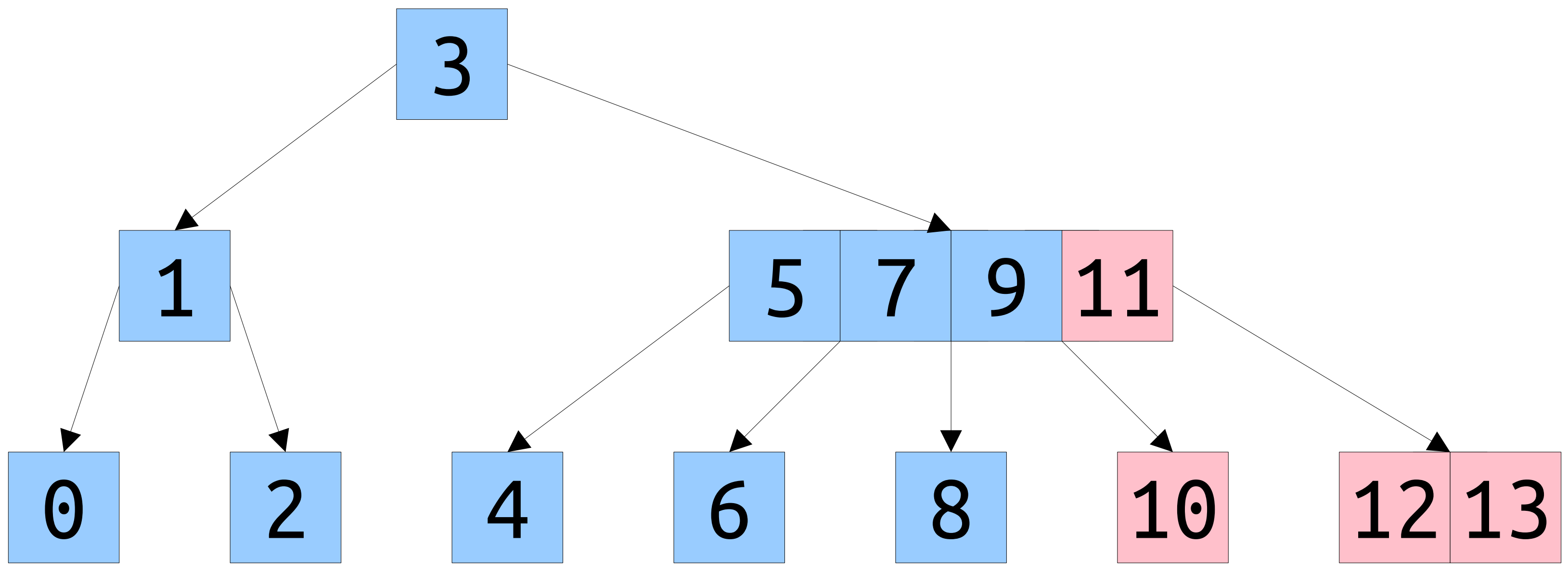

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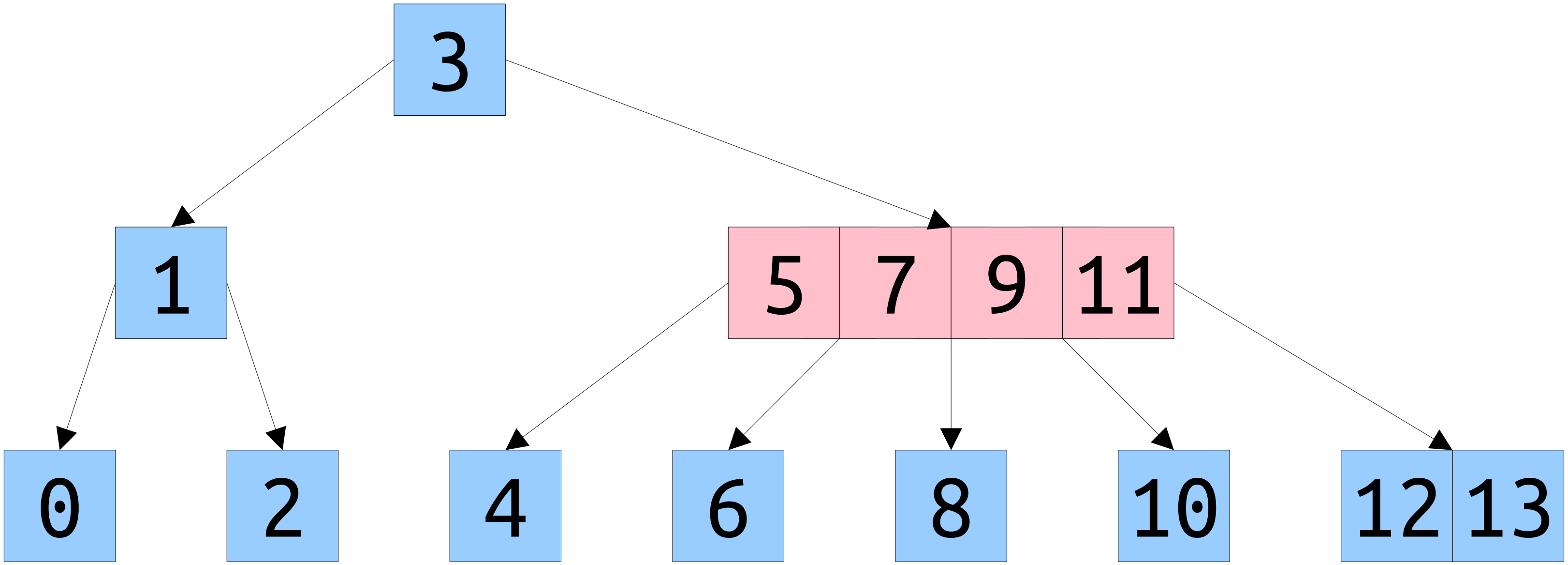

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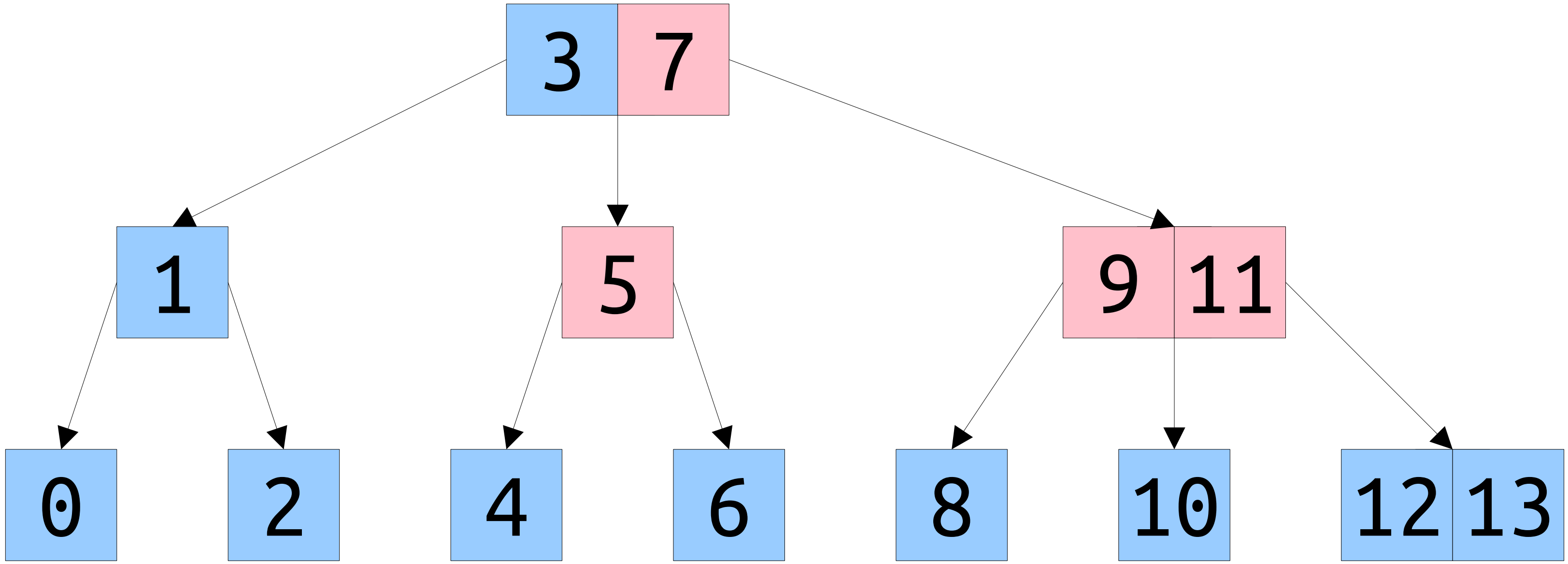

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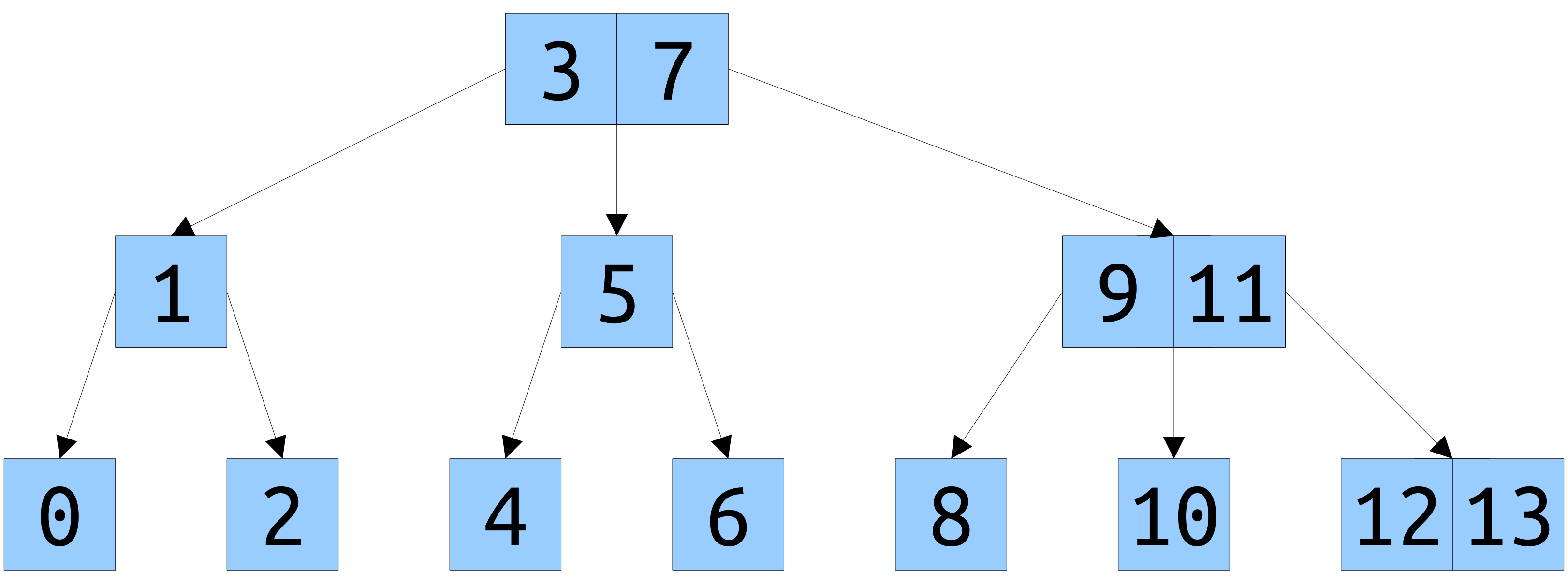
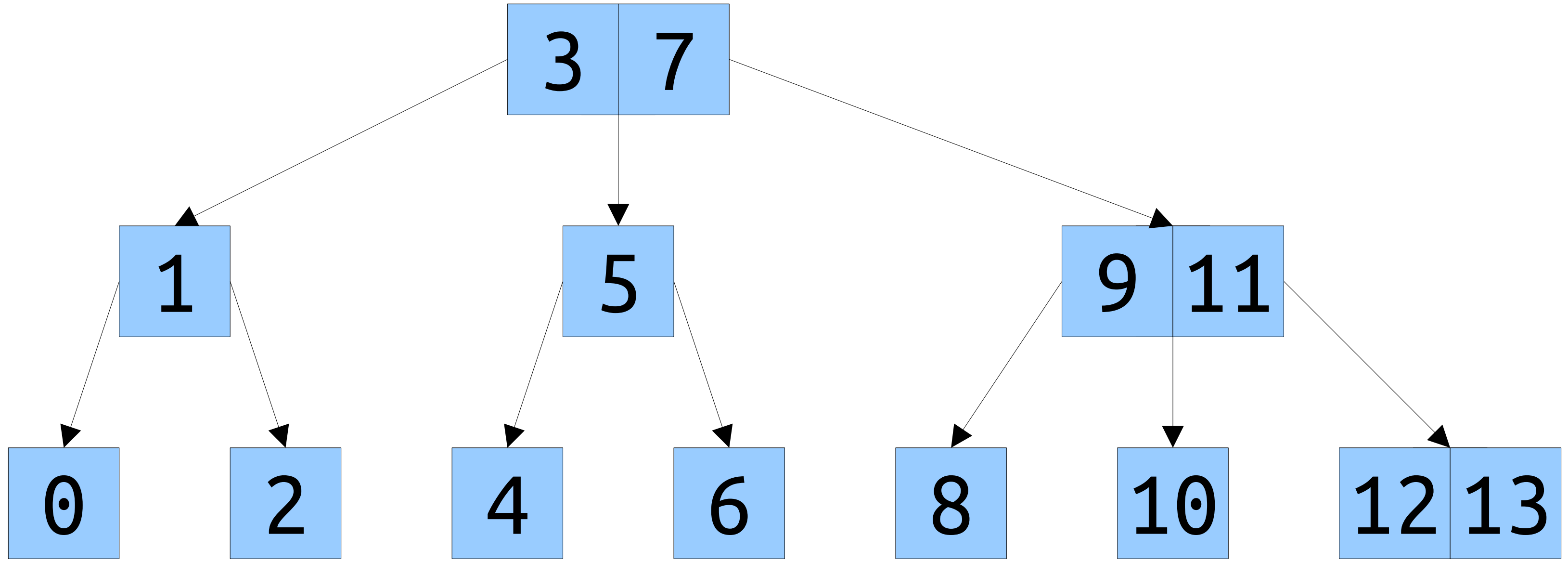


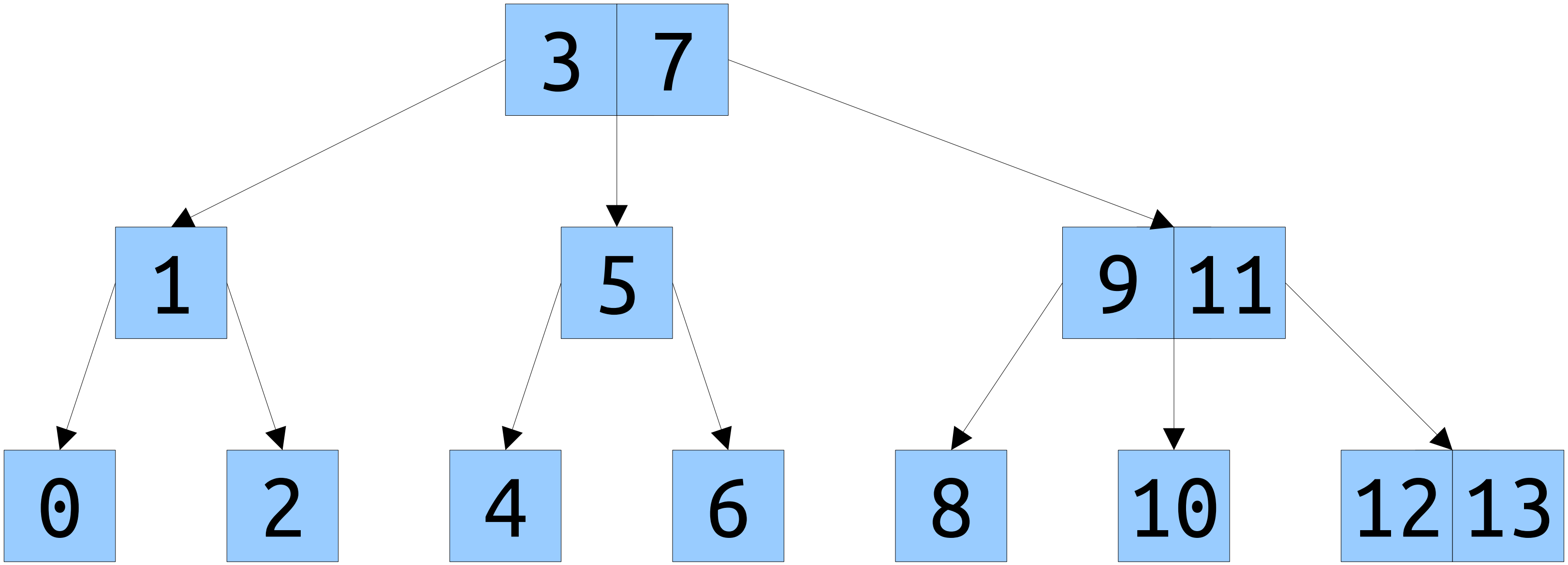
Building B-Trees

- ***Idea 2:*** Since all insertions will happen at the rightmost leaf, store a pointer to that leaf. Add new values by appending to this leaf, then doing any necessary splits.
- ***Question:*** How fast is this?



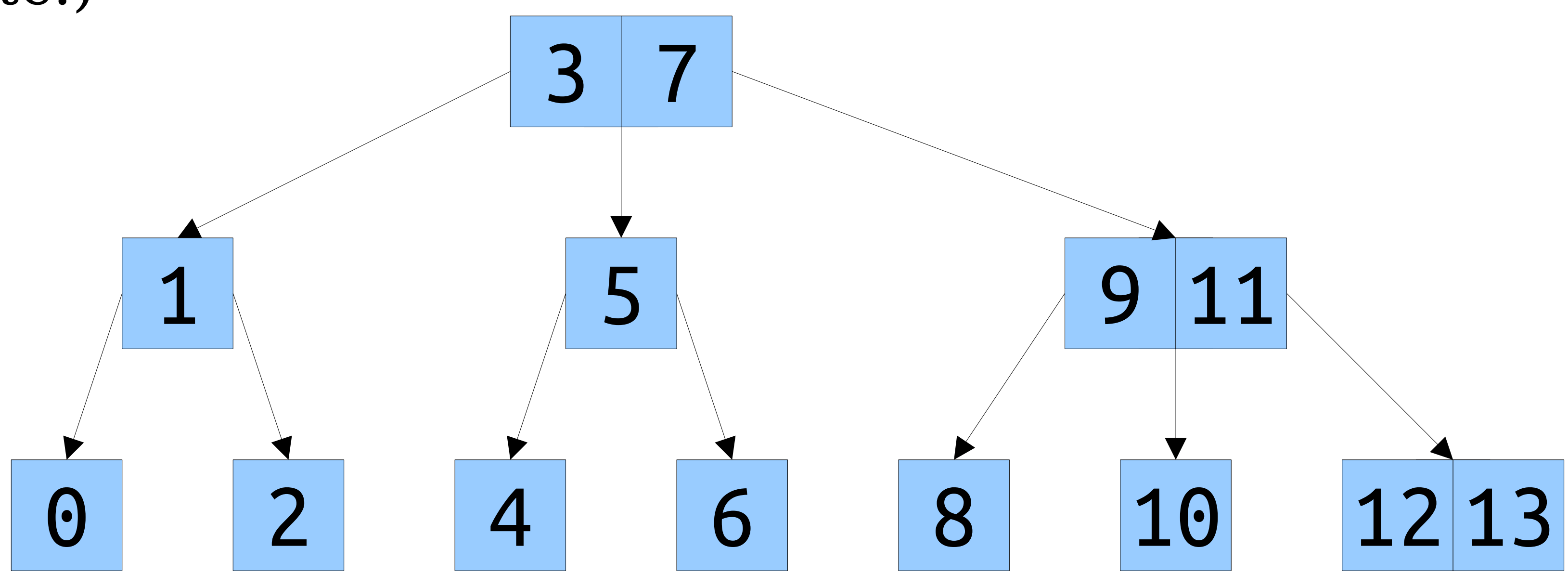
Building B-Trees

- The cost of an insert varies based on the shape of the tree.
  - If no splits are required, the cost is O(1).
  - If one split is required, the cost is O(*b*).
  - If we have to split all the way up, the cost is $O(b \log_b n)$.
- Using our worst-case cost across *n* inserts gives a runtime bound of $O(nb \log_b n)$
- ***Claim:*** The cost of *n* inserts is always O(*n*).



Building B-Trees

- Of all the n insertions into the tree, a roughly $^1/_b$ fraction will split a node in the bottom layer of the tree (a leaf).
- Of those, roughly a $^1/_b$ fraction will split a node in the layer above that.
- Of those, roughly a $^1/_b$ fraction will split a node in the layer above that.
- (etc.)



Building B-Trees

- Total number of splits:

Building B-Trees

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$$\frac{n}{b} \cdot \left(1 + \frac{1}{b} \cdot \left(1 + \frac{1}{b} \cdot \left(1 + \frac{1}{b} \cdot \left(\ldots\right)\right)\right)\right)$$

Building B-Trees

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$$= \quad \frac{n}{b} \cdot \Theta(1)$$

Building B-Trees

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$$= \quad \frac{n}{b}\cdot\left(1+\frac{1}{b}+\frac{1}{b^2}+\frac{1}{b^3}+\frac{1}{b^4}+\ldots\right)$$

$$= \quad \frac{n}{b}\cdot\Theta\left(1\right)$$

$$= \quad \Theta\left(\frac{n}{b}\right)$$

Building B-Trees

- Total number of splits:

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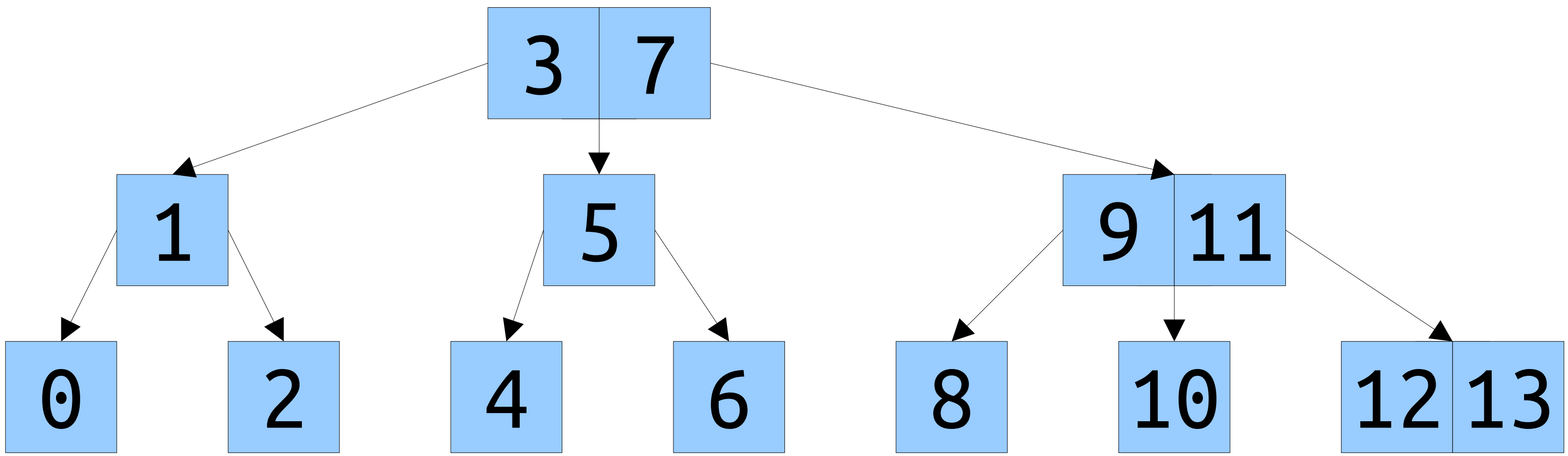
$$= \quad \frac{n}{b} \cdot \Theta(1)$$

$$= \quad \Theta\left(\frac{n}{b}\right)$$

- Total cost of those splits: $\boldsymbol{\Theta(n)}$.

Building B-Trees

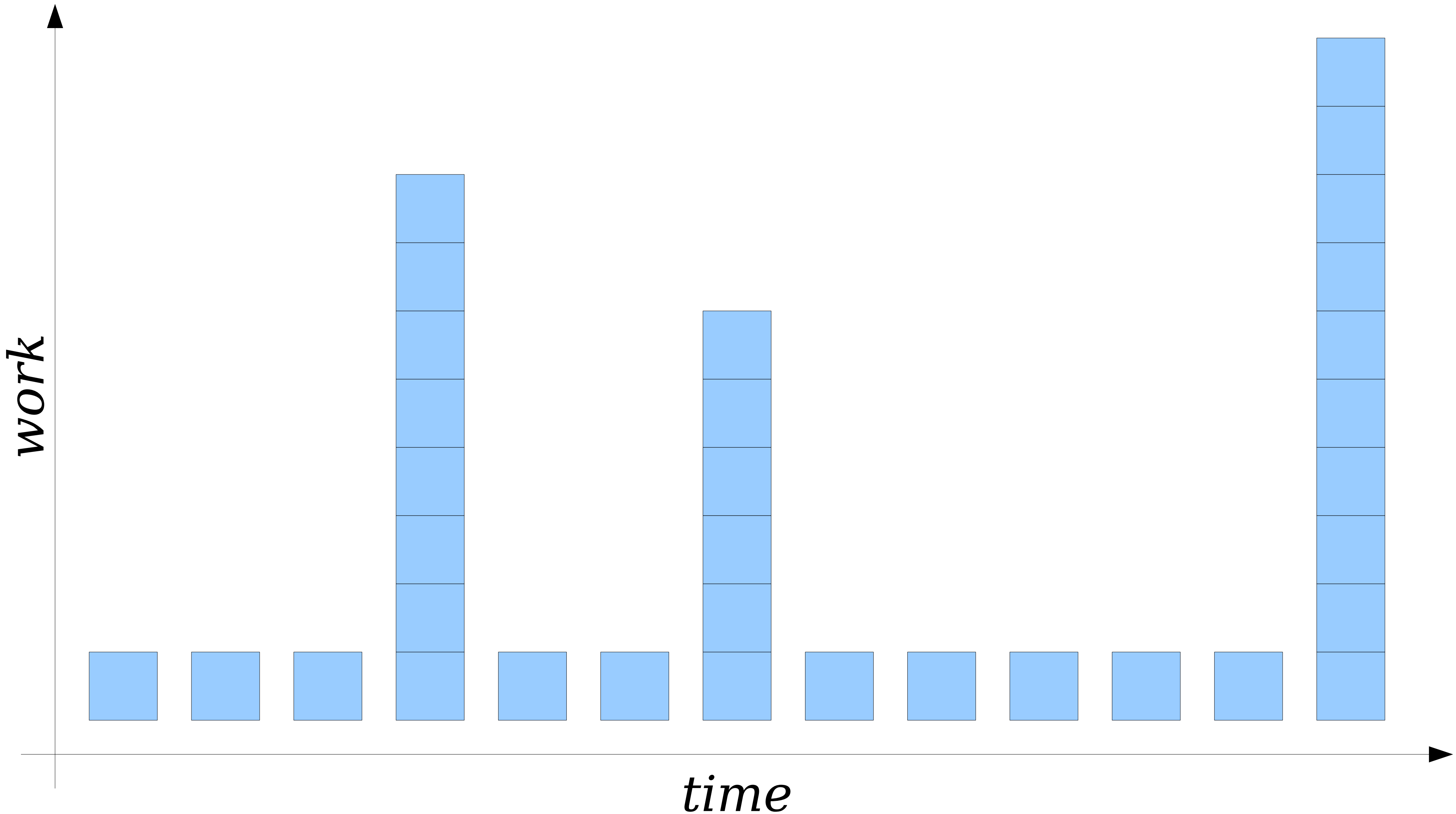
- It is correct but misleading to say the cost of an insert is $O(b \log_b n)$.
  - This is comparatively rare.
- It is wrong, but useful, to pretend that the cost of an insert is O(1).
  - Some operations take more time than this.
  - However, pretending each insert takes time O(1) never underestimates the total amount of work done across all operations.
- ***Question:*** What's an honest, accurate way to describe the cost of inserting one more value?



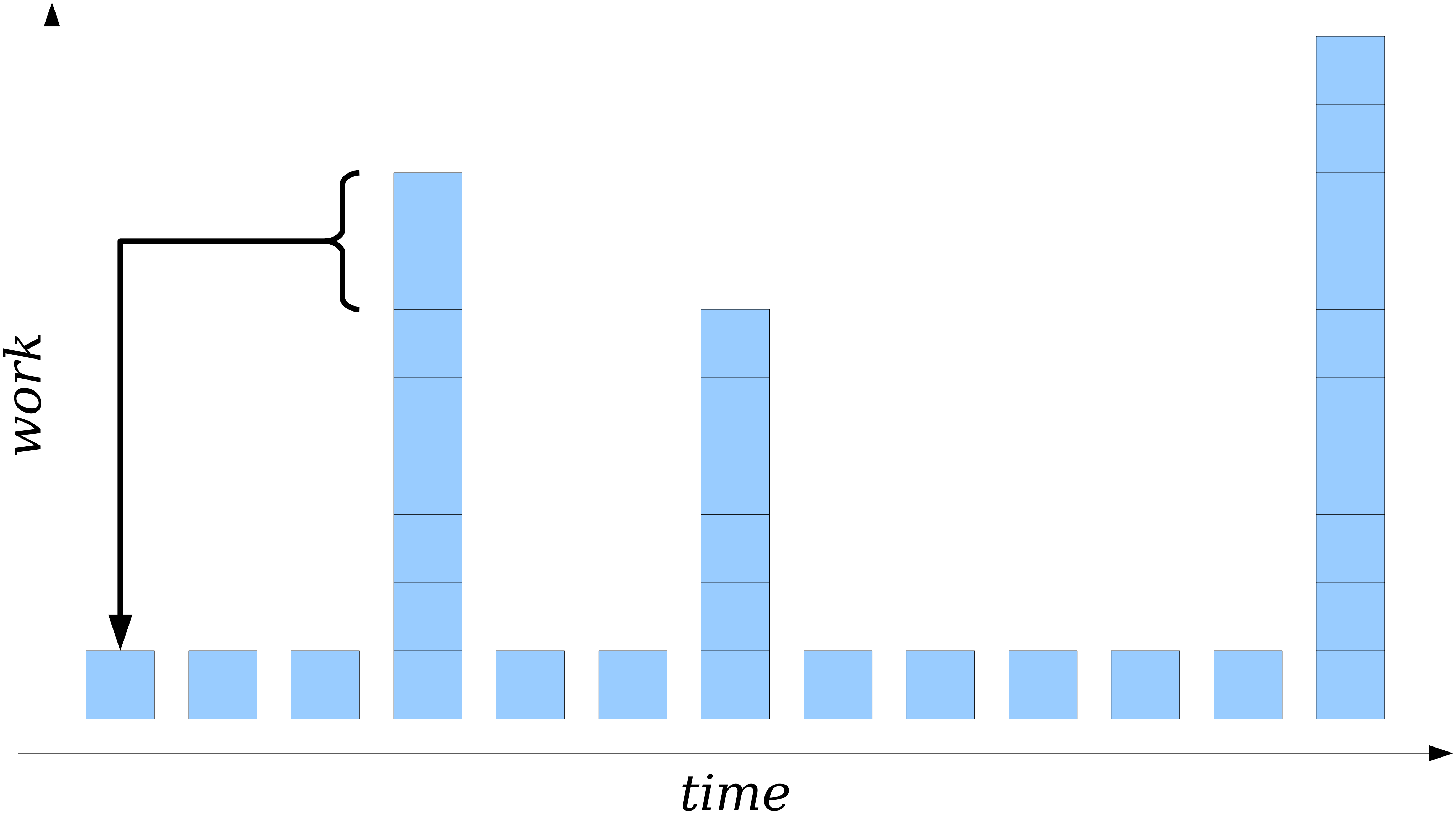
Amortized Analysis

The Setup

- We now have three examples of data structures where
  - ***individual operations may be slow***, but
  - ***any series of operations is fast***.
- Giving weak upper bounds on the cost of each operation is not useful for making predictions.
- How can we clearly communicate when a situation like this one exists?



Key Idea: Backcharge expensive operations to cheaper ones.



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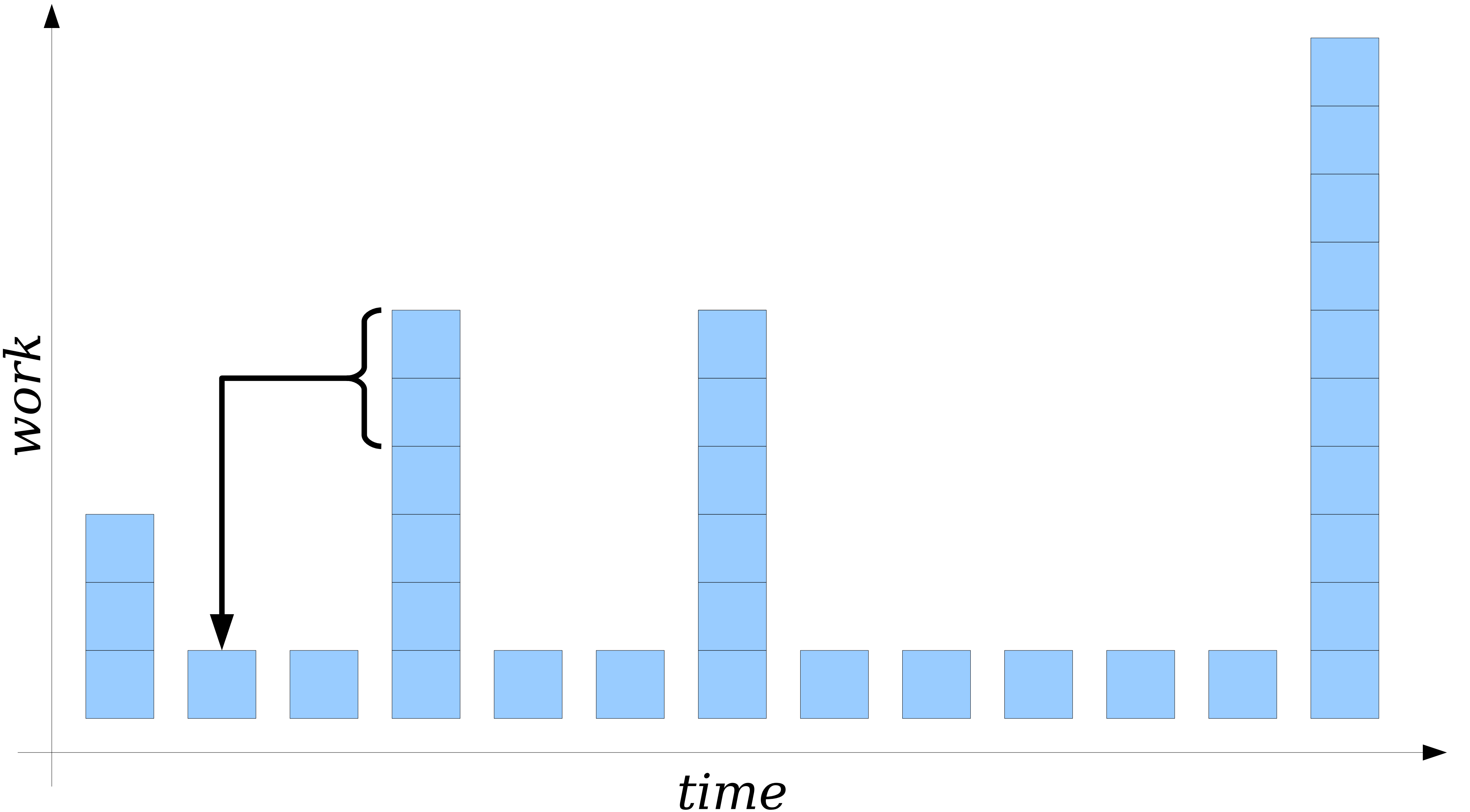

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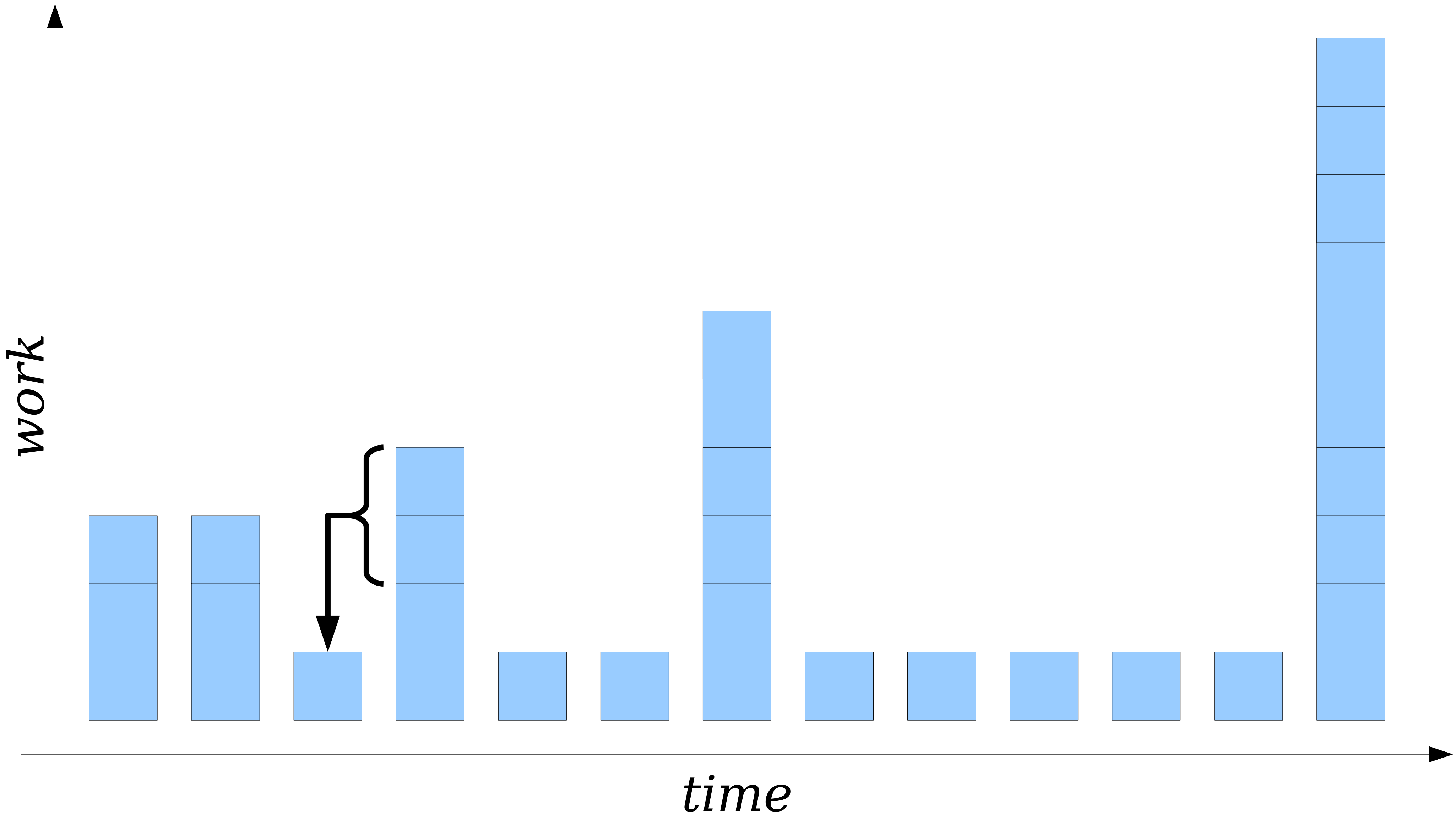

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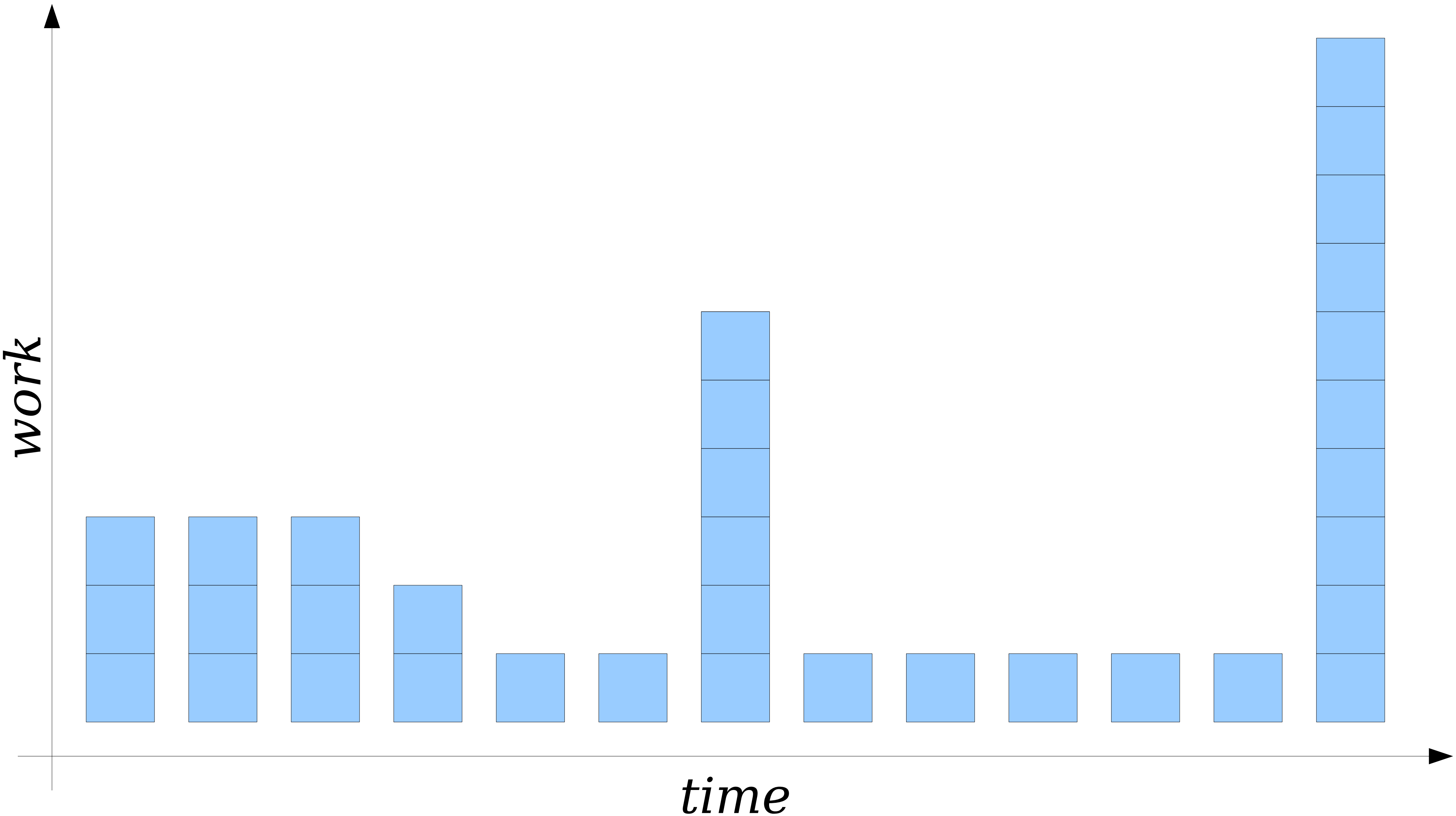

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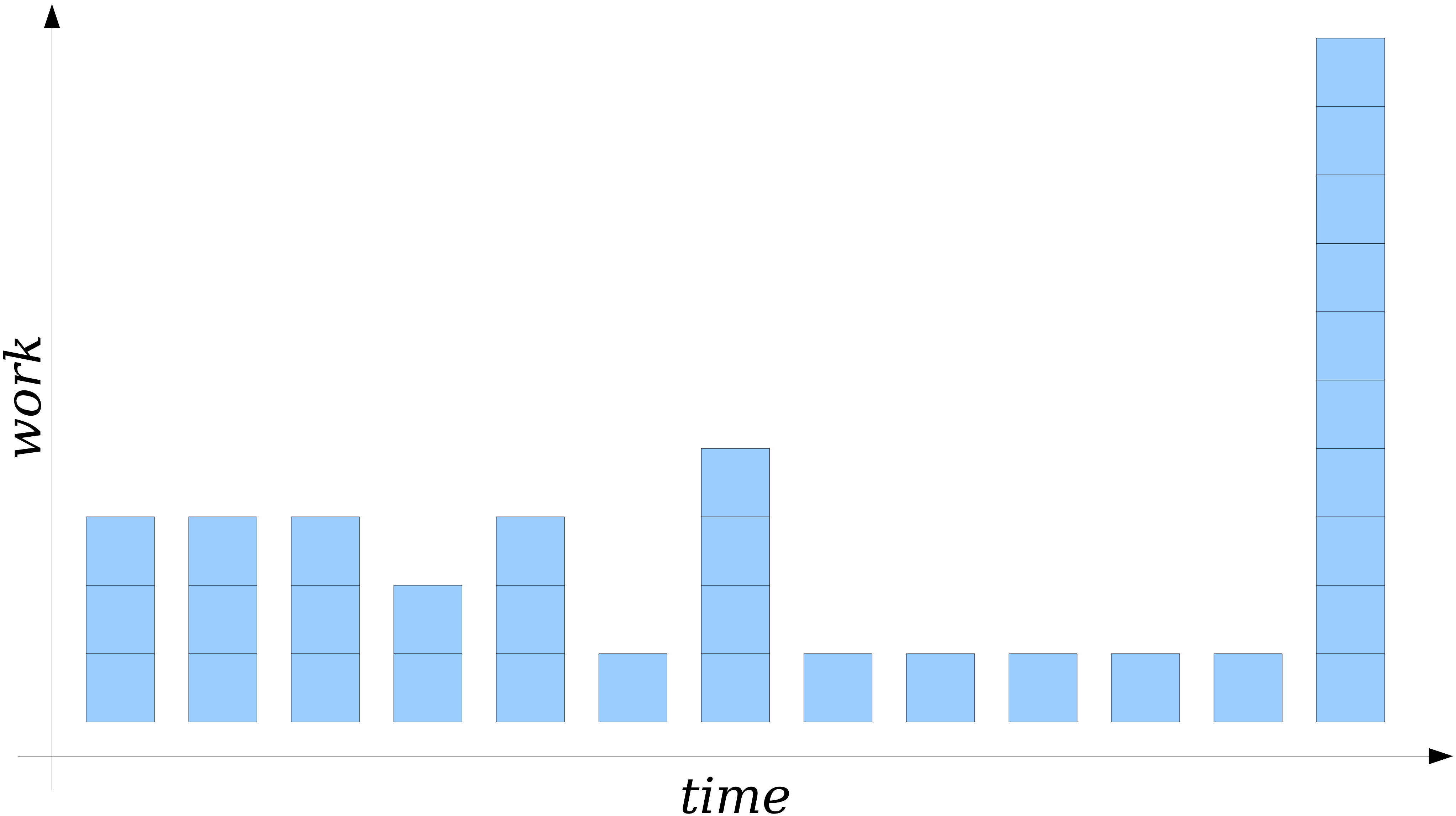

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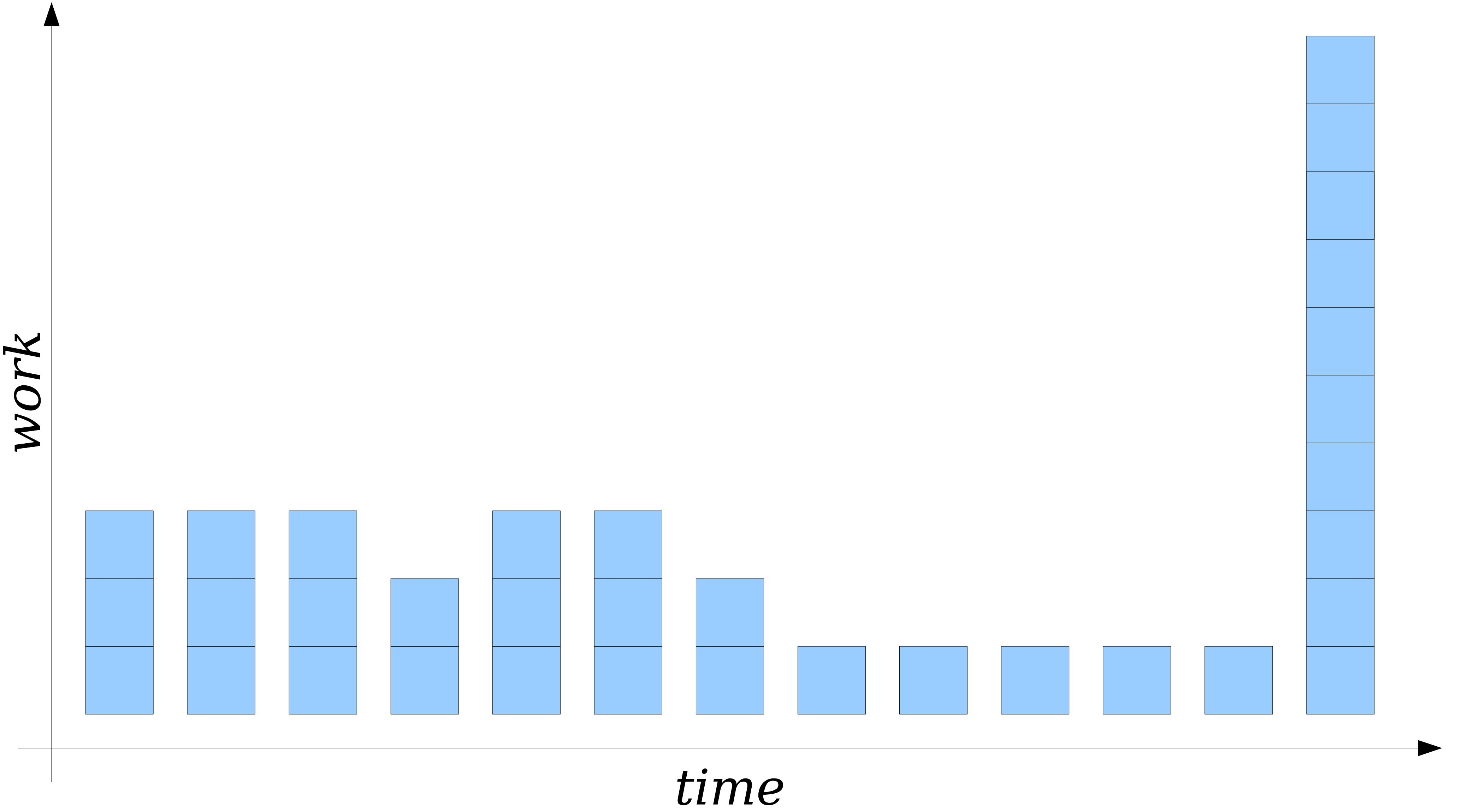

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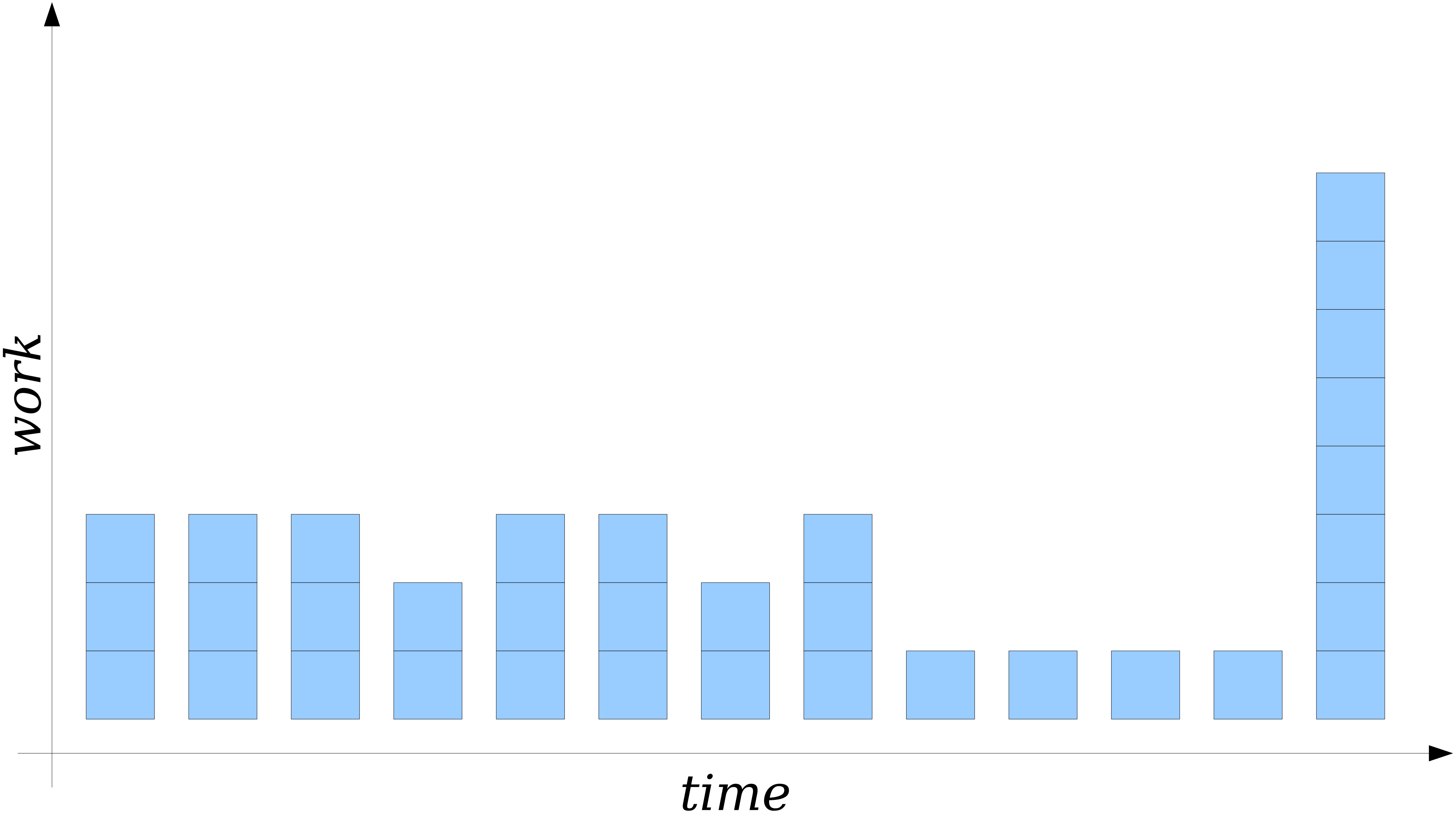

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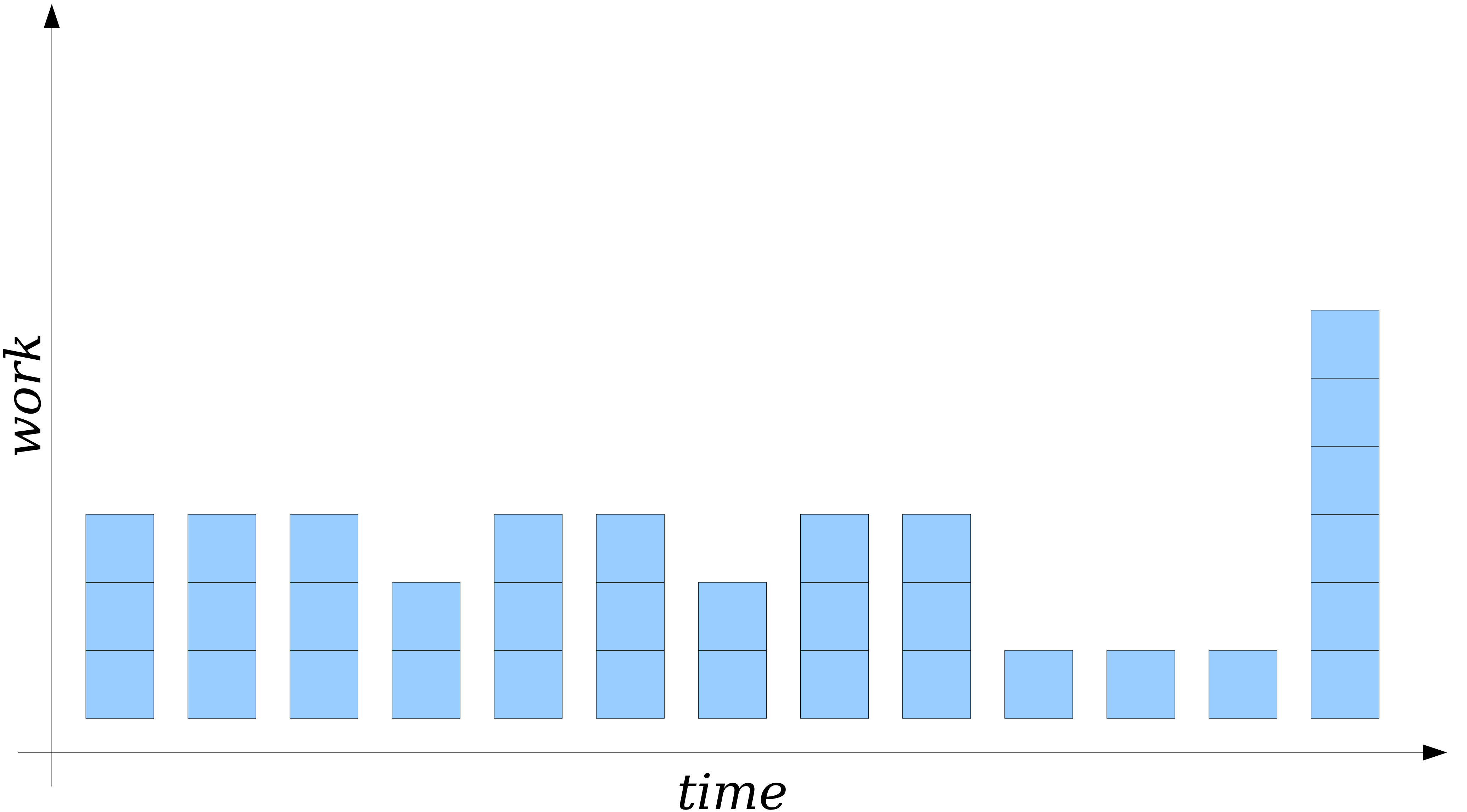

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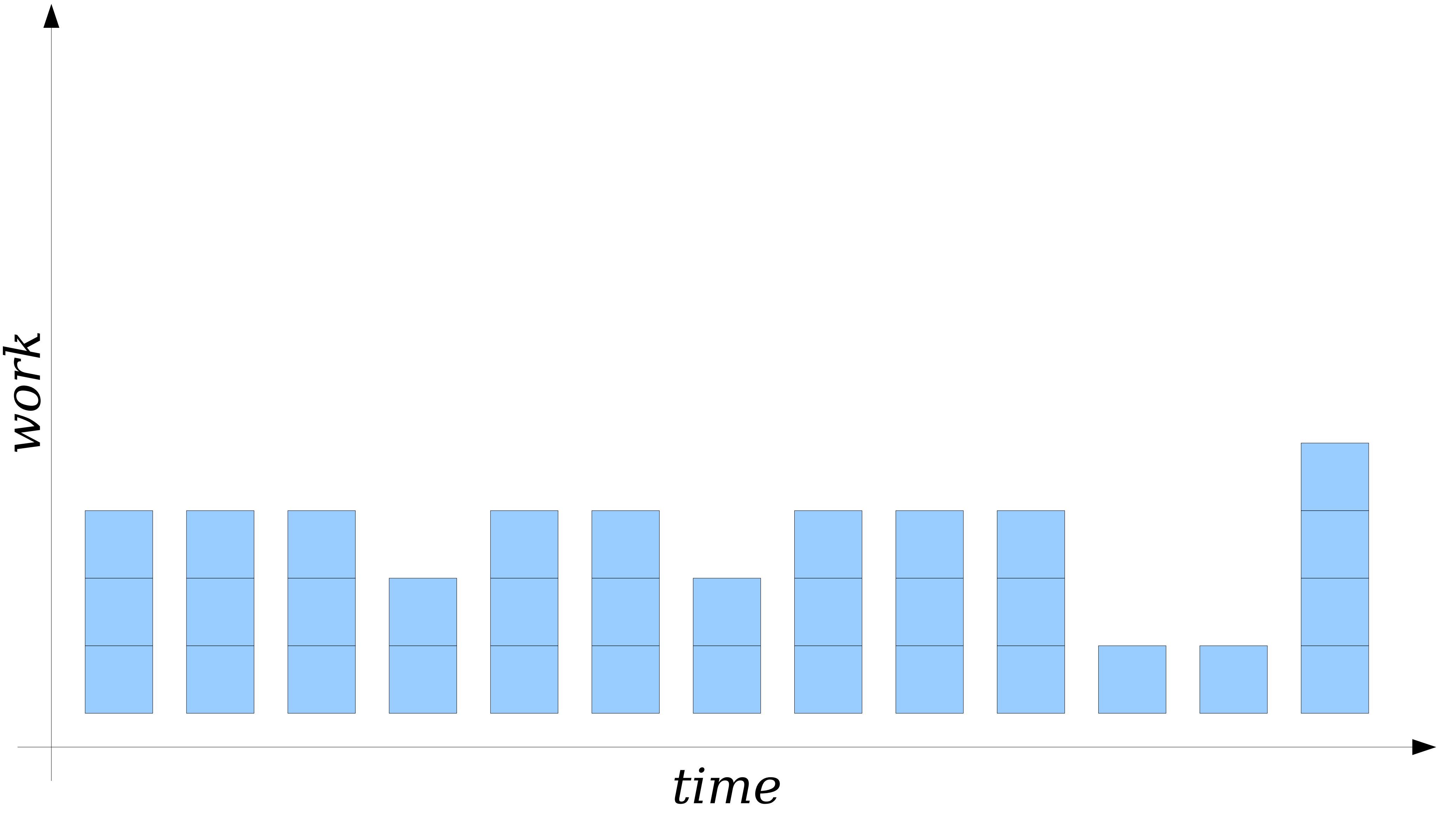

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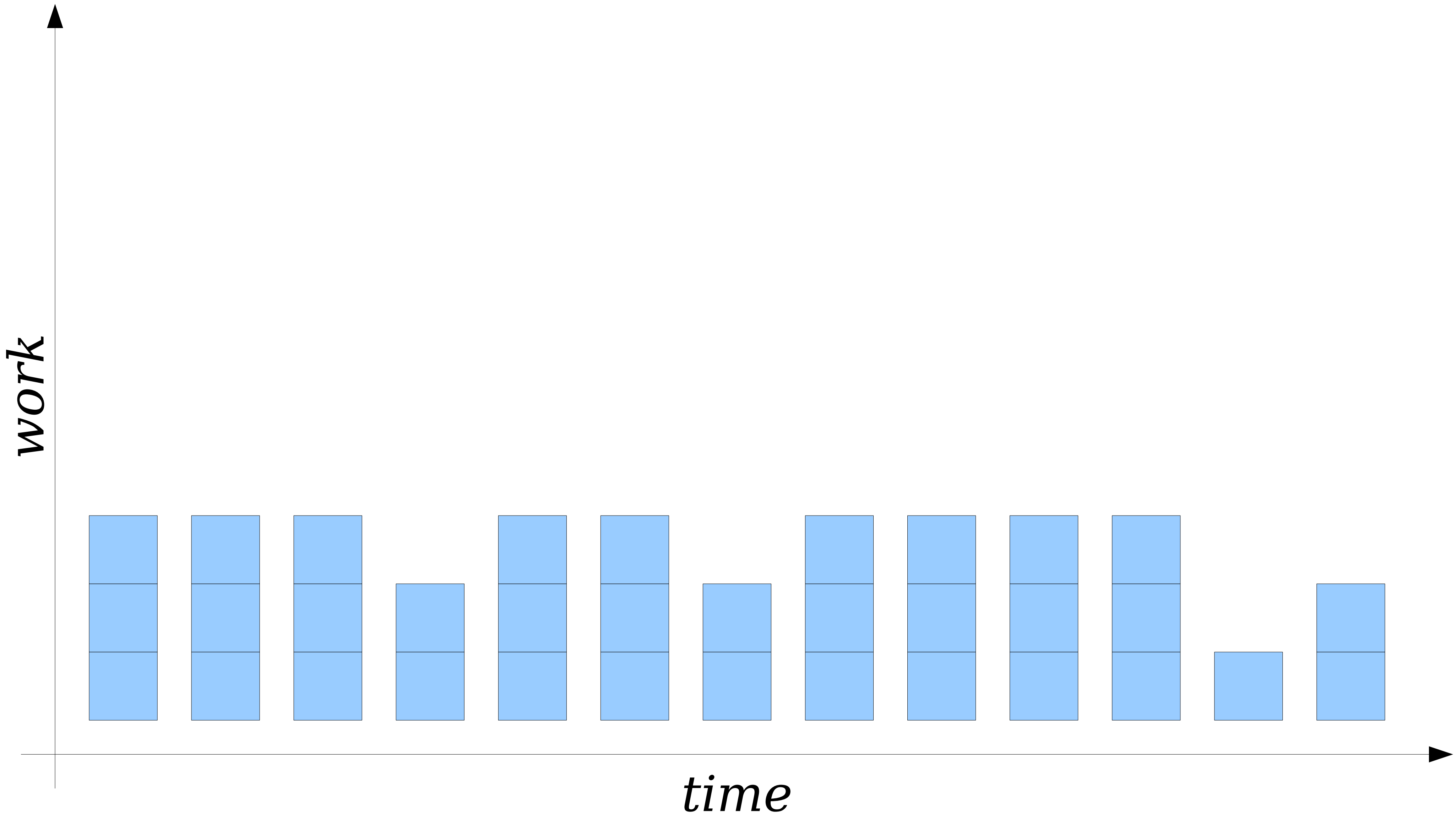

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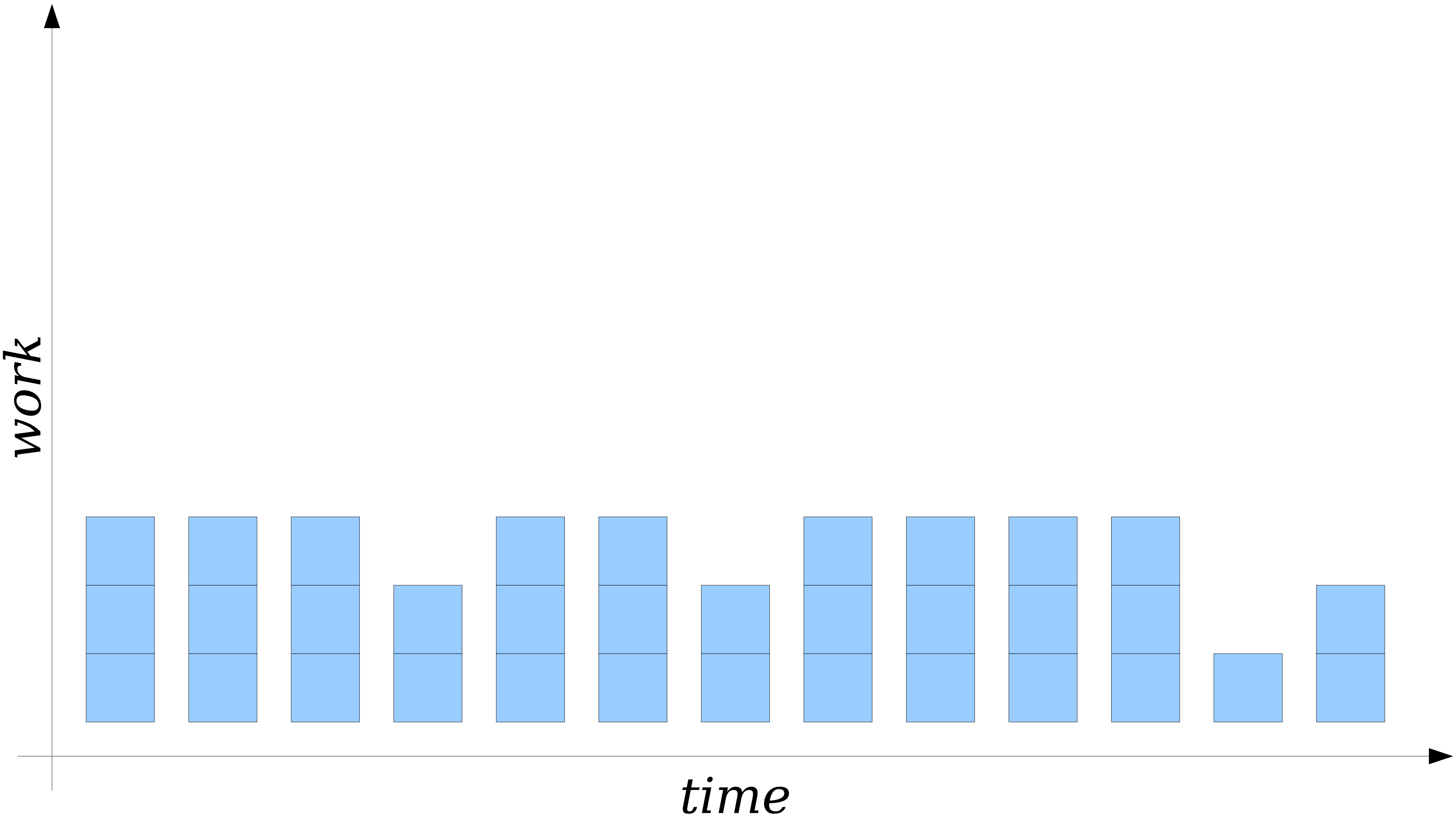
work
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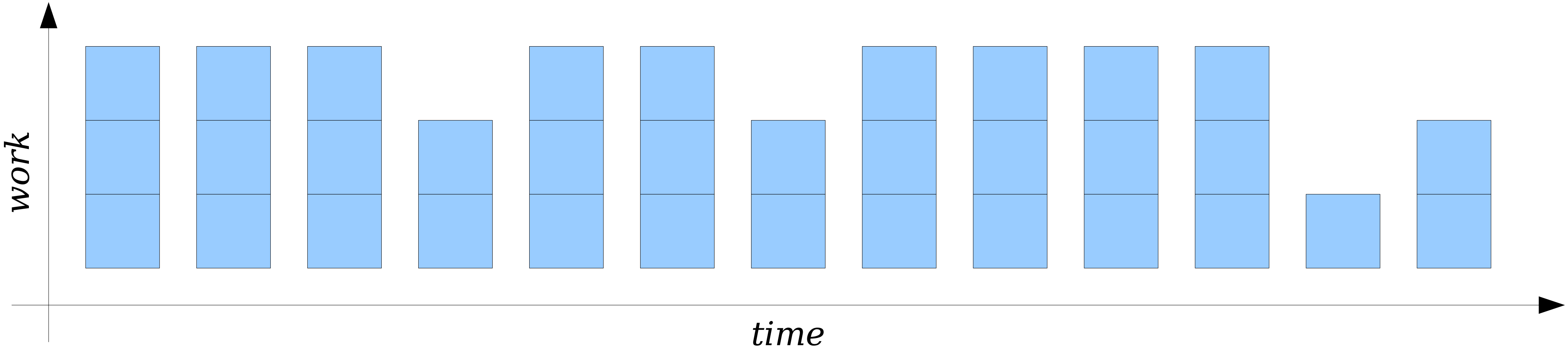
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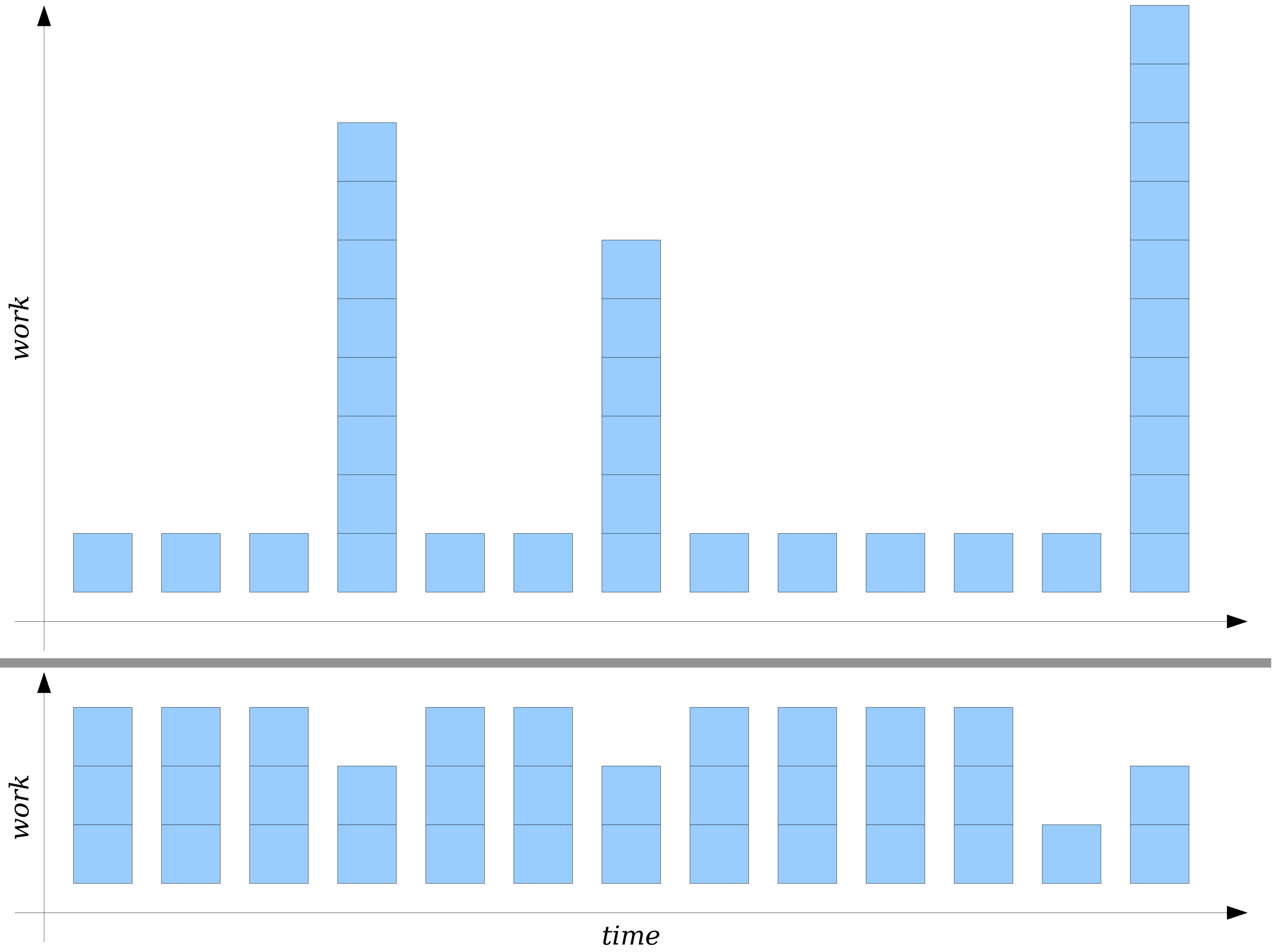
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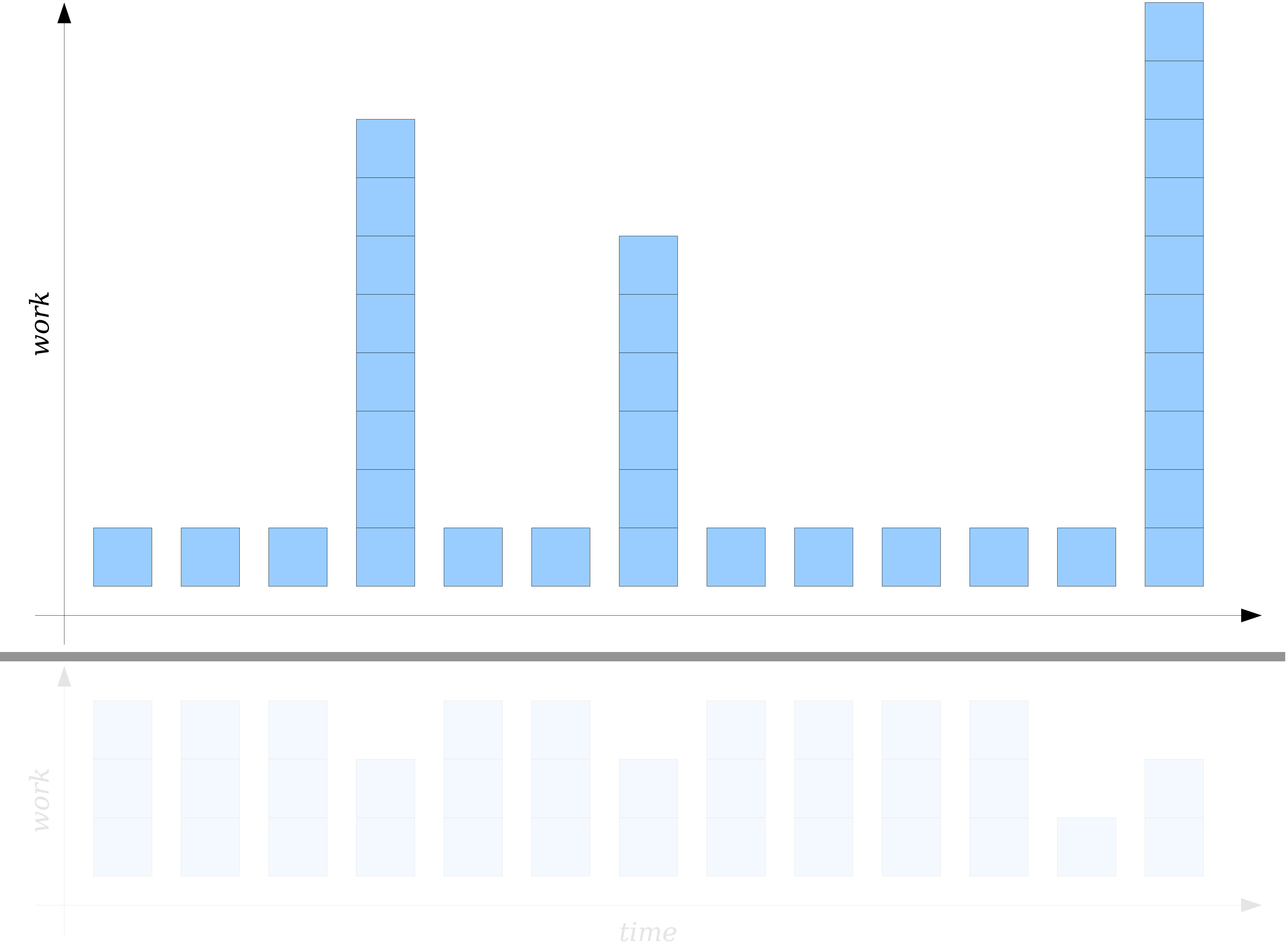
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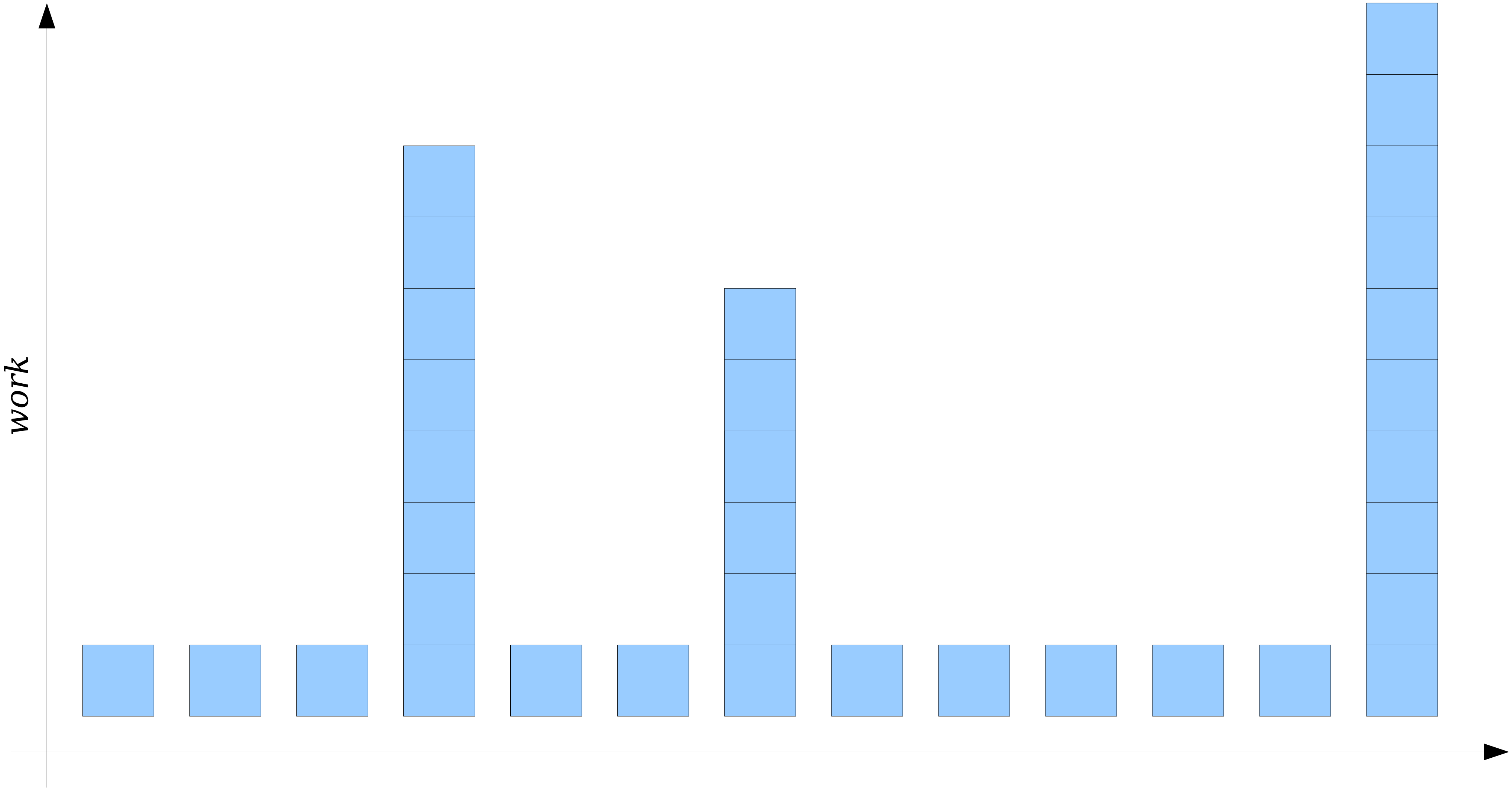
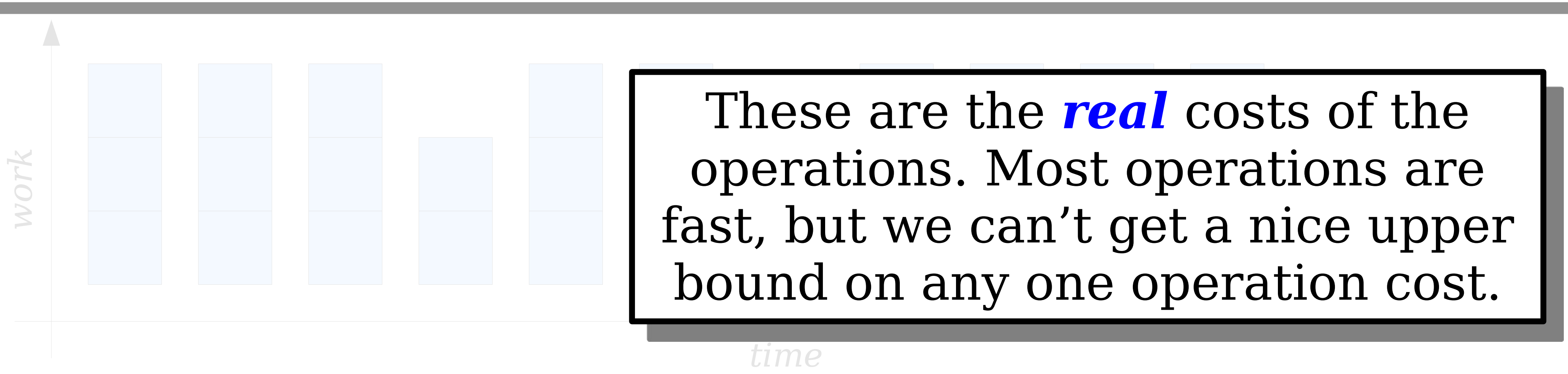
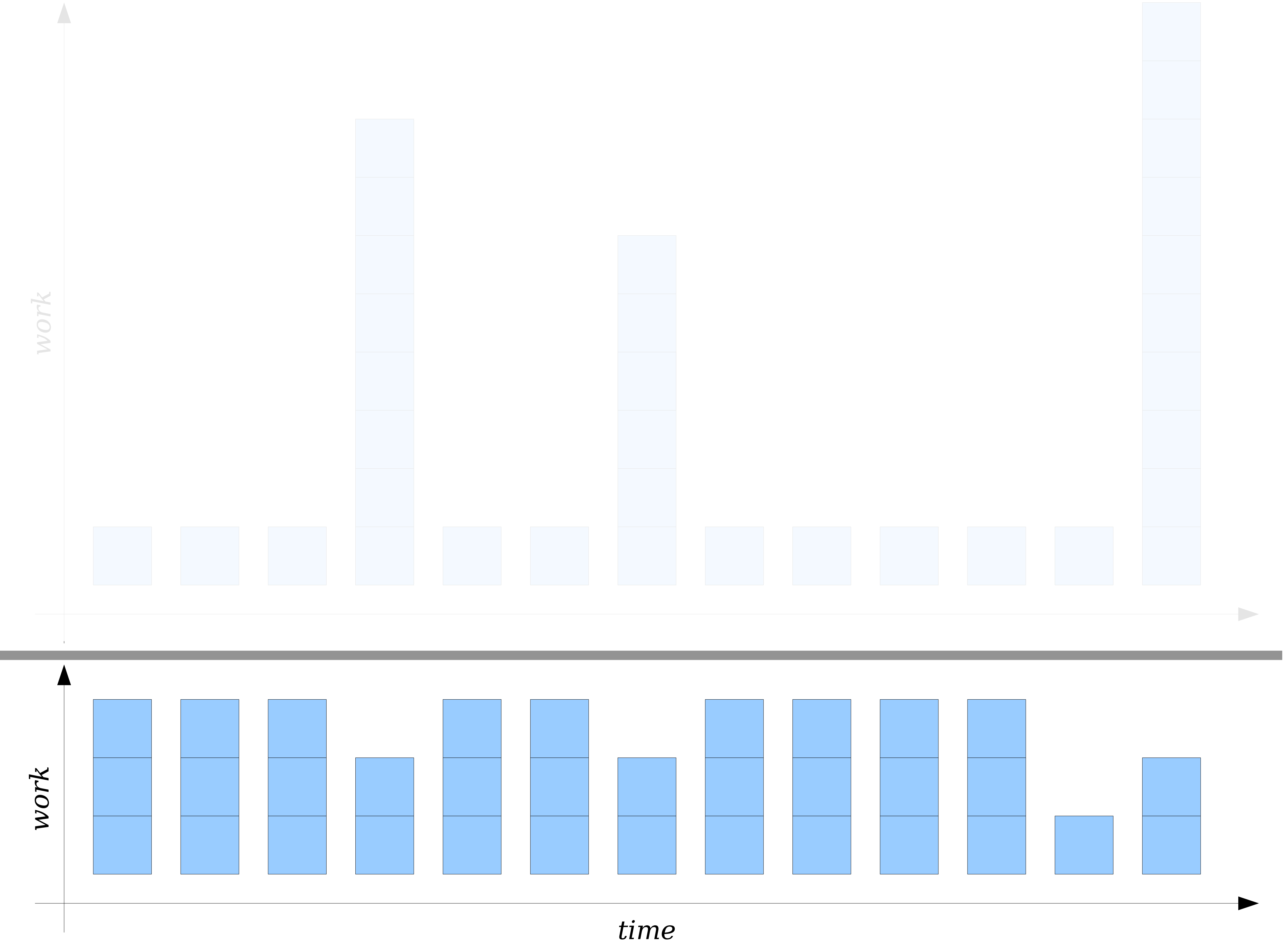




These are the ***real*** costs of the operations. Most operations are fast, but we can't get a nice upper bound on any one operation cost.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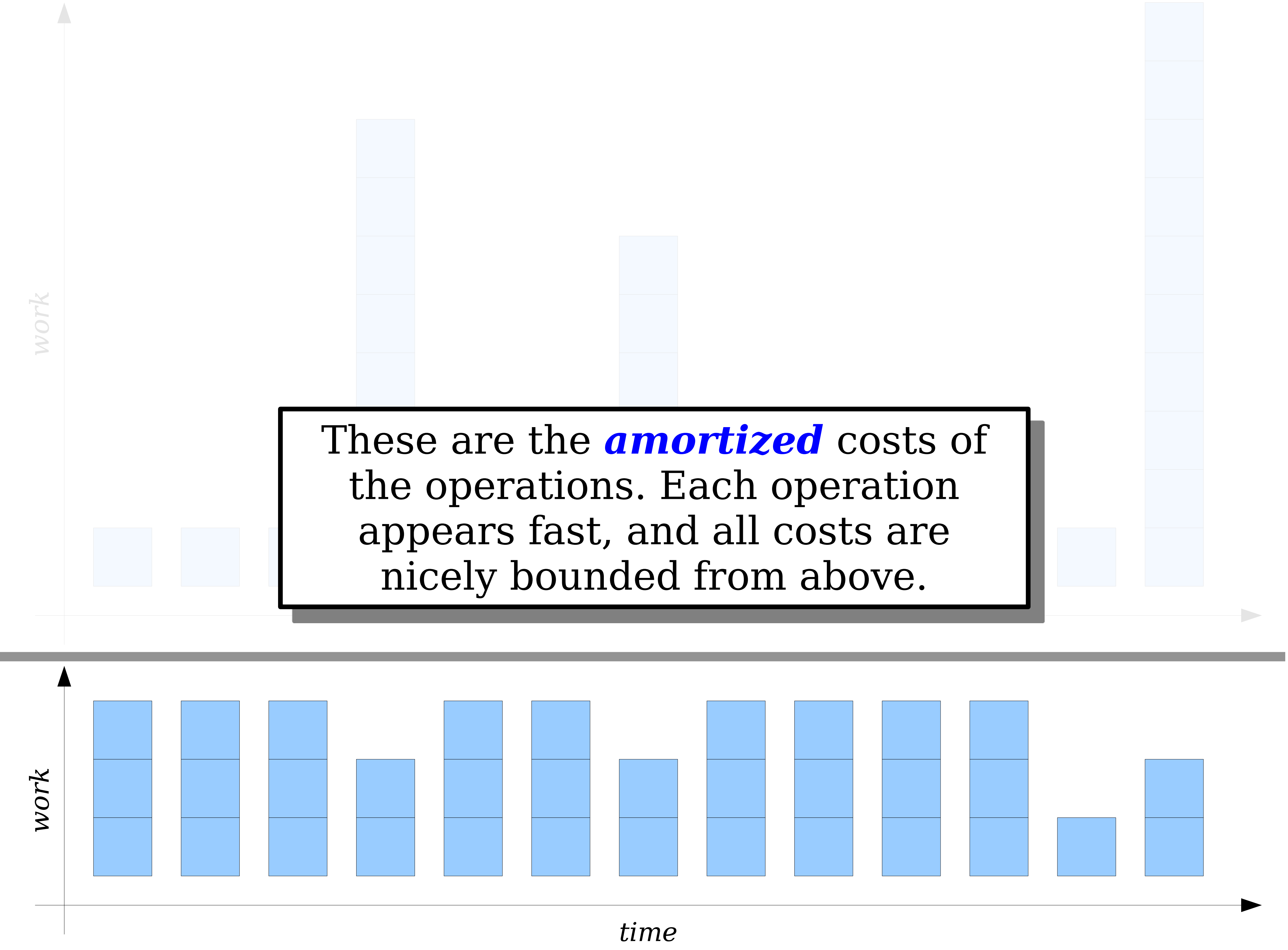

work
These are the *amortized* costs of the operations. Each operation appears fast, and all costs are nicely bounded from above.
work
time

Amortized Analysis

- ***Key Idea:*** Assign each operation a (fake!) cost called its ***amortized cost*** such that, *for any series of operations performed,* the following is true:

$$\sum \textit{amortized-cost} \geq \sum \textit{real-cost}$$

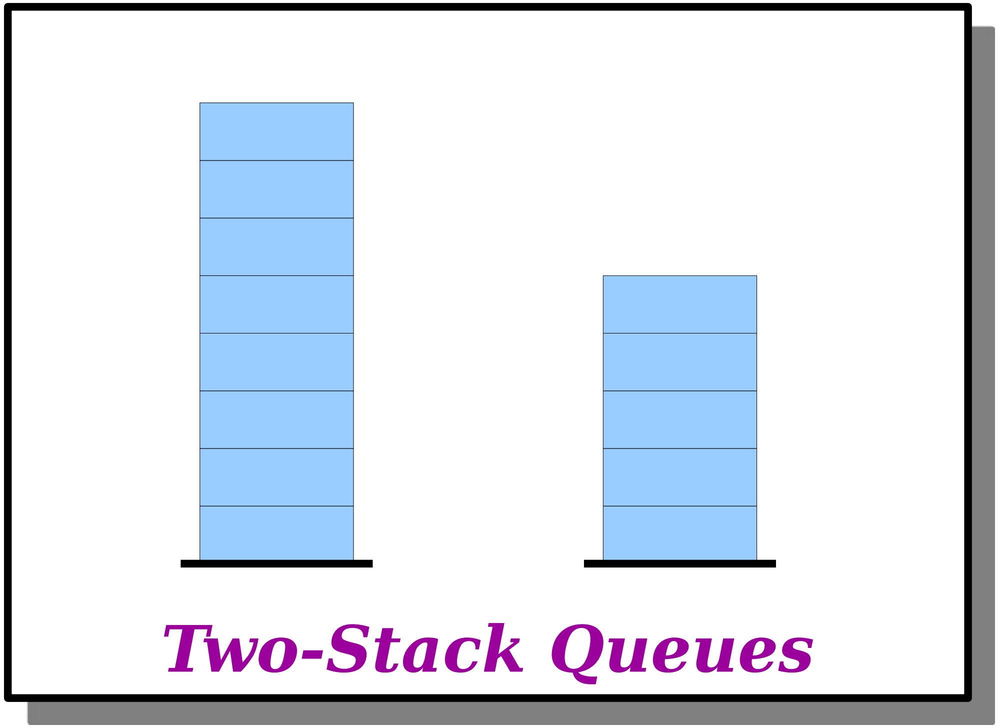
- Amortized costs shift work backwards from expensive operations onto cheaper ones.
  - Cheap operations are artificially made more expensive to pay for future cleanup work.
  - Expensive operations are artificially made cheaper by shifting the work backwards.

Where We're Going

- The ***amortized*** cost of an enqueue or dequeue into a two-stack queue is O(1).
- Any sequence of n operations on a two-stack queue will take time

$$n \cdot O(1) = O(n).$$

- However, each individual operation may take more than O(1) time to complete.

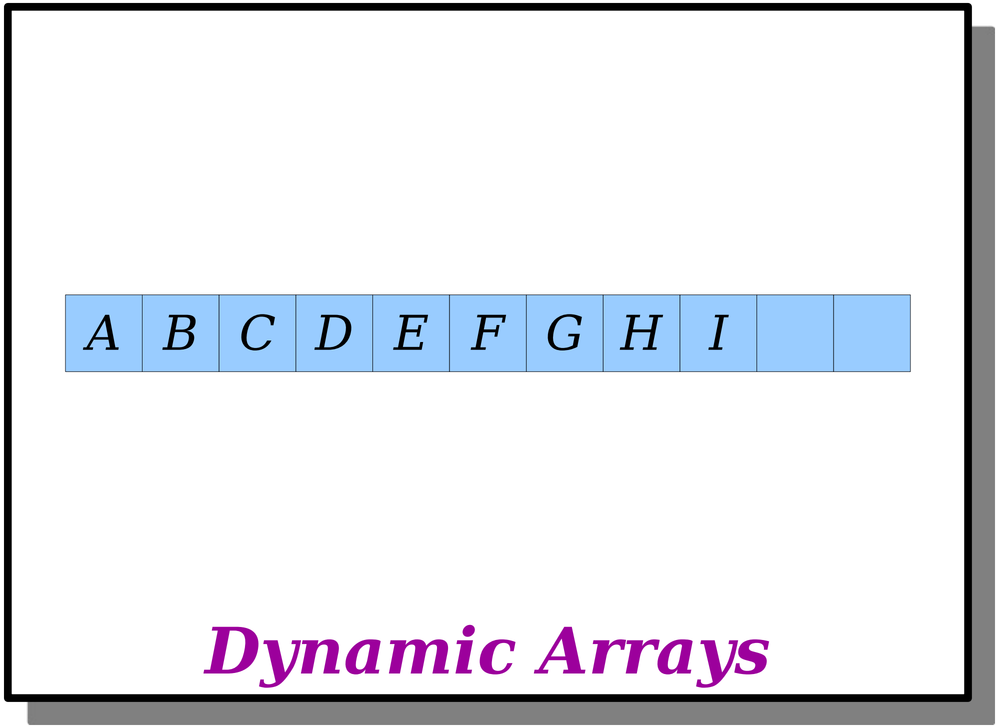


Where We're Going

- The ***amortized*** cost of appending to a dynamic array is O(1).
- Any sequence of n appends to a dynamic array will take time

$$n \cdot O(1) = O(n).$$

- However, each individual operation may take more than O(1) time to complete.

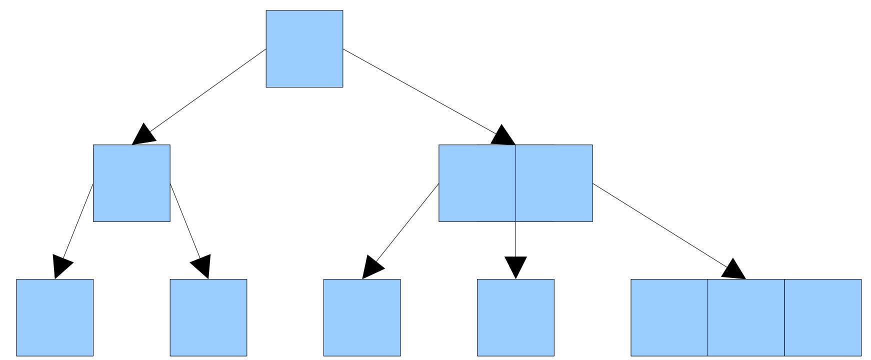


Where We're Going

- The ***amortized*** cost of inserting a new element at the end of a B-tree, assuming we have a pointer to the rightmost leaf, is O(1).
- Any sequence of *n* appends will take time

$$n \cdot O(1) = O(n).$$

- However, each individual operation may take more than O(1) time to complete.



Building B-Trees

Formalizing This Idea

Assigning Amortized Costs

- The approach we've taken so far for assigning amortized costs is called an ***aggregate analysis***.
  - Directly compute the maximum possible work done across any sequence of operations, then divide that by the number of operations.
- This approach works well here, but it doesn't scale well to more complex data structures.
  - What if different operations contribute to / clean up messes in different ways?
  - What if it's not clear what sequence is the worst-case sequence of operations?
- In practice, we tend to use a different strategy called the ***potential method*** to assign amortized costs.

Potential Functions

- To assign amortized costs, we'll need to measure how "messy" the data structure is.
- For each data structure, we define a ***potential function*** Φ that, in a sense, "quantifies messiness."
  - Φ is small when the data structure is "clean," and
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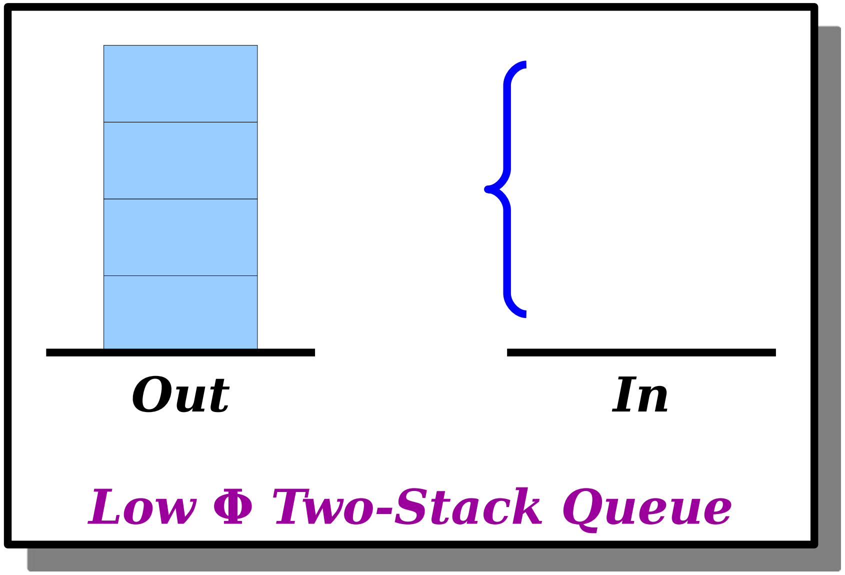

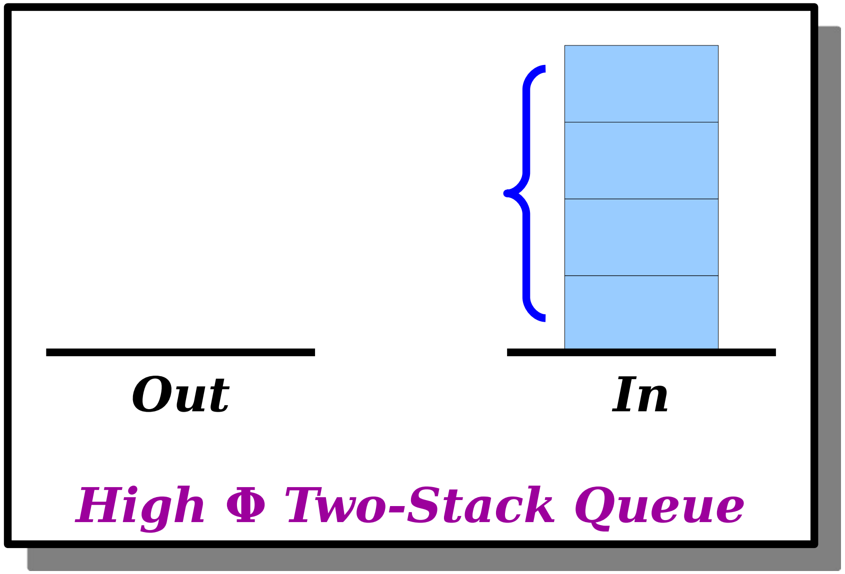

Potential Functions

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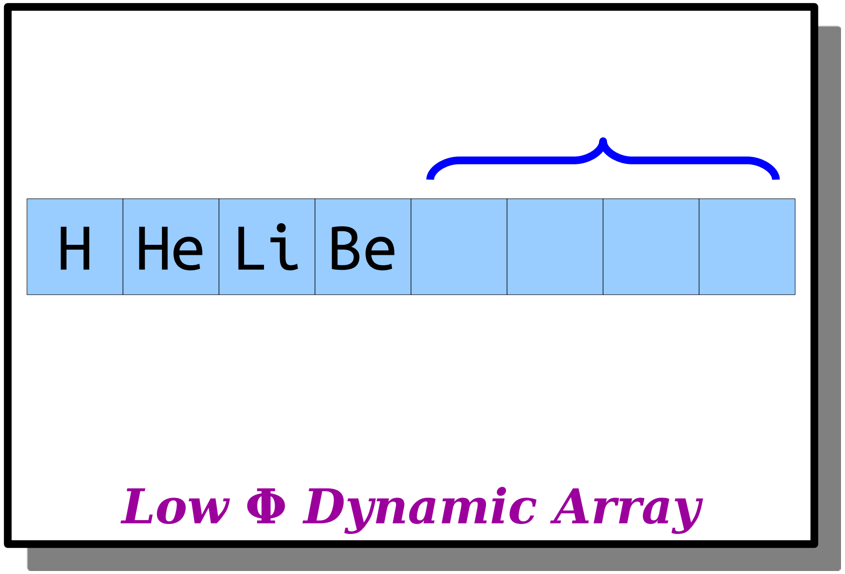

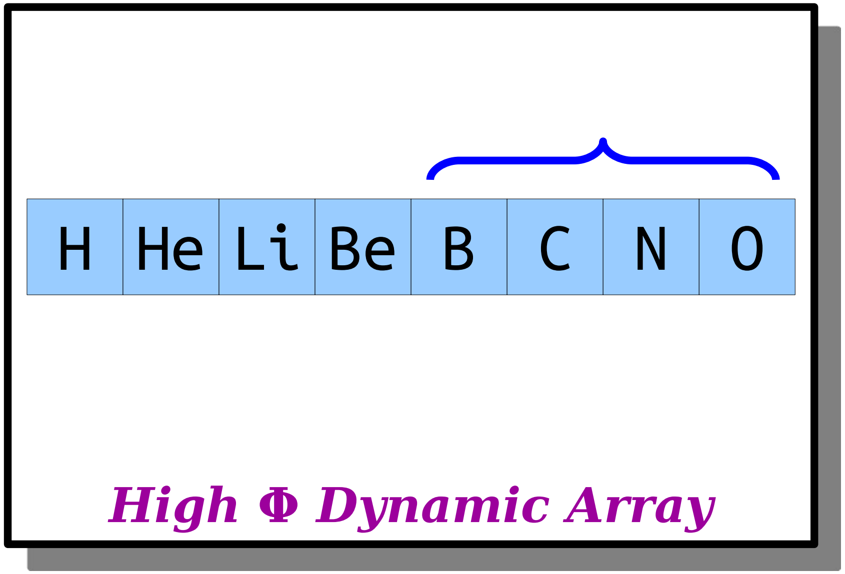

Potential Functions

- Once we have Φ, we can start looking, for each operation, at how Φ changes.
  - If an operation makes things "messier," then Φ increases.
  - If an operation makes things "cleaner," then Φ decreases.
- What we want to have happen:
  - If an operation increases Φ, we artificially raise its cost.
  - If an operation decreases Φ, we artificially lower its cost.
- ***Why?***

Answer at
https://pollev.com/cs166spr23

Potential Functions

- Define the amortized cost of an operation to be

  $$\textbf{\textit{amortized-cost = real-cost + k}} \cdot \Delta\Phi$$

  where k is a constant under our control and $\Delta\Phi$ is the difference between Φ just after the operation finishes and Φ just before the operation started:

  $$\Delta\Phi = \Phi_{after} - \Phi_{before}$$

- Intuitively:
  - If Φ increases, the data structure got "messier," and the amortized cost is *higher* than the real cost to account for future cleanup costs.
  - If Φ decreases, the data structure got "cleaner," and the amortized cost is *lower* than the real cost

Why This Works

$$\sum \textit{amortized-cost} \;=\; \sum \left(\textit{real-cost} + k \cdot \Delta\Phi\right)$$

Why This Works

$$\begin{aligned}\sum amortized\text{-}cost &= \sum \left(real\text{-}cost + k \cdot \Delta\Phi\right)\\ &= \sum real\text{-}cost + k \cdot \sum \Delta\Phi\end{aligned}$$

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Think "fundamental theorem of calculus,"
but for discrete derivatives!

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\sum_{x=a}^{b} \Delta f(x) = f(b+1) - f(a)$$

Look up ***finite calculus*** if you're curious to learn more!

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Let's make two assumptions:

$$\Phi \geq 0.$$

$$\Phi_{start} = 0.$$

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$$\geq \sum real\text{-}cost$$

Assigning costs this way will never, in any circumstance, overestimate the total amount of work done.

The Story So Far

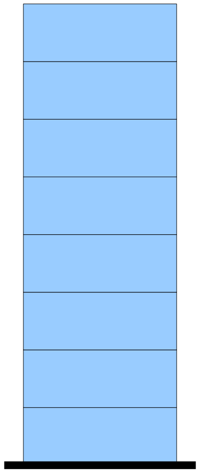
- We will assign amortized costs to each operation such that

$$\sum \textit{amortized-cost} \geq \sum \textit{real-cost}$$

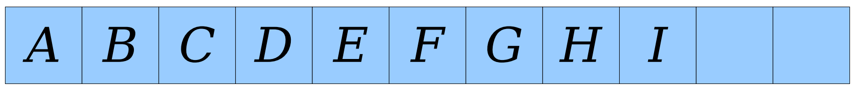
- To do so, define a ***potential function*** Φ such that
  - Φ measures how "messy" the data structure is,
  - $\Phi_{start} = 0$, and
  - $\Phi \geq 0$.
- Then, define amortized costs of operations as

$$\textit{amortized-cost} = \textit{real-cost} + k \cdot \Delta\Phi$$

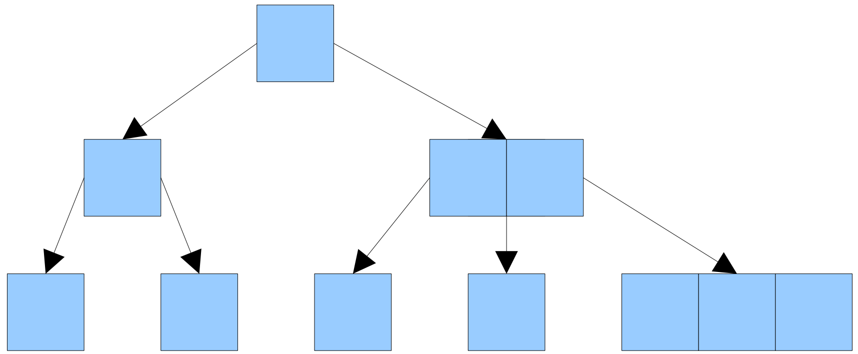
  for a choice of k under our control.



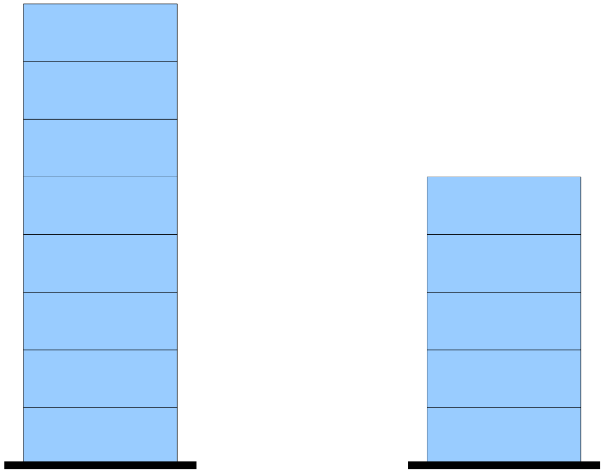
Two-Stack Queues



Dynamic Arrays



Building B-Trees



Two-Stack Queues

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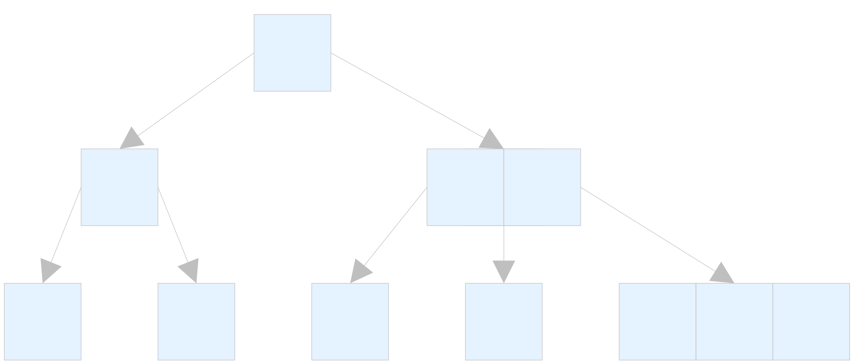

Building B-Trees

The Two-Stack Queue

Out **In**

The Two-Stack Queue

Φ = height of ***In*** stack

Out

In

The Two-Stack Queue

Φ = height of ***In*** stack

The Two-Stack Queue

Φ = height of ***In*** stack

$$\textit{amortized-cost} = \textit{real-cost} + k \cdot \Delta\Phi$$

The Two-Stack Queue

Φ = height of ***In*** stack

$$\begin{aligned} \textit{amortized-cost} &= \textit{real-cost} + k \cdot \Delta\Phi \\ &= O(1) + k \cdot 1 \end{aligned}$$

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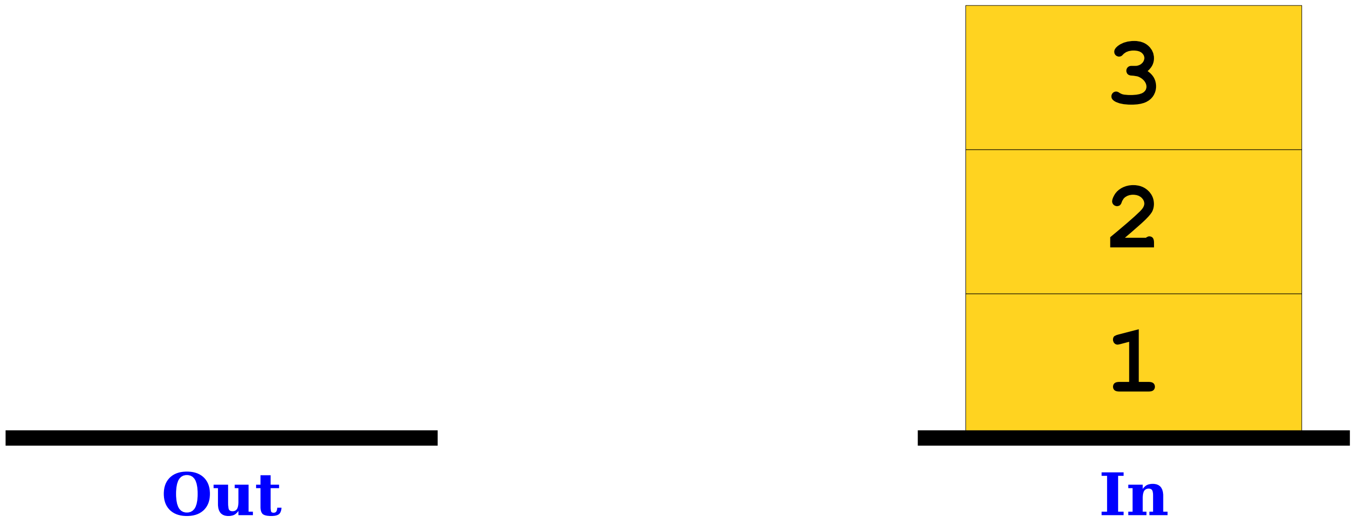

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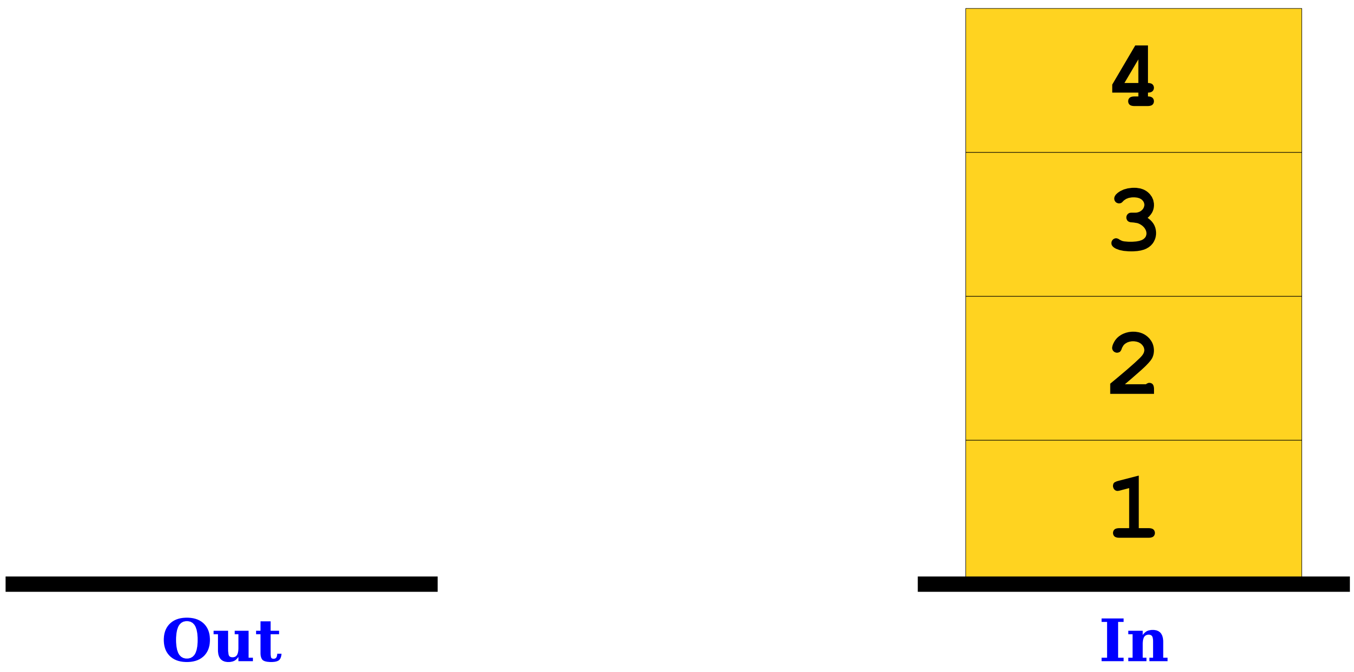

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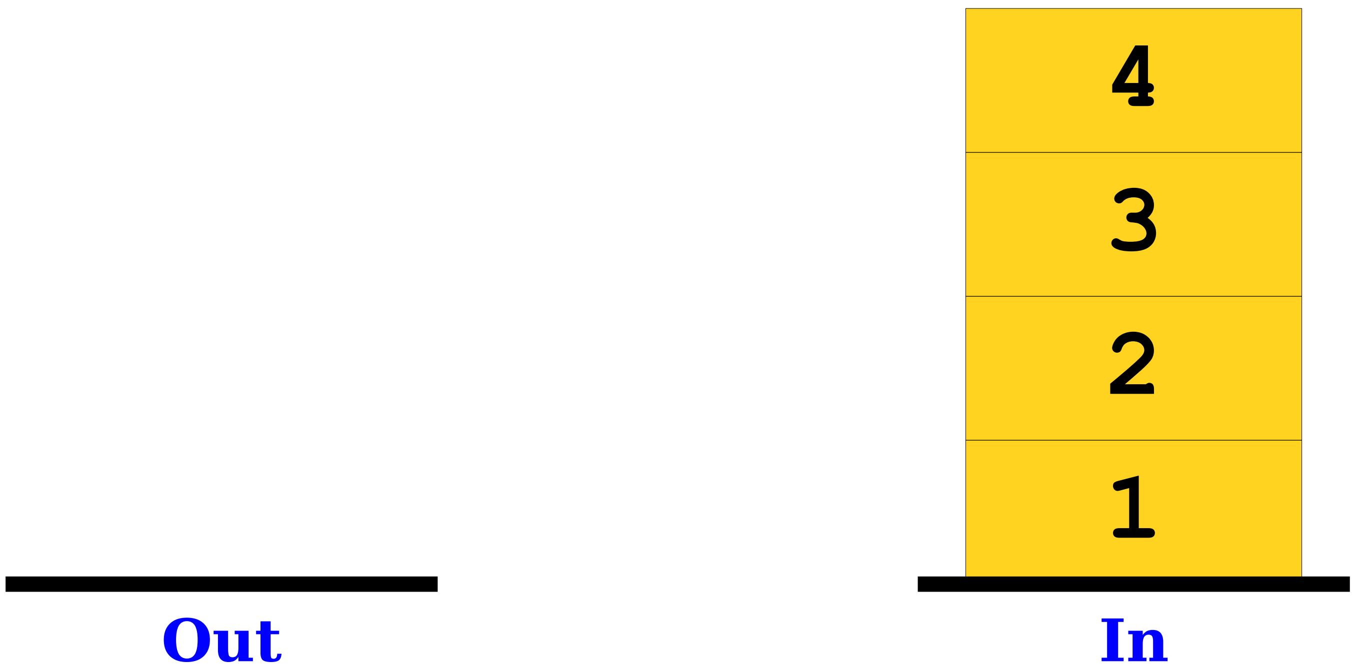

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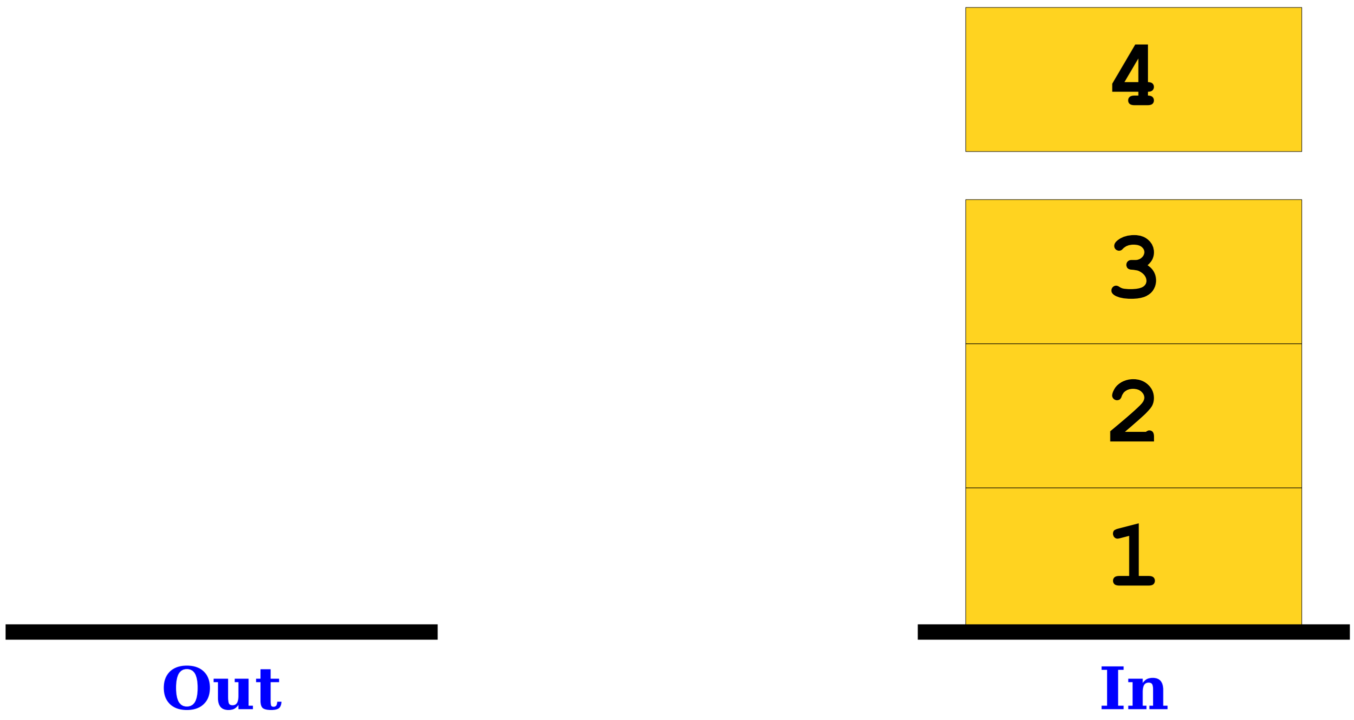

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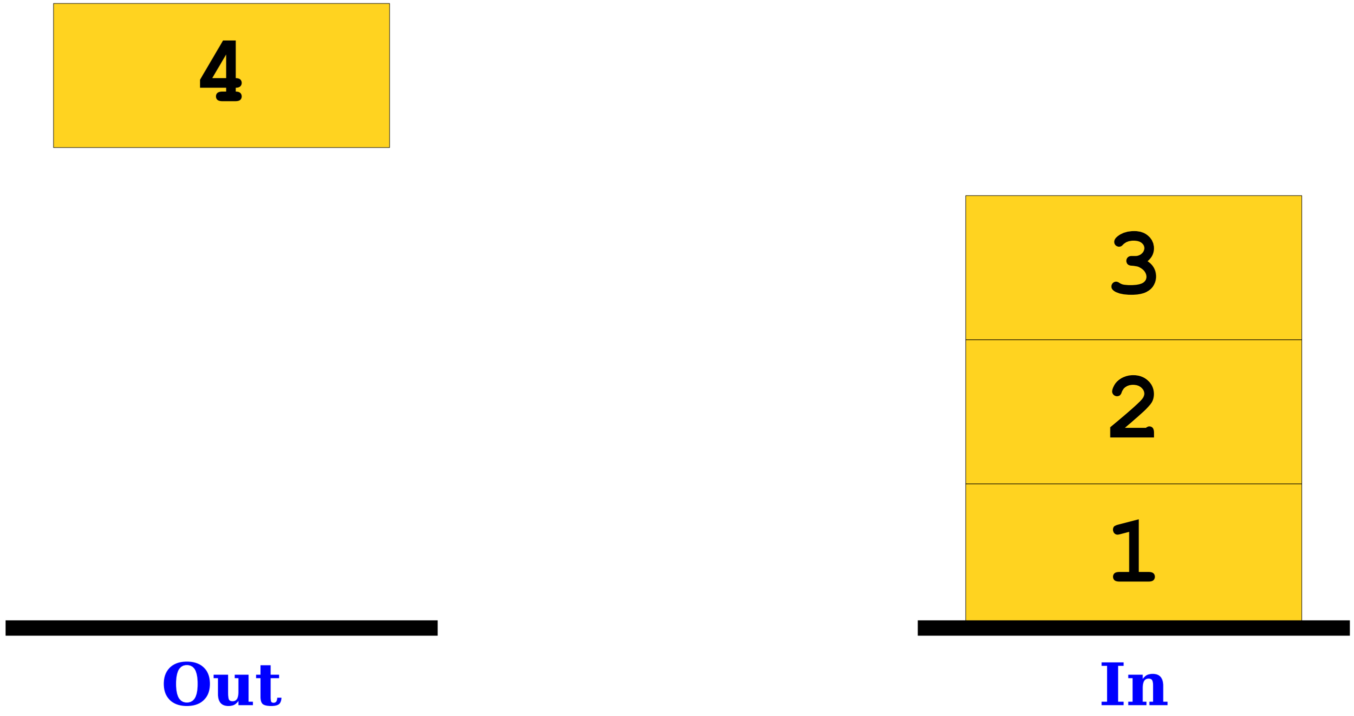

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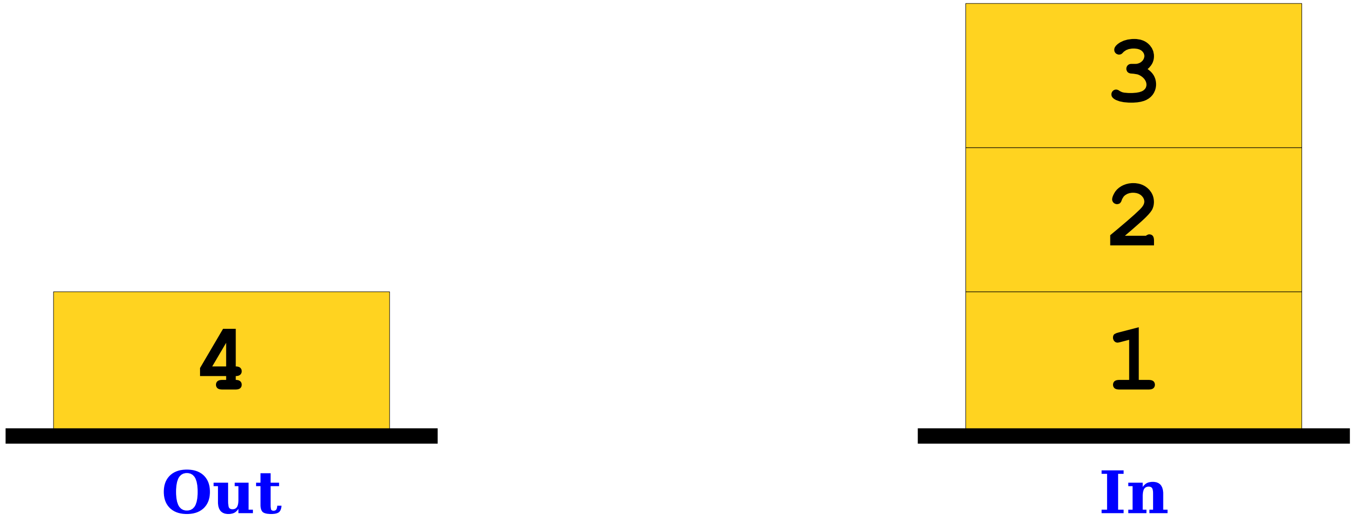

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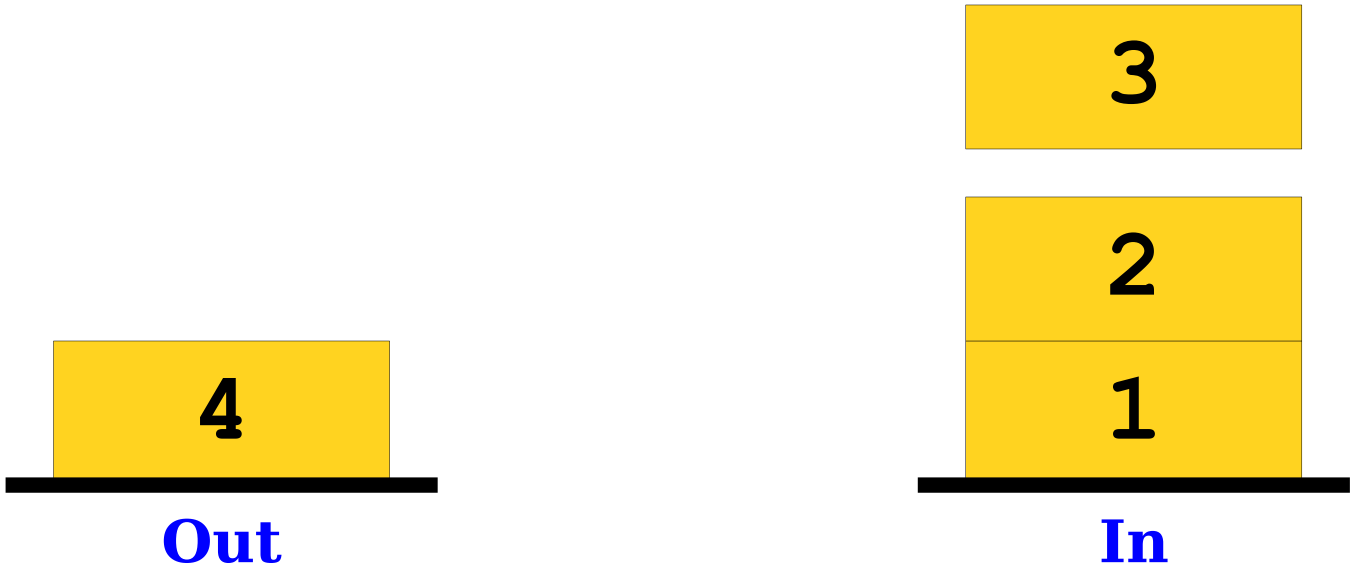

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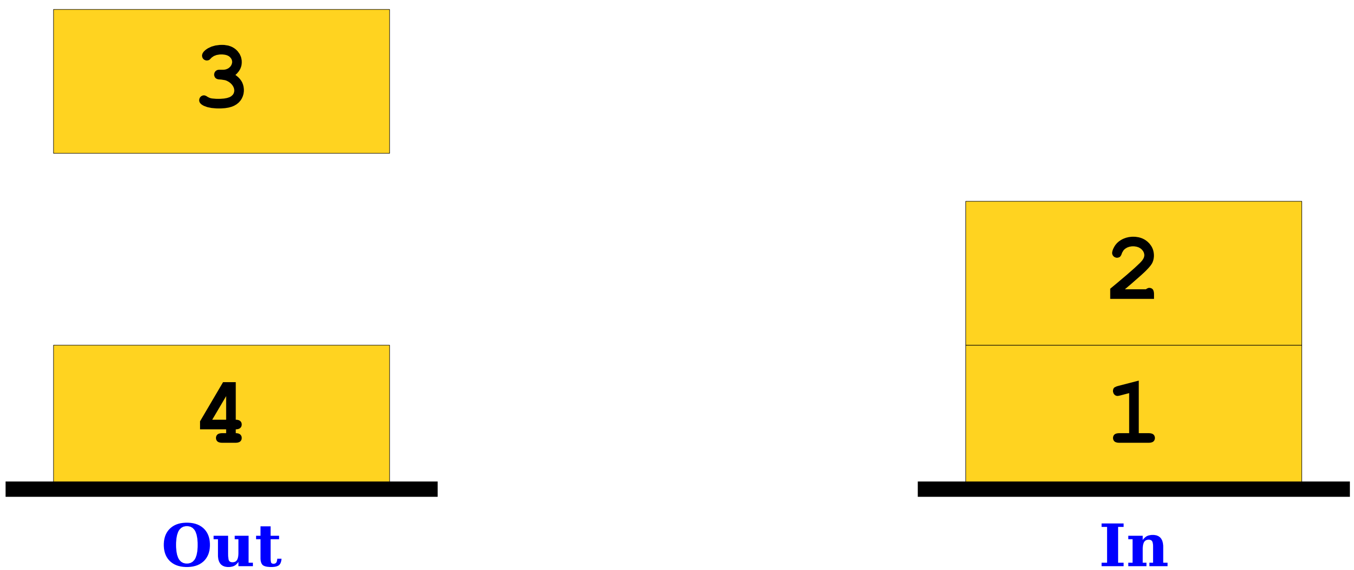

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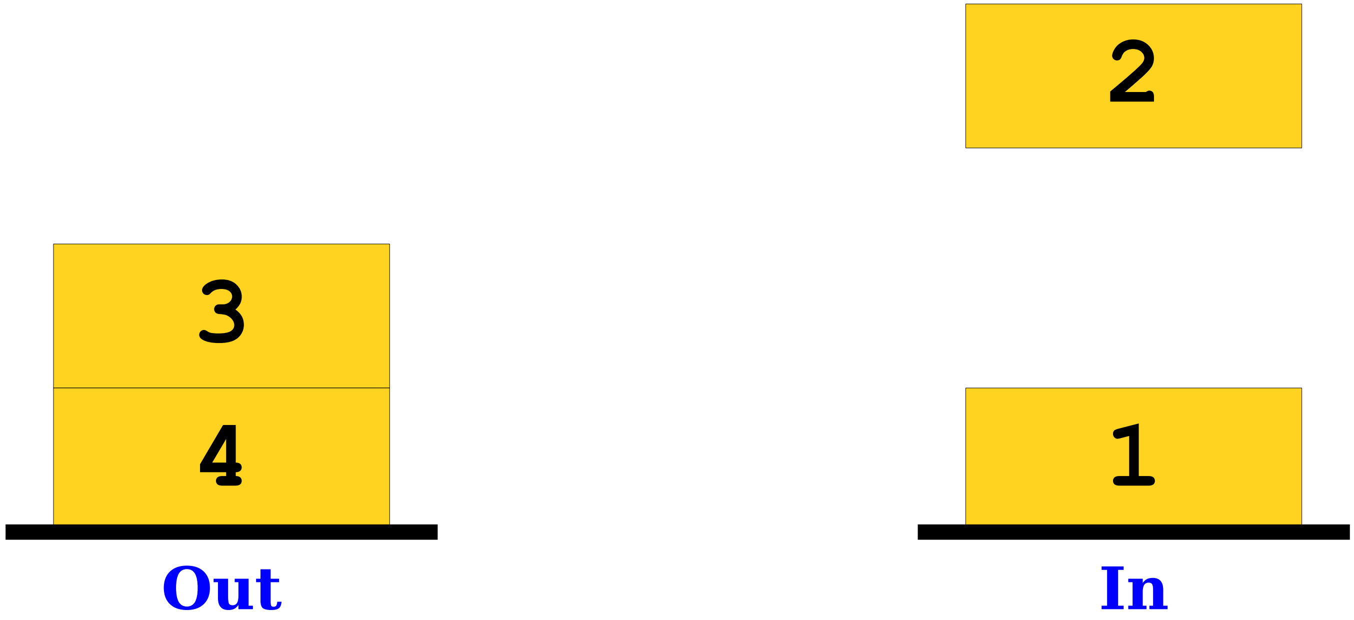

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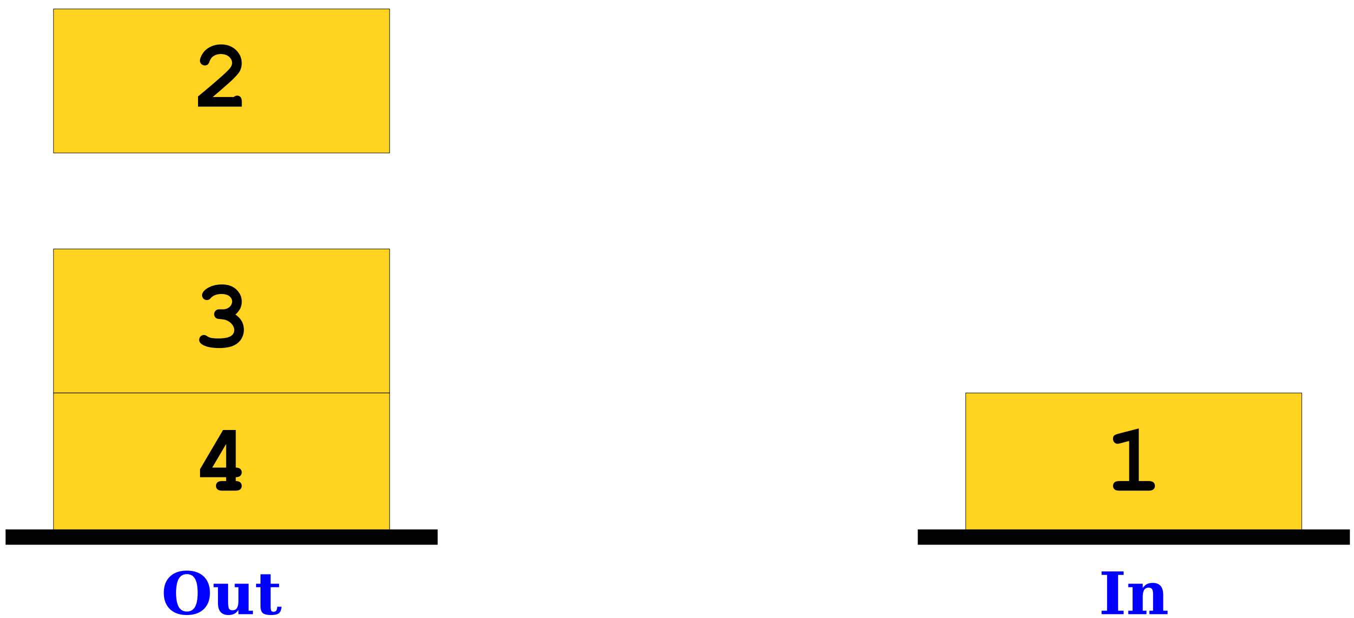

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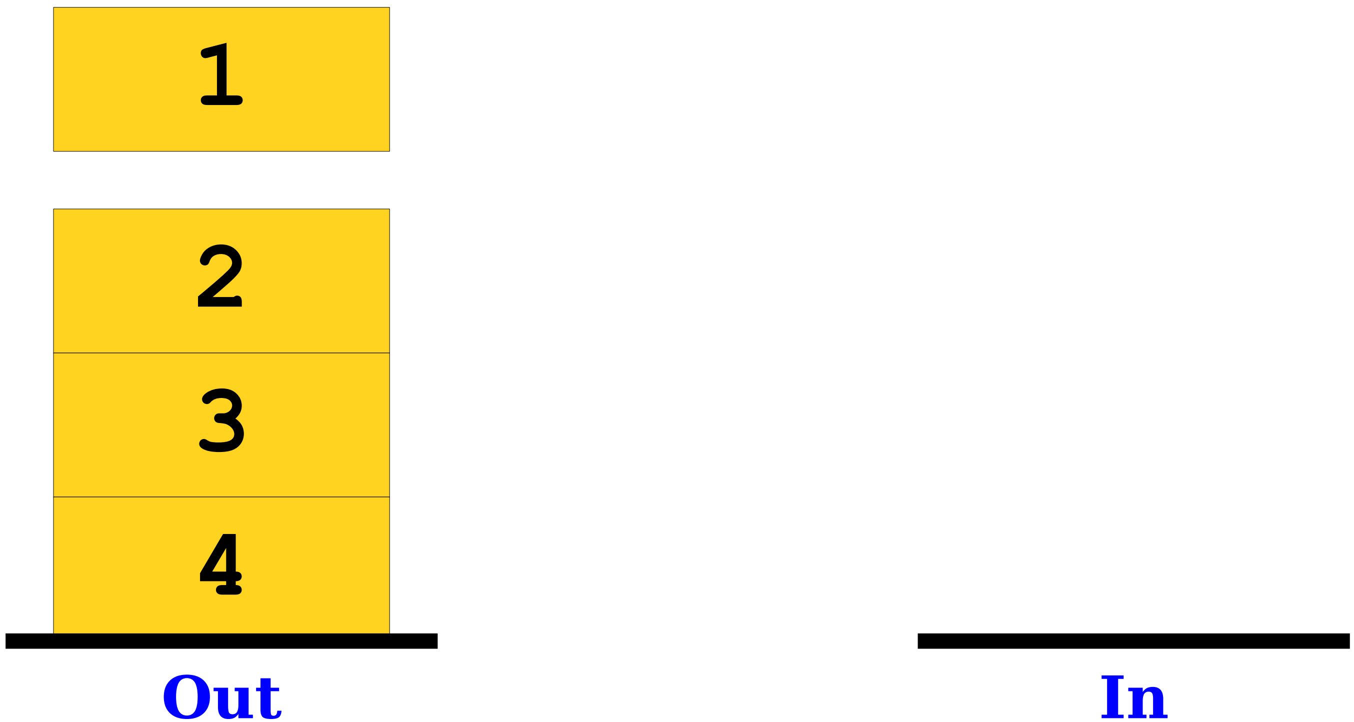

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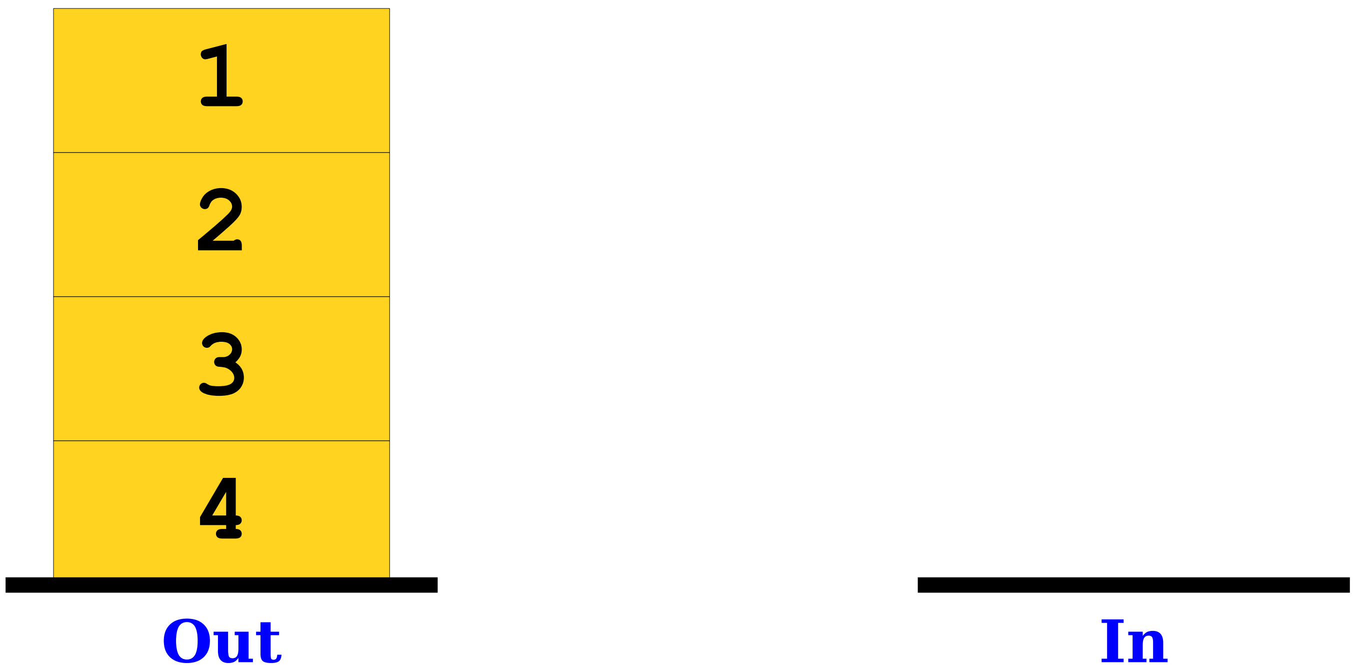

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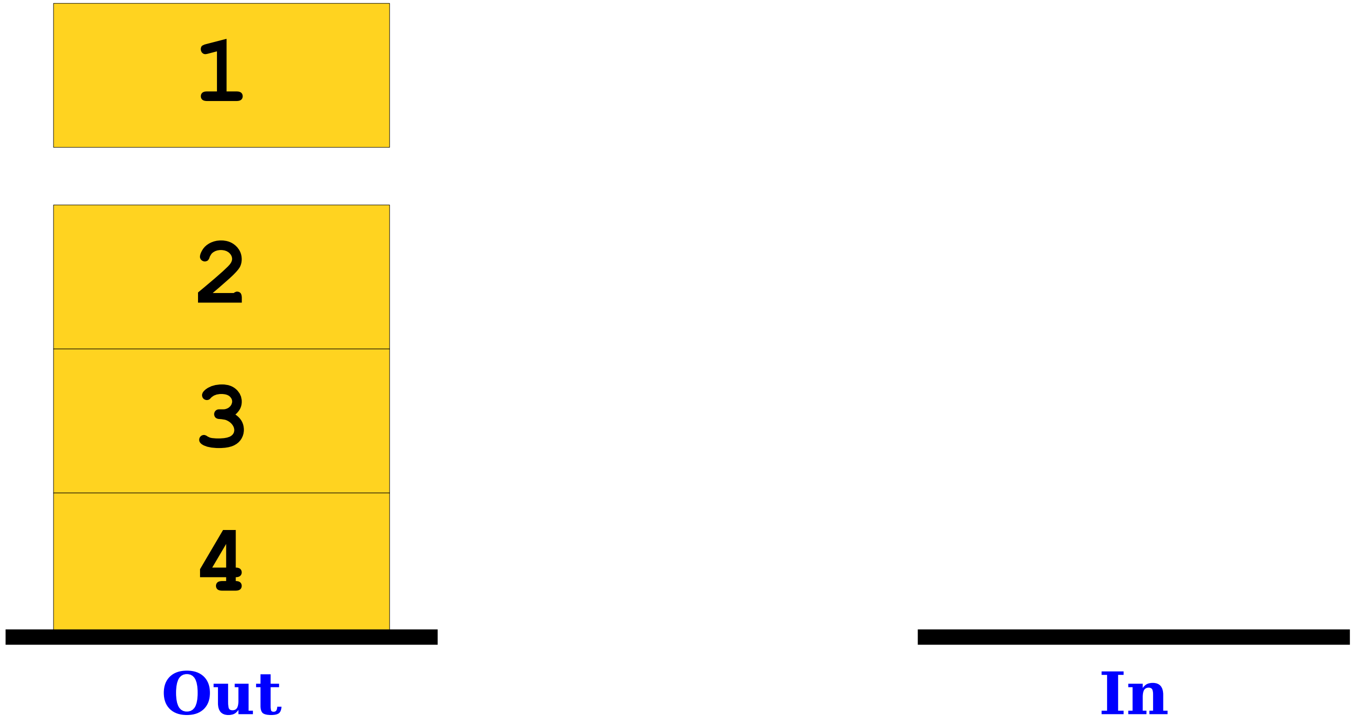

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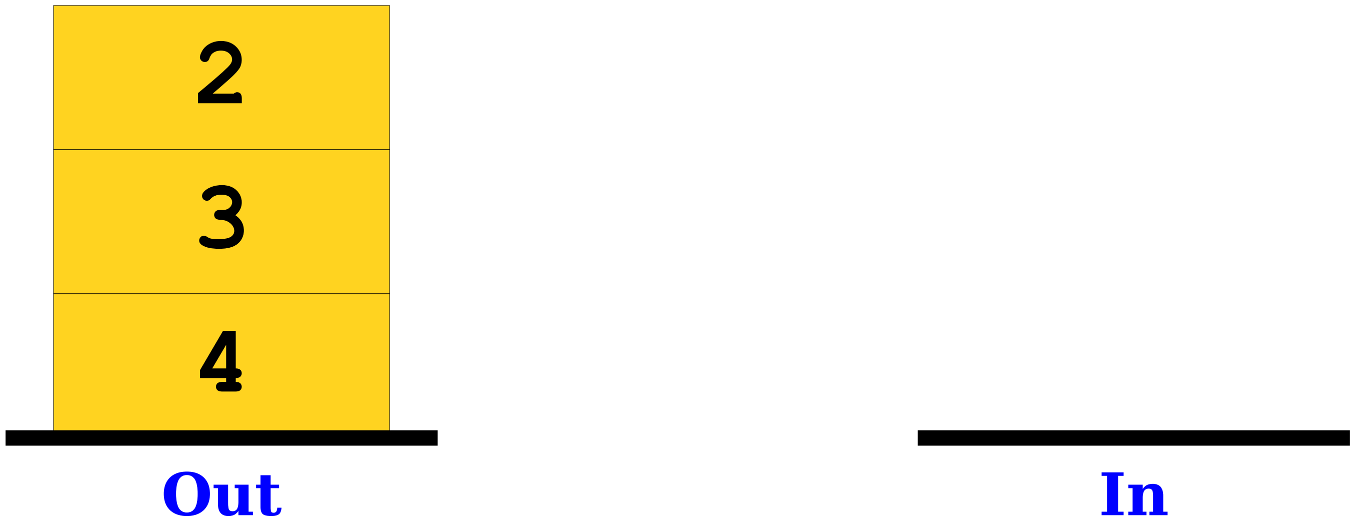

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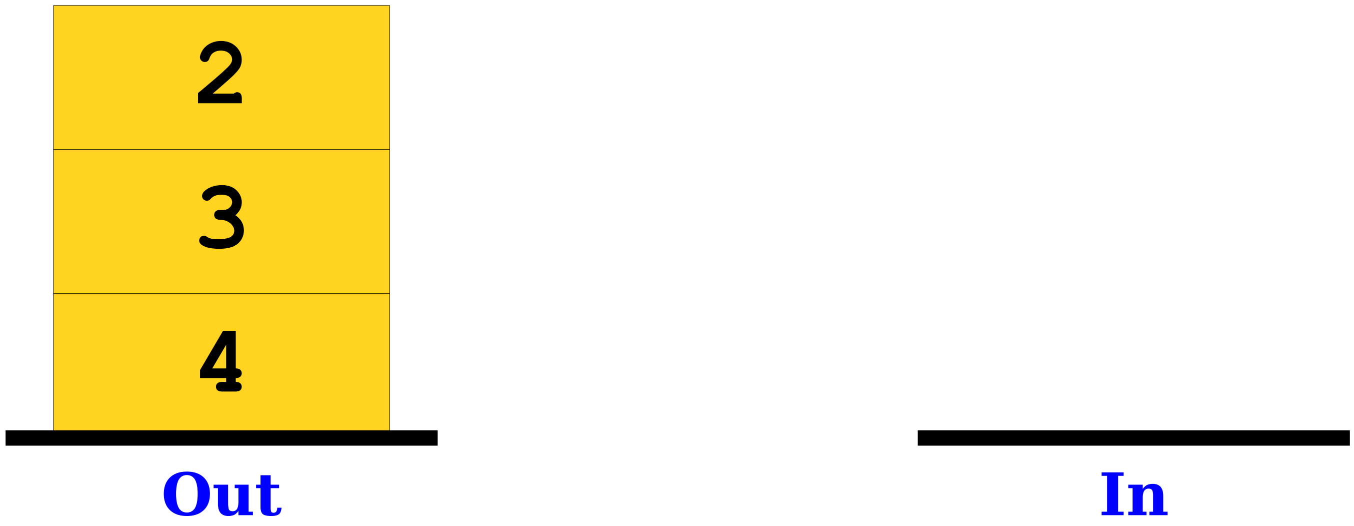

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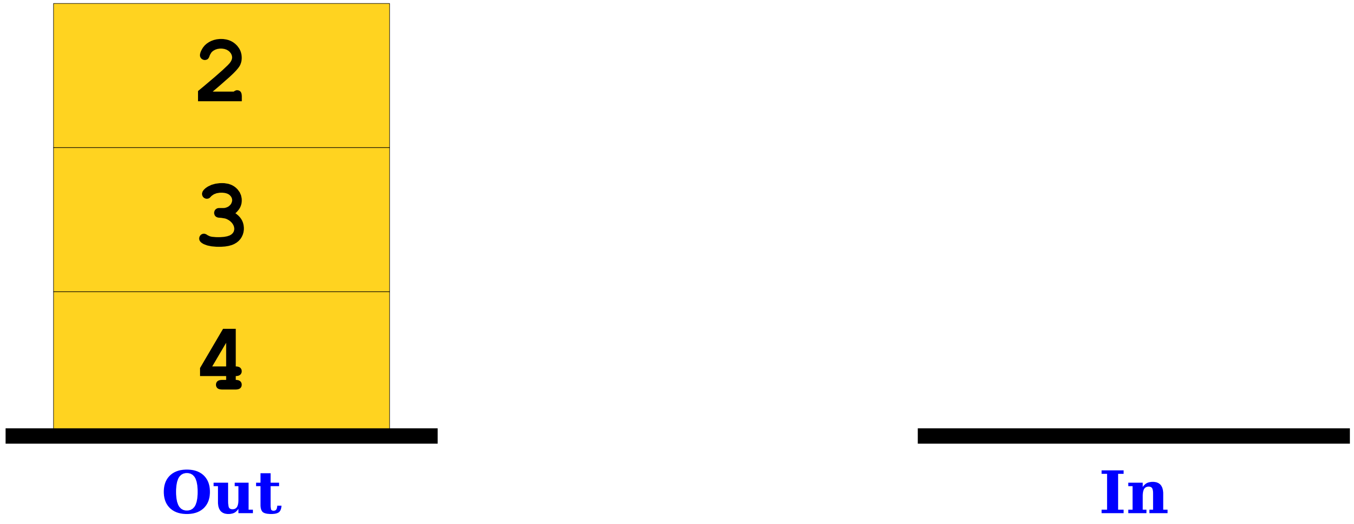

amortized-cost $= \textit{real-cost} + k \cdot \Delta\Phi$
$= O(h) + k \cdot -h$ // h = height of ***In*** stack

The Two-Stack Queue

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amortized-cost = *real-cost* + $k \cdot \Delta\Phi$
= $O(h) + k \cdot -h$ // h = height of ***In*** stack
= **O(1)** // Choose k strategically

The Two-Stack Queue

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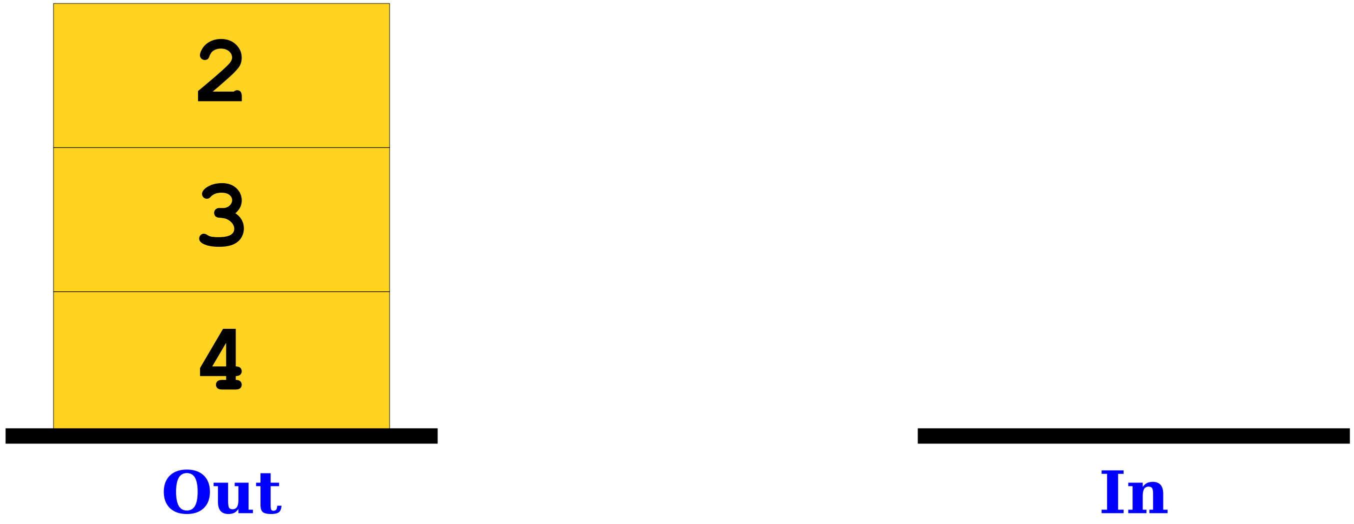

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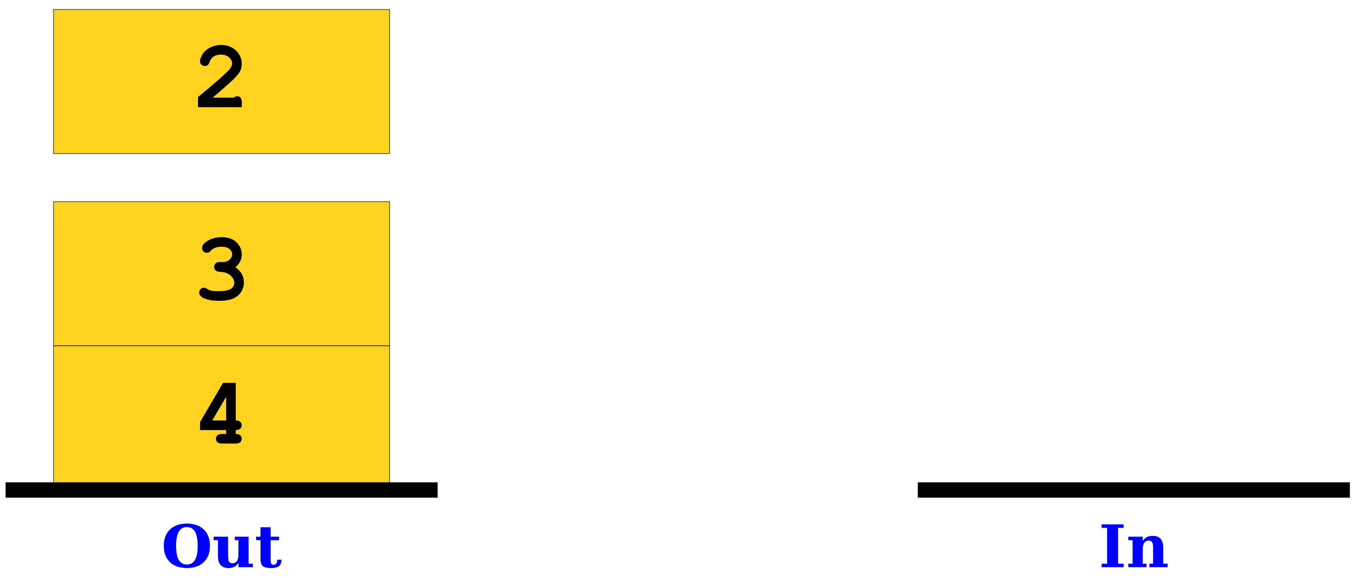

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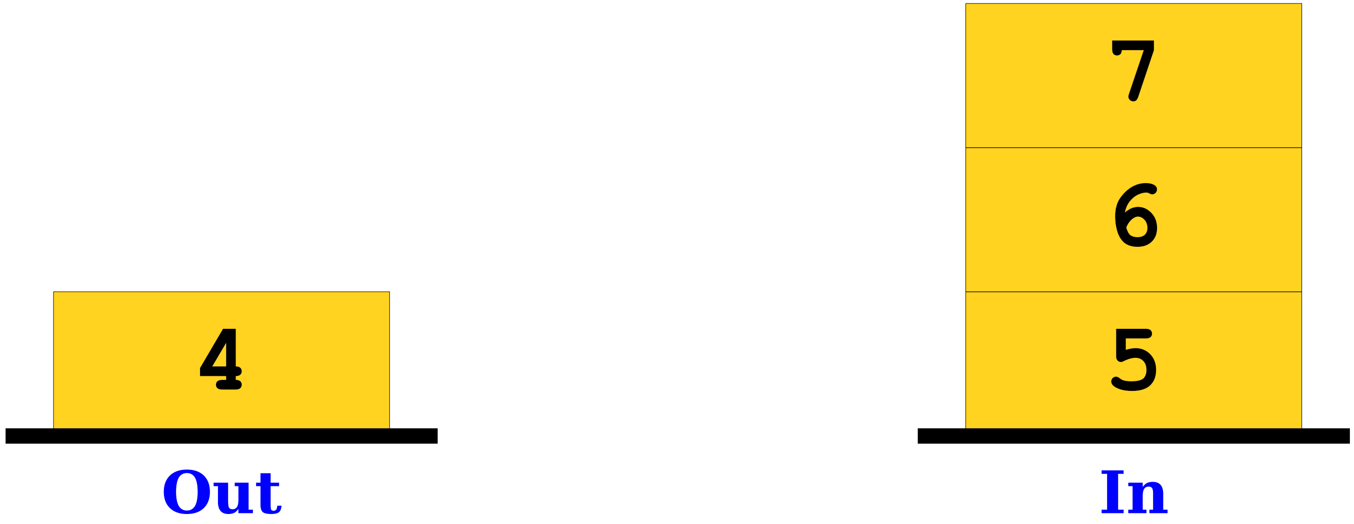

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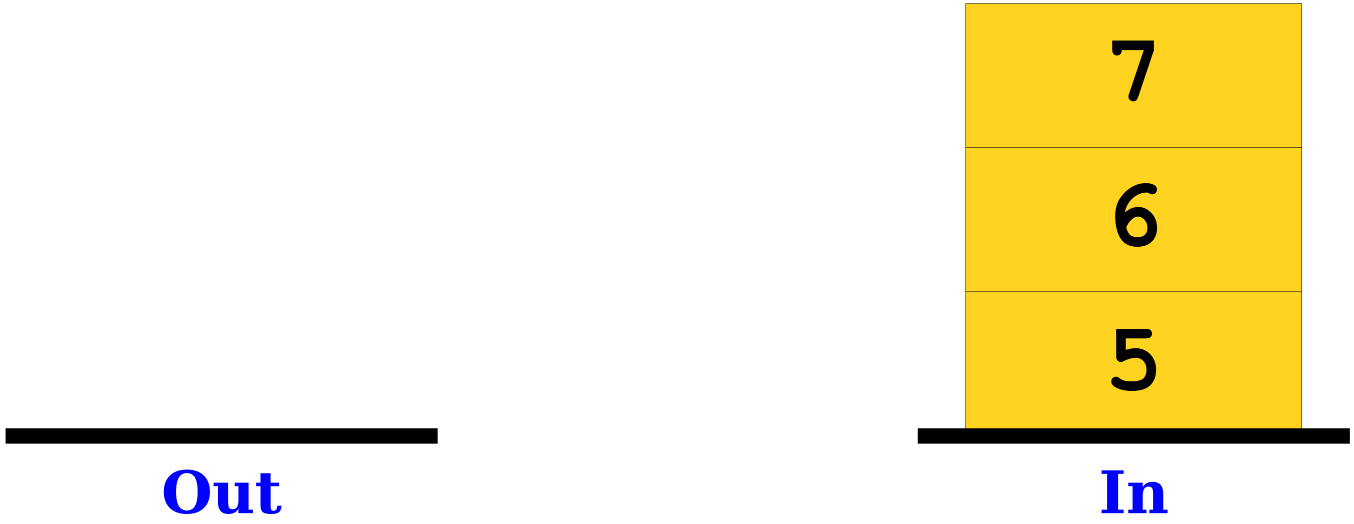

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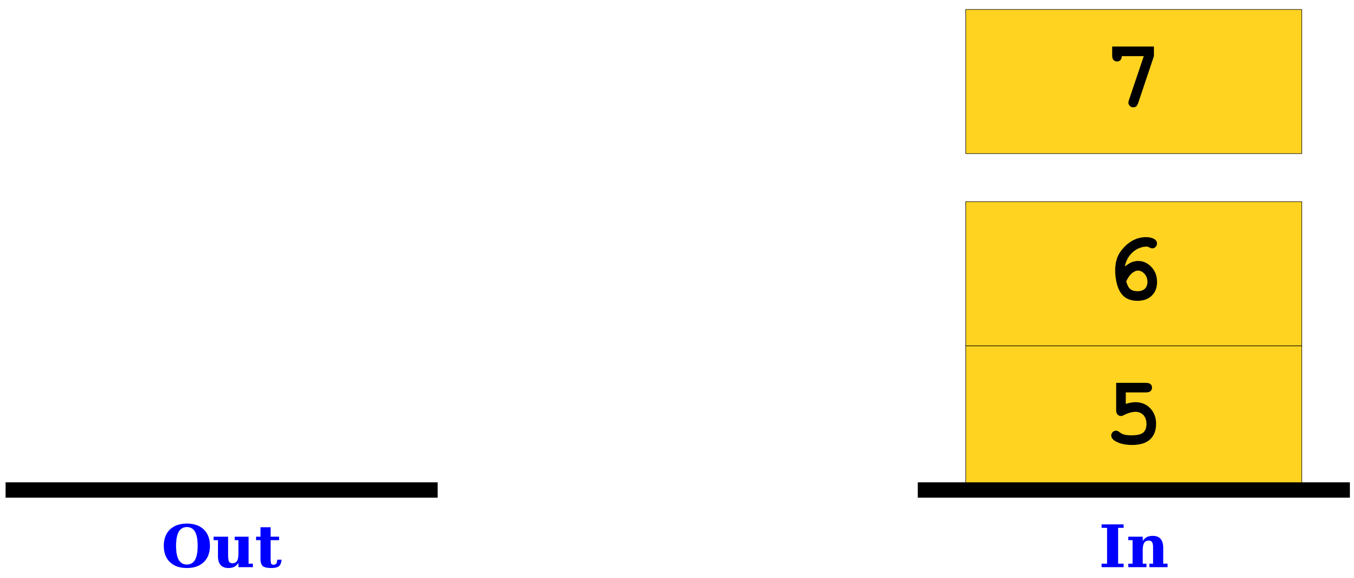

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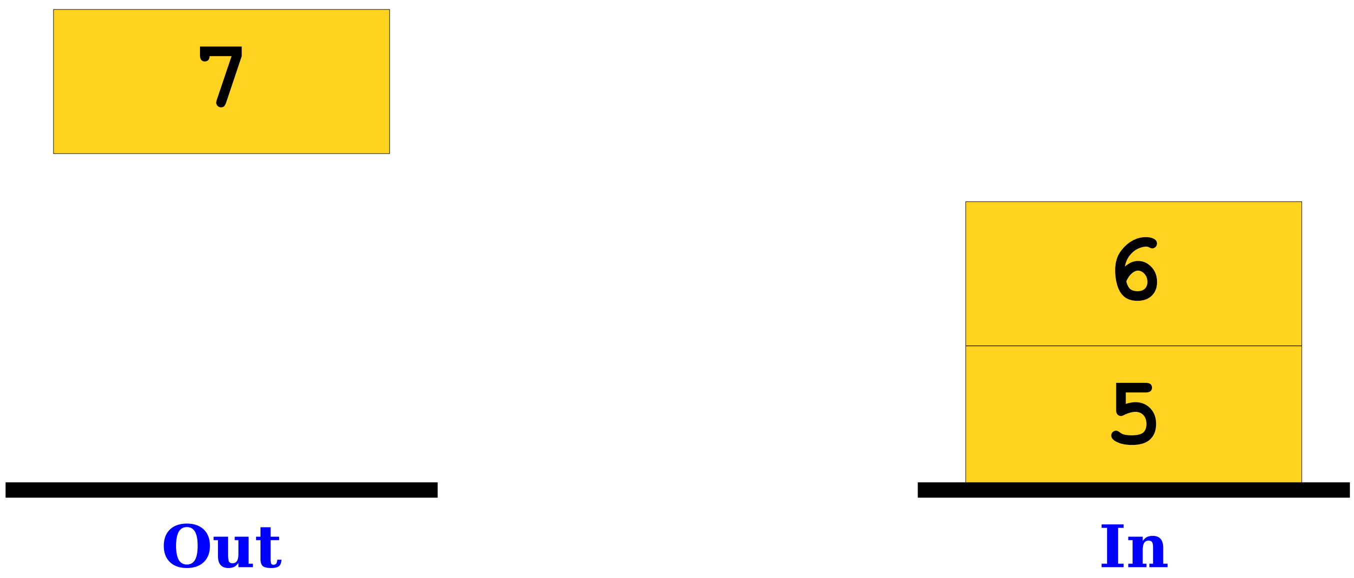

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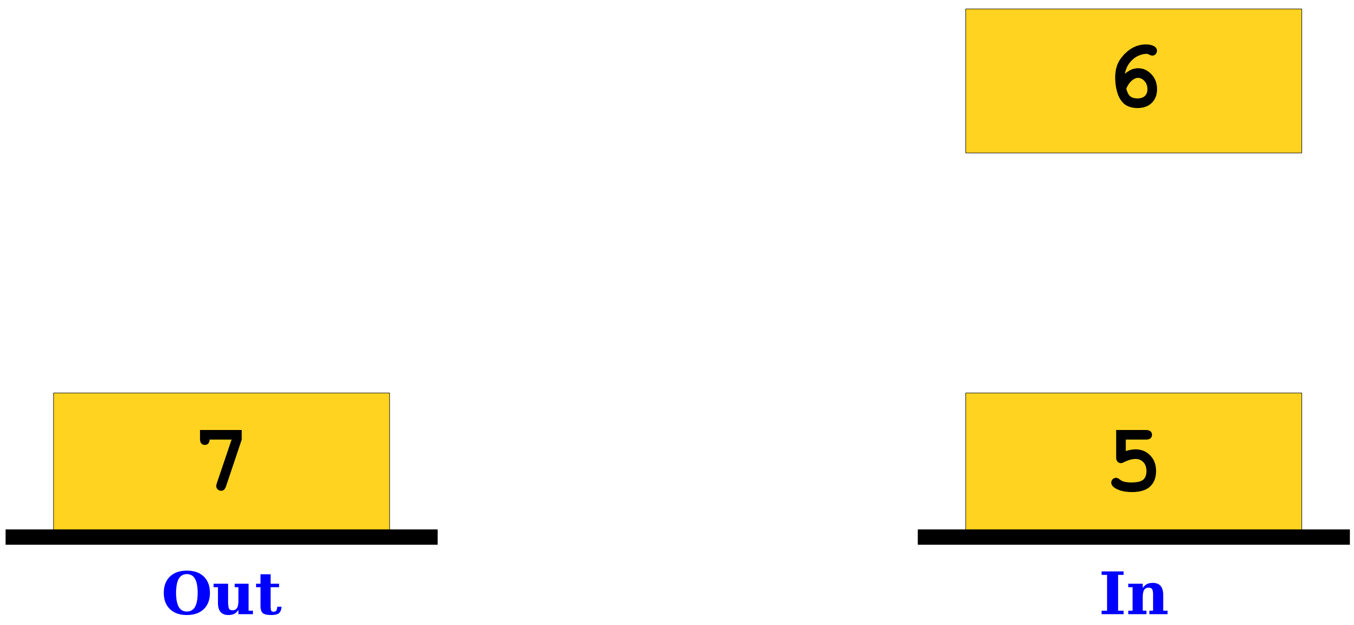

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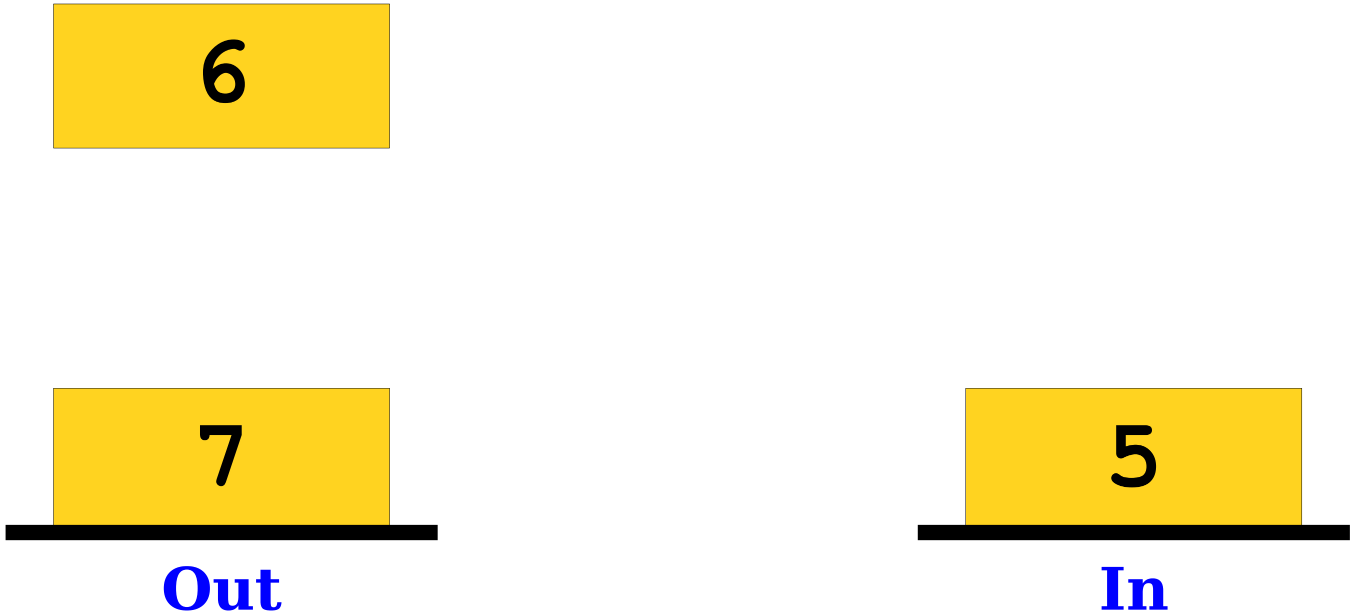

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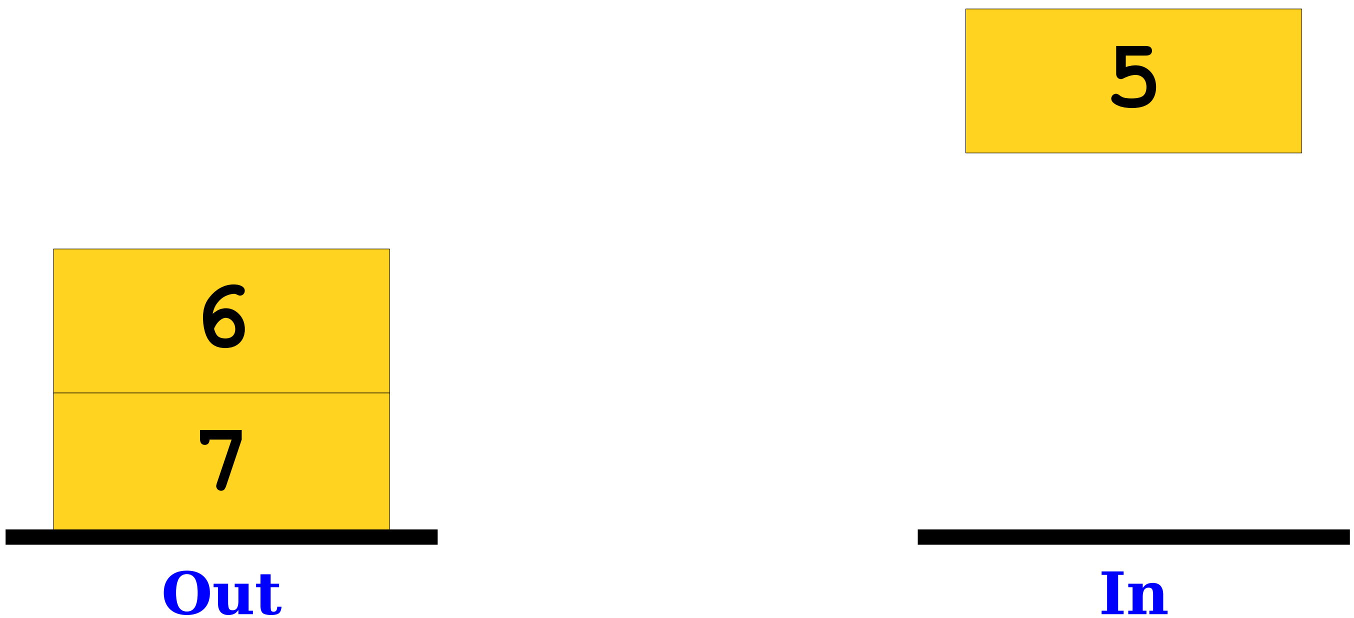

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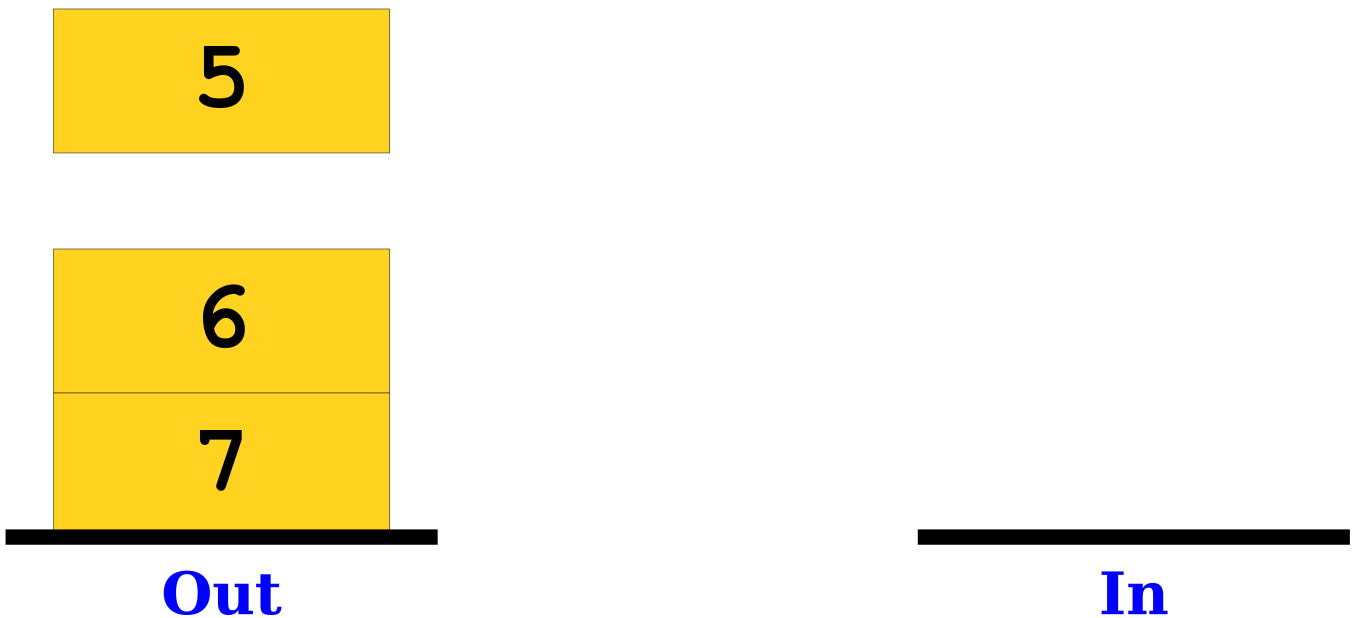

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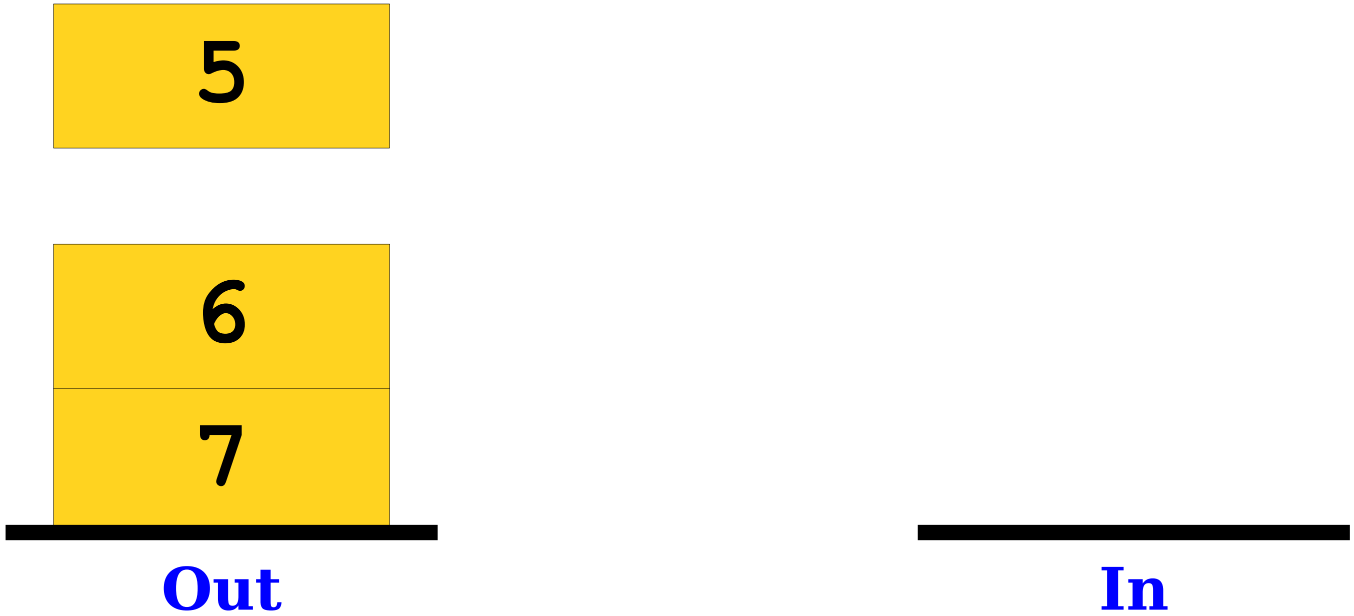

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Φ = height of ***In*** stack

$$\begin{aligned} \textit{amortized-cost} &= \textit{real-cost} + k \cdot \Delta\Phi \\ &= O(h) + k \cdot -h \quad \text{// } h = \text{height of } \textbf{\textit{In}} \text{ stack} \\ &= \mathbf{O(1)} \quad \text{// Choose } k \text{ strategically} \end{aligned}$$

Theorem: The amortized cost of any enqueue or dequeue operation on a two-stack queue is O(1).

Proof: Let Φ be the height of the *In* stack in the two-stack queue. Each enqueue operation does a single push and increases the height of the *In* stack by one. Therefore, its amortized cost is

$$O(1) + k \cdot \Delta\Phi = O(1) + k \cdot 1 = O(1).$$

Now, consider a dequeue operation. If the *Out* stack is nonempty, then the dequeue does O(1) work and does not change Φ. Its cost is therefore

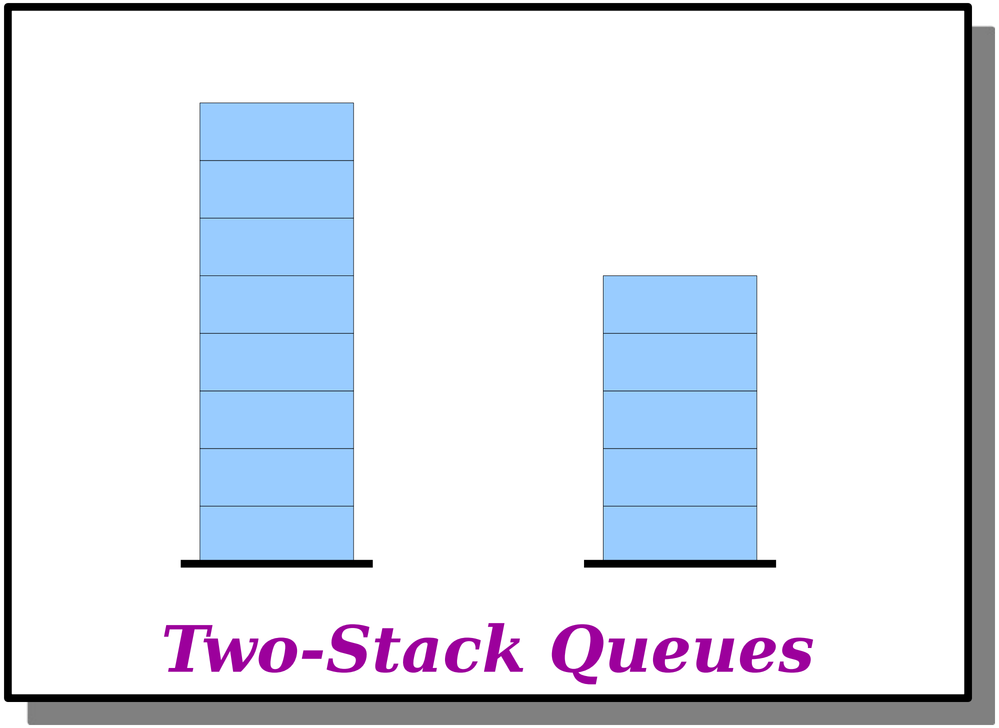
$$O(1) + k \cdot \Delta\Phi = O(1) + k \cdot 0 = O(1).$$

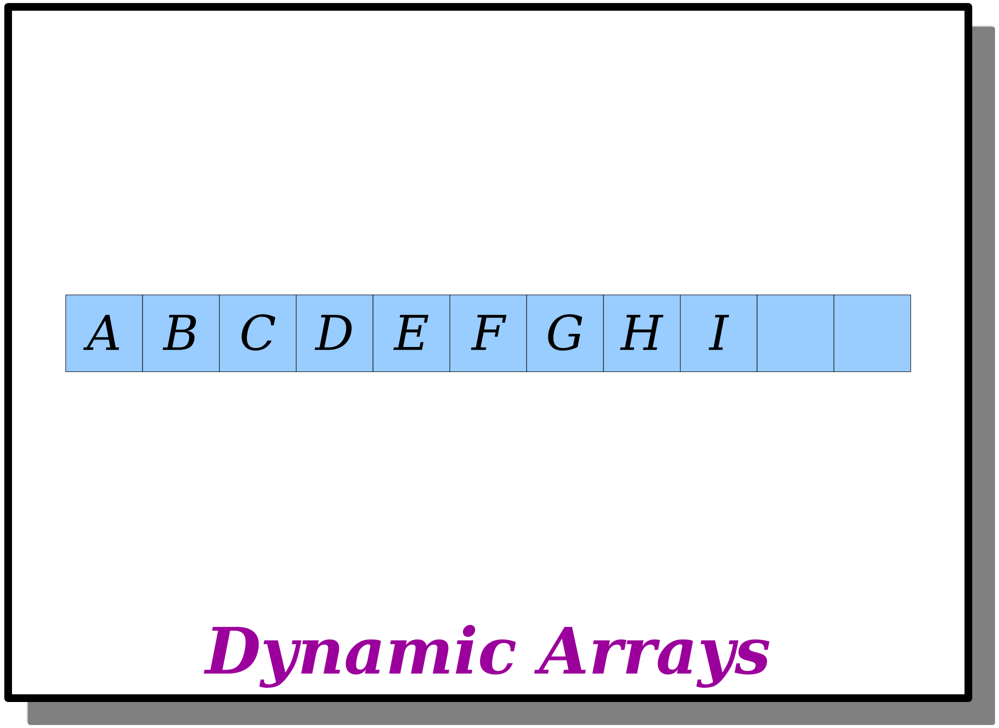
Otherwise, the *Out* stack is empty. Suppose the *In* stack has height h. The dequeue does $O(h)$ work to pop the elements from the *In* stack and push them onto the *Out* stack, followed by one additional pop for the dequeue. This is $O(h)$ total work.

At the beginning of this operation, we have $\Phi = h$. At the end of this operation, we have $\Phi = 0$. Therefore, $\Delta\Phi = -h$, so the amortized cost of the operation is

$$O(h) + k \cdot -h = O(1),$$

assuming we pick k to cancel out the constant factor hidden in the $O(h)$ term. ■


Two-Stack Queues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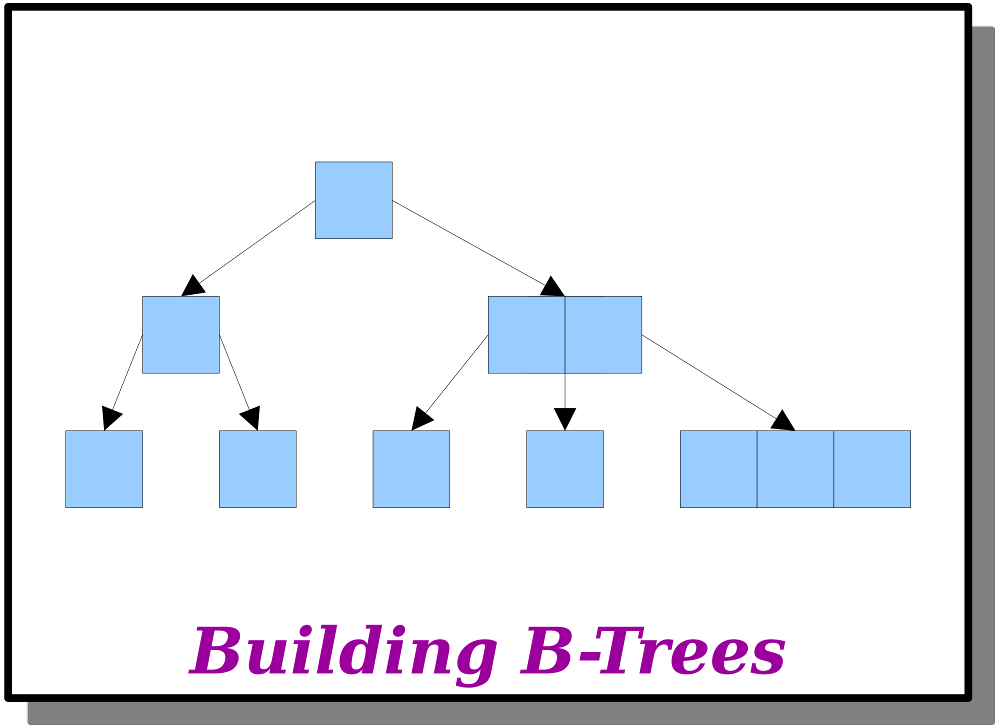
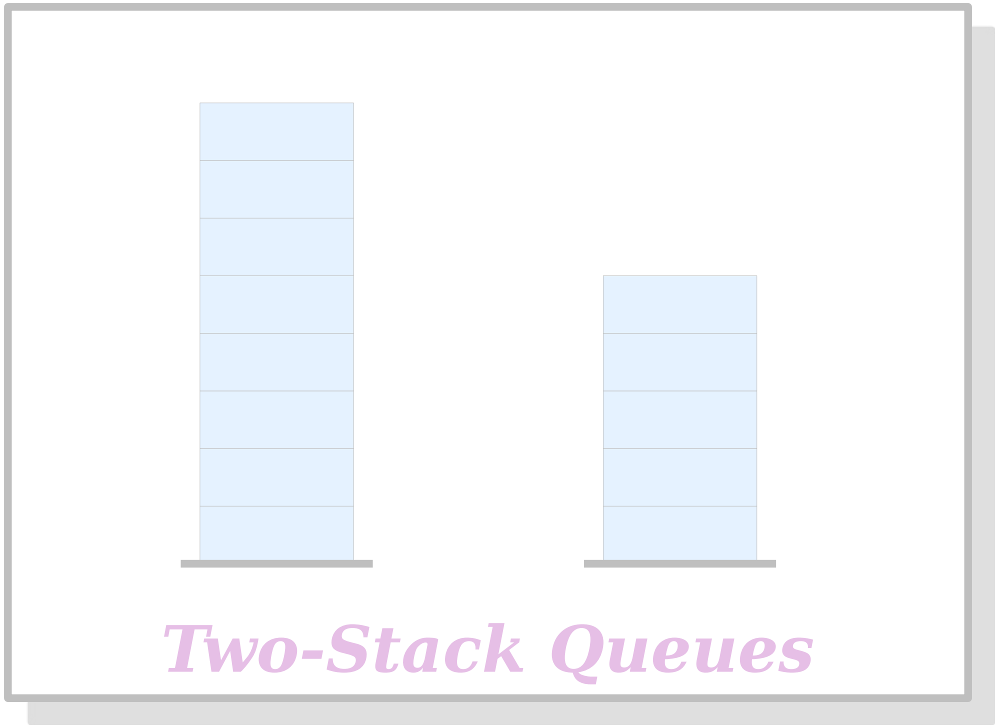
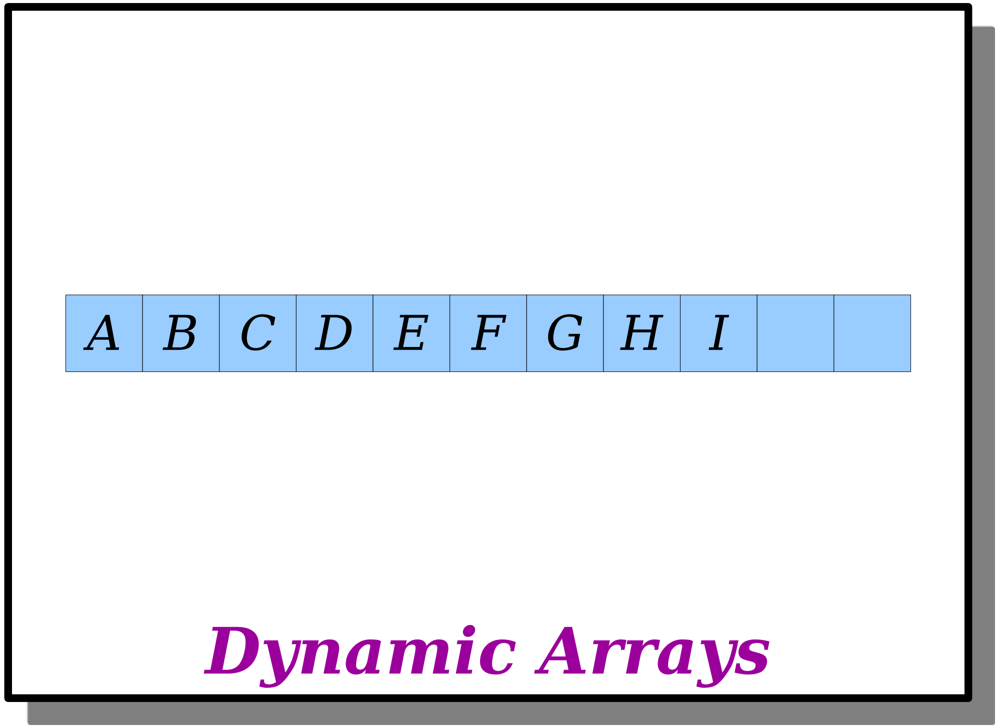
Dynamic Arrays


Building B-Trees


Two-Stack Queues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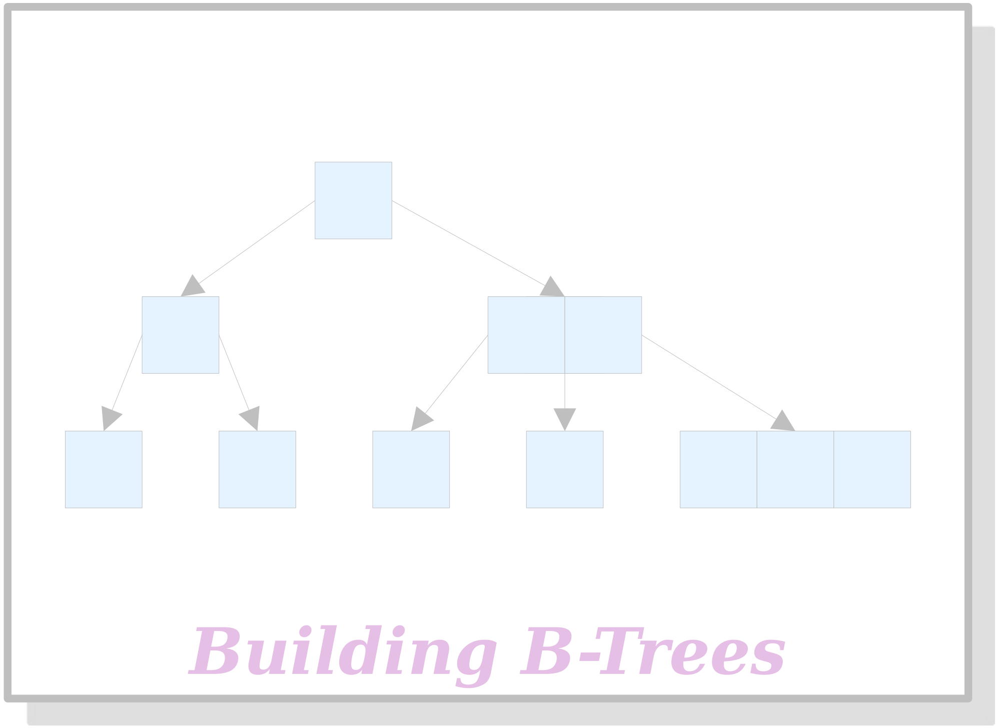
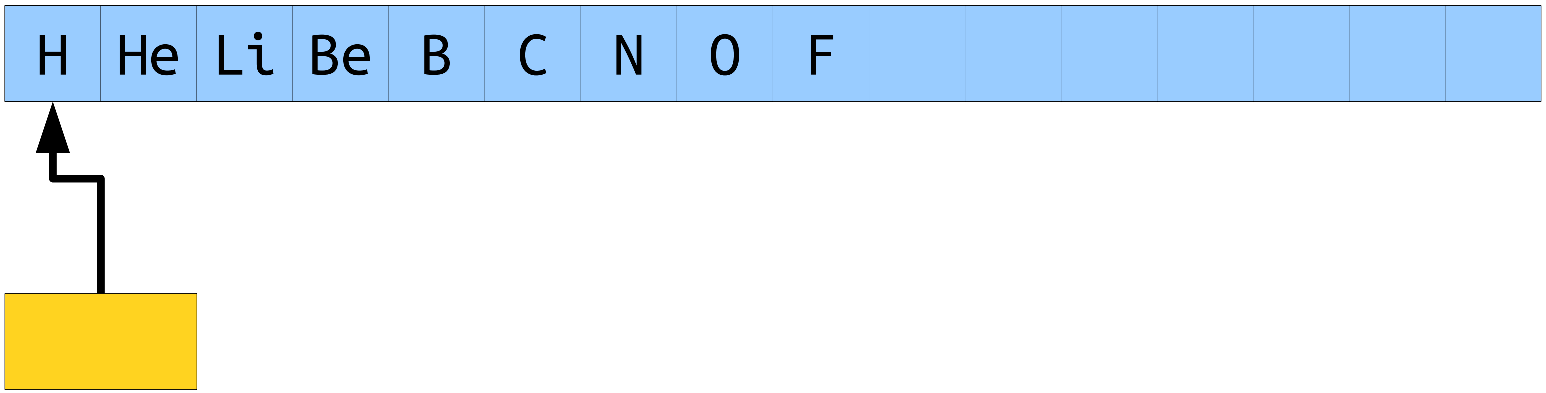

Building B-Trees

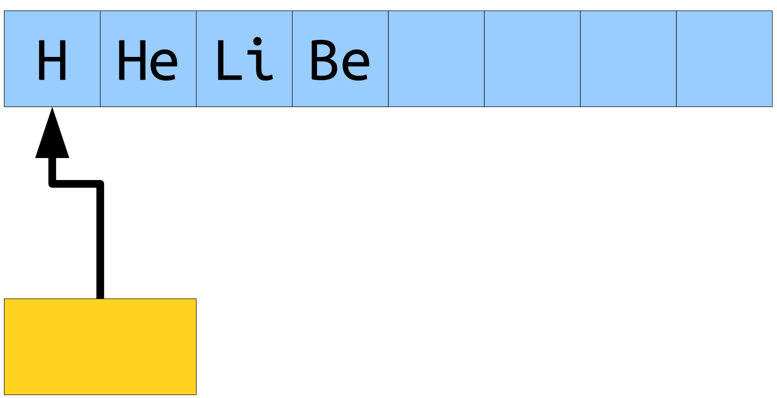
Analyzing Dynamic Arrays

- ***Goal:*** Choose a potential function Φ such that the amortized cost of an append is O(1).
- ***Initial (wrong!) guess:*** Set Φ to be the number of free slots left in the array.



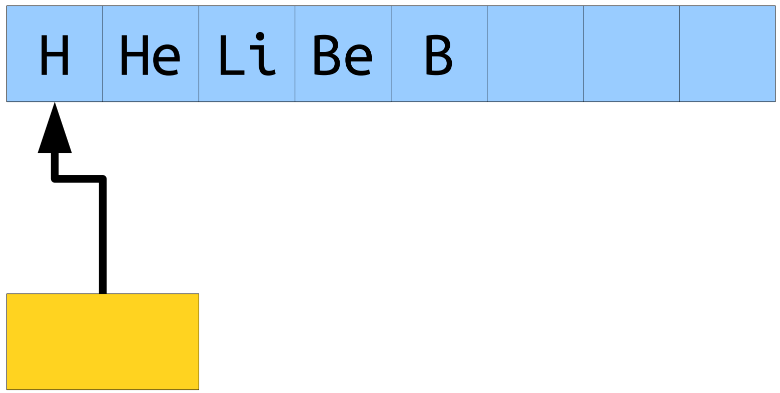
Dynamic Arrays

Φ = number of free slots



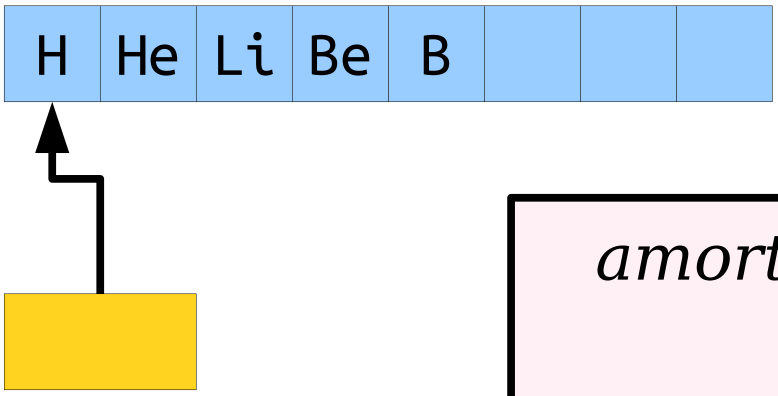
Dynamic Arrays

Φ = number of free slots



Dynamic Arrays

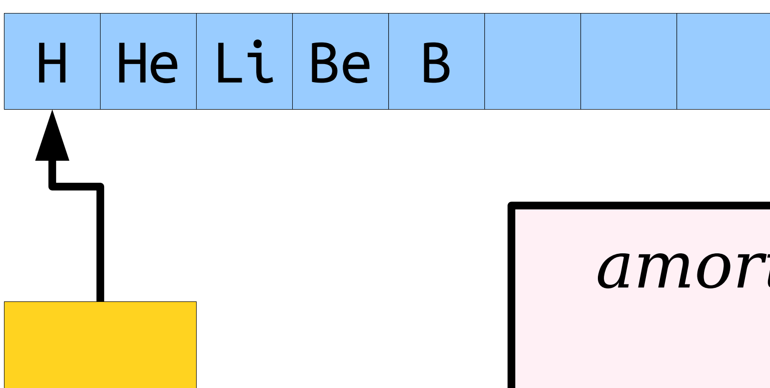
Φ = number of free slots



$$\textit{amortized-cost} = \textit{real-cost} + k \cdot \Delta\Phi$$

Dynamic Arrays

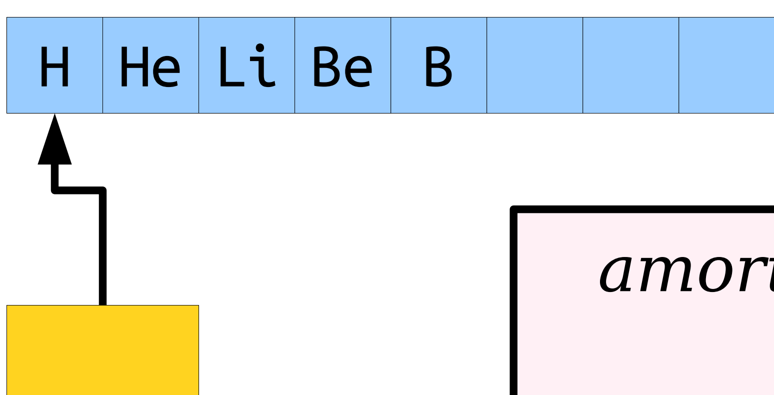
Φ = number of free slots



$$
\begin{aligned}
\textit{amortized-cost} &= \textit{real-cost} + k \cdot \Delta\Phi \\
&= O(1) + k \cdot -1
\end{aligned}
$$

Dynamic Arrays

Φ = number of free slots



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Dynamic Arrays

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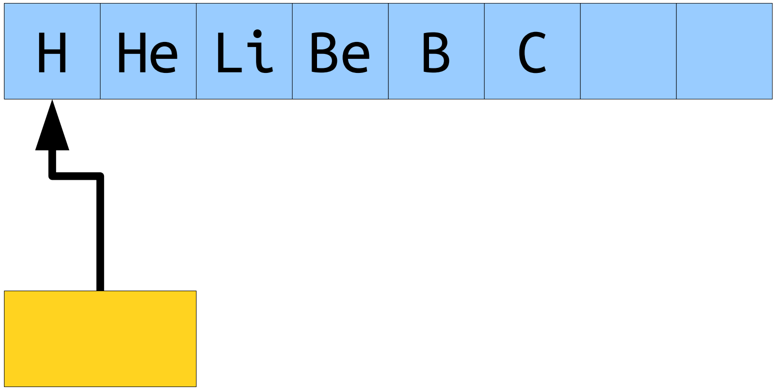
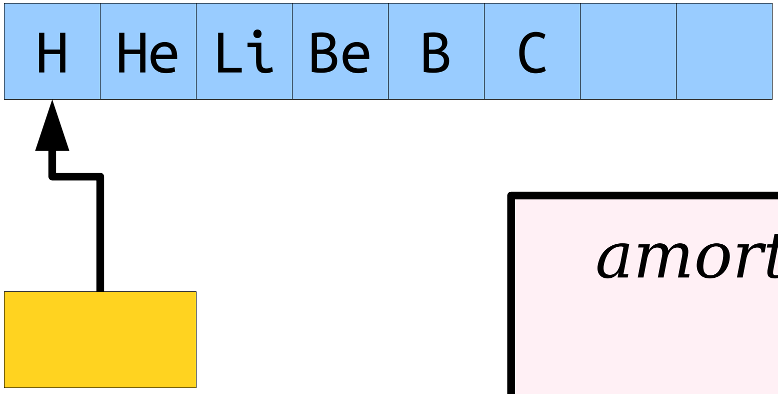


Dynamic Arrays

Φ = number of free slots



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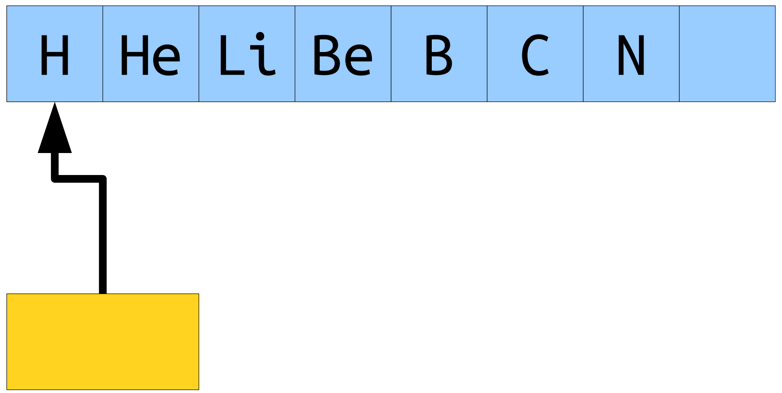
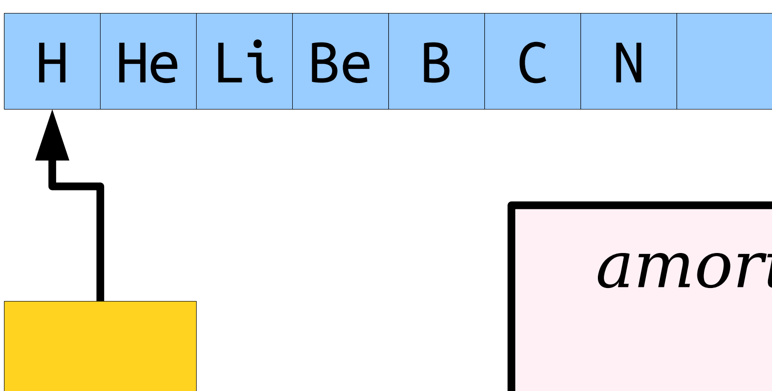
Dynamic Arrays

Φ = number of free slots



Dynamic Arrays

Φ = number of free slots



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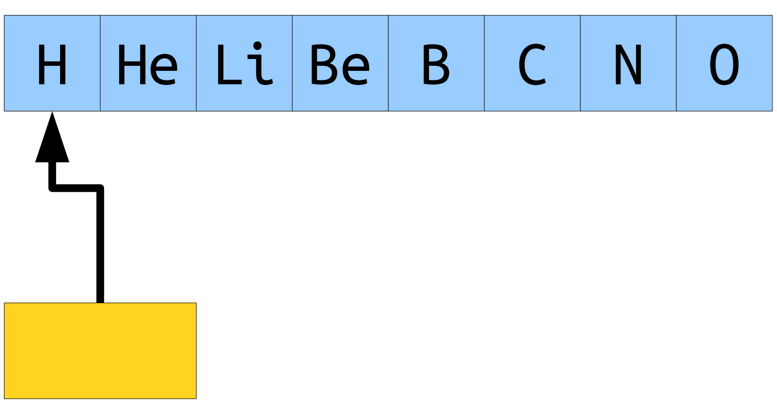
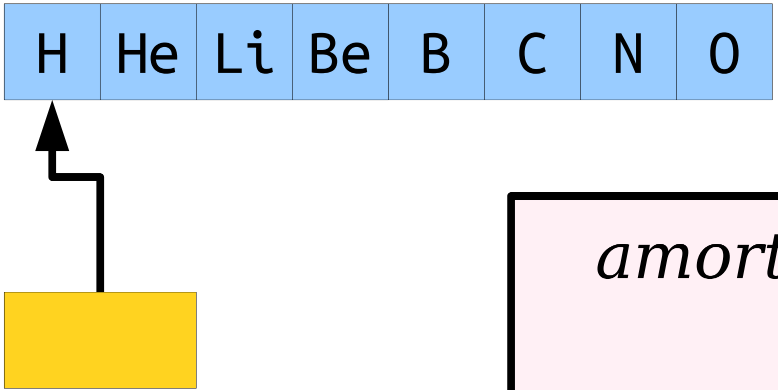
Dynamic Arrays

Φ = number of free slots



Dynamic Arrays

Φ = number of free slots



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Dynamic Arrays

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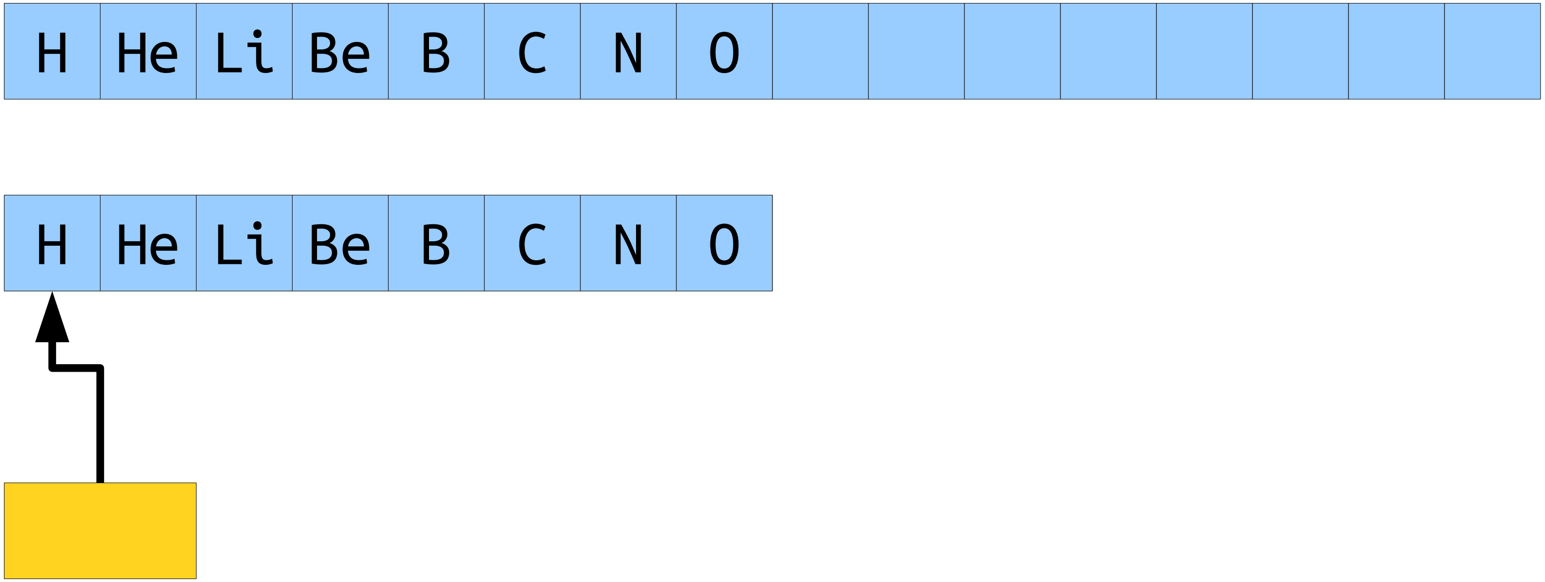
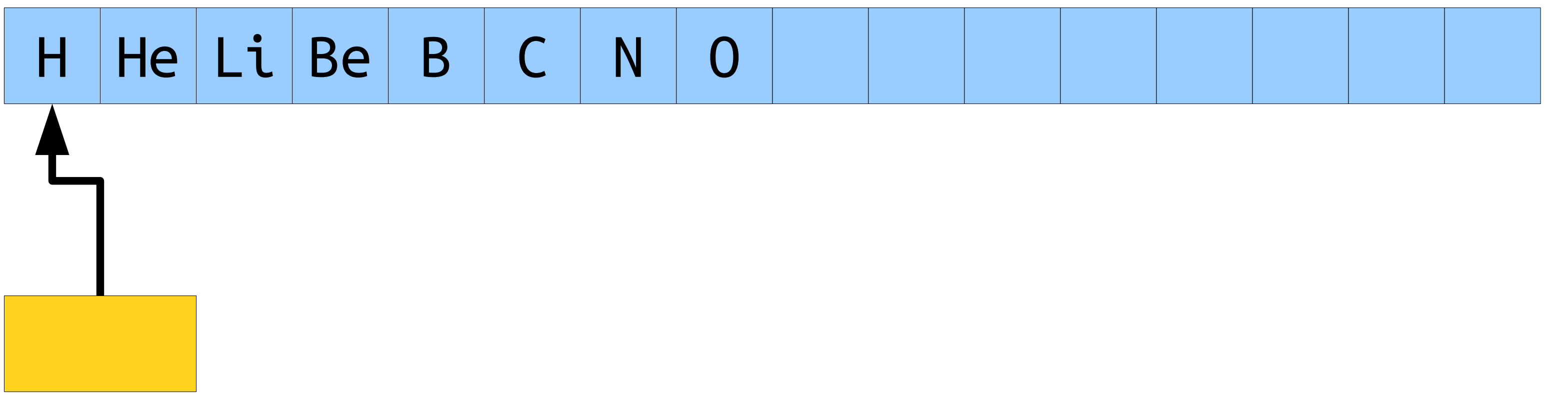


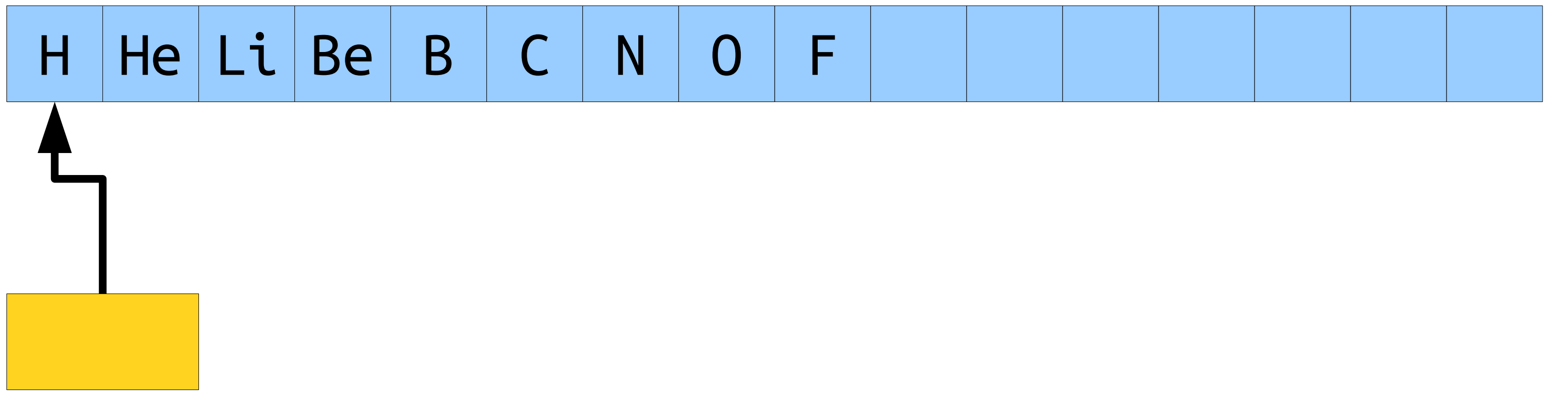
Dynamic Arrays

Φ = number of free slots



Dynamic Arrays

Φ = number of free slots

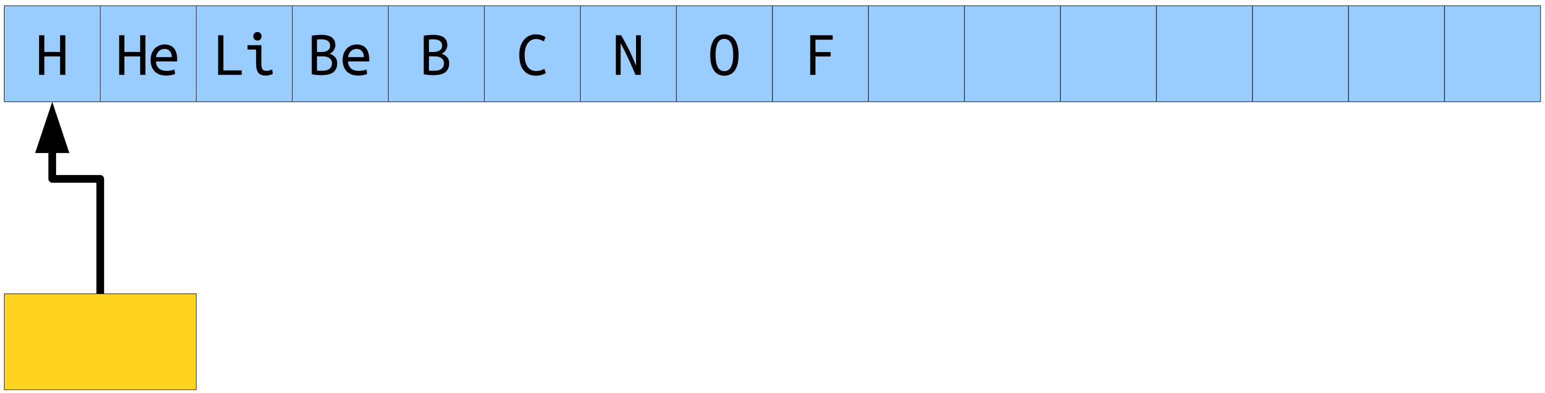


Dynamic Arrays

Φ = number of free slots

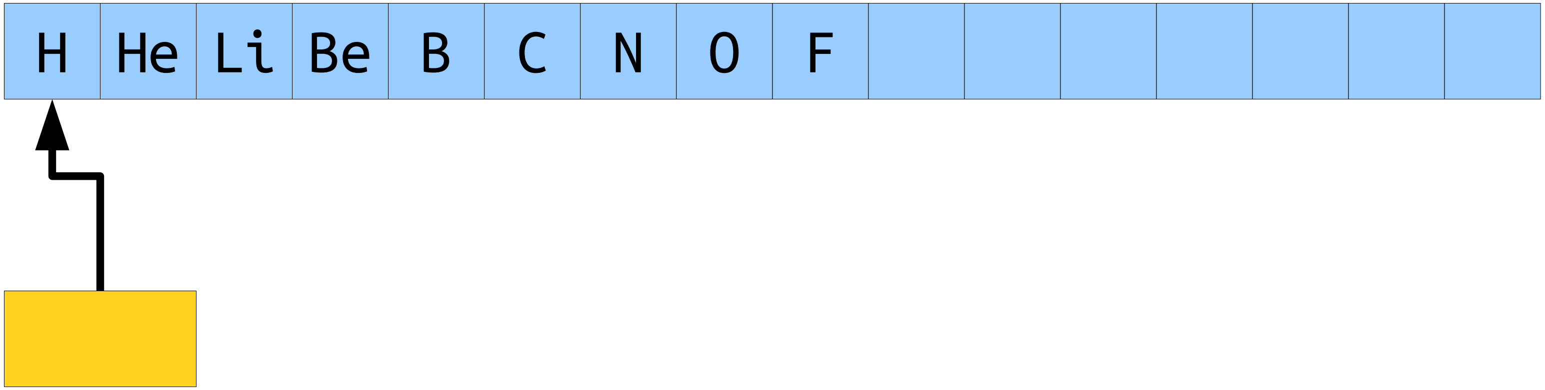
With this choice of Φ, what is the amortized cost of an append to an array of size *n* when no free slots are left?

Answer at **https://pollev.com/cs166spr23**



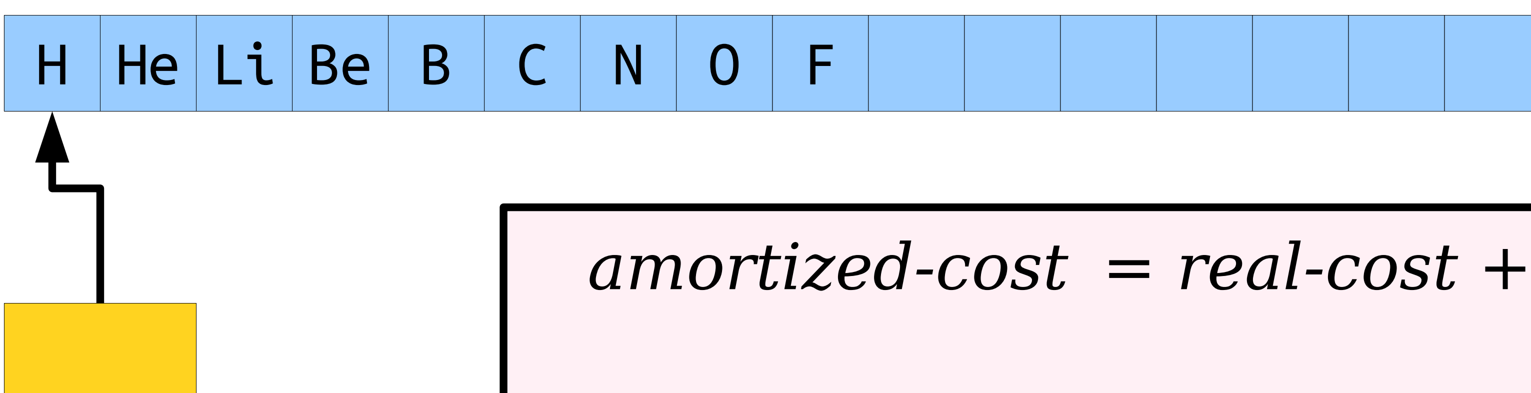
Dynamic Arrays

Φ = number of free slots



Dynamic Arrays

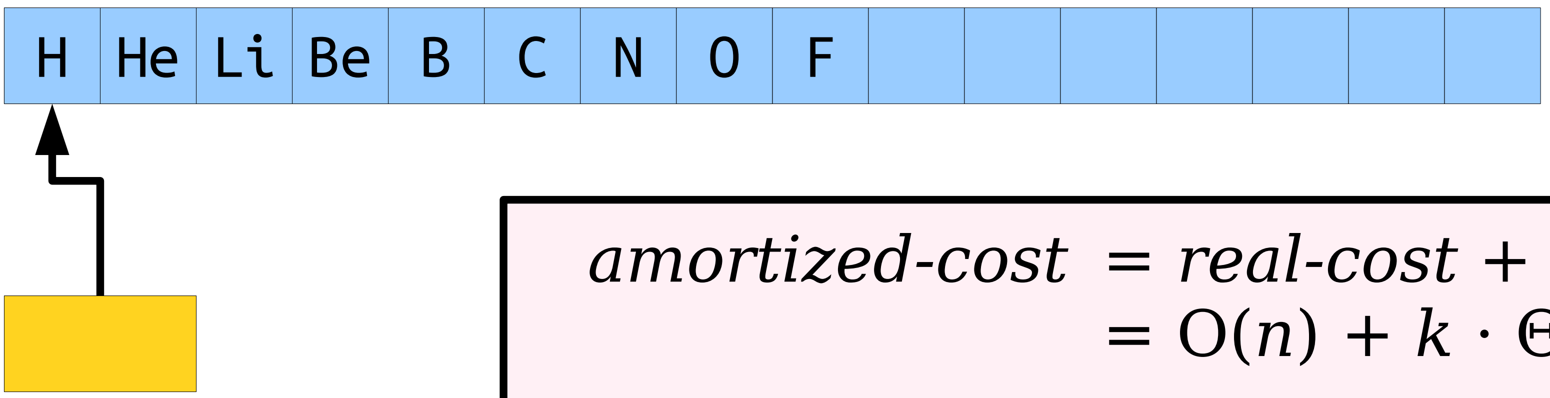
Φ = number of free slots



$$\textit{amortized-cost} = \textit{real-cost} + k \cdot \Delta\Phi$$

Dynamic Arrays

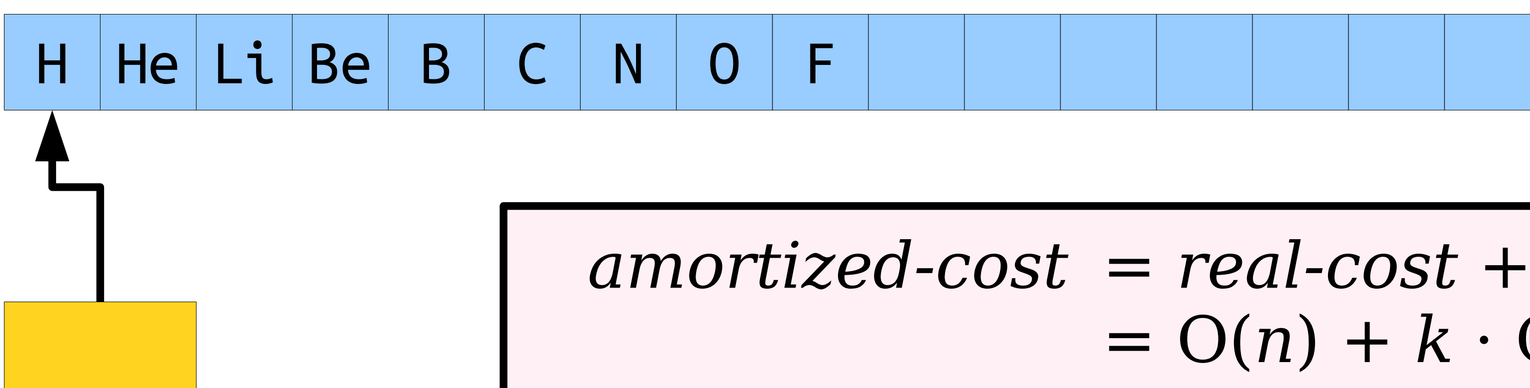
Φ = number of free slots



$$\begin{aligned} \textit{amortized-cost} &= \textit{real-cost} + k \cdot \Delta\Phi \\ &= \mathrm{O}(n) + k \cdot \Theta(n) \end{aligned}$$

Dynamic Arrays

Φ = number of free slots



$$
\begin{aligned}
\textit{amortized-cost} &= \textit{real-cost} + k \cdot \Delta\Phi \\
&= O(n) + k \cdot \Theta(n) \\
&= \mathbf{O(\boldsymbol{n})}
\end{aligned}
$$

Analyzing Dynamic Arrays

- ***Intuition:*** Φ should measure how "messy" the data structure is.
  - Having lots of free slots means there's very little mess.
  - Having few free slots means there's a lot of mess.
- We basically got our potential function backwards. Oops.
- ***Question:*** What should Φ be?

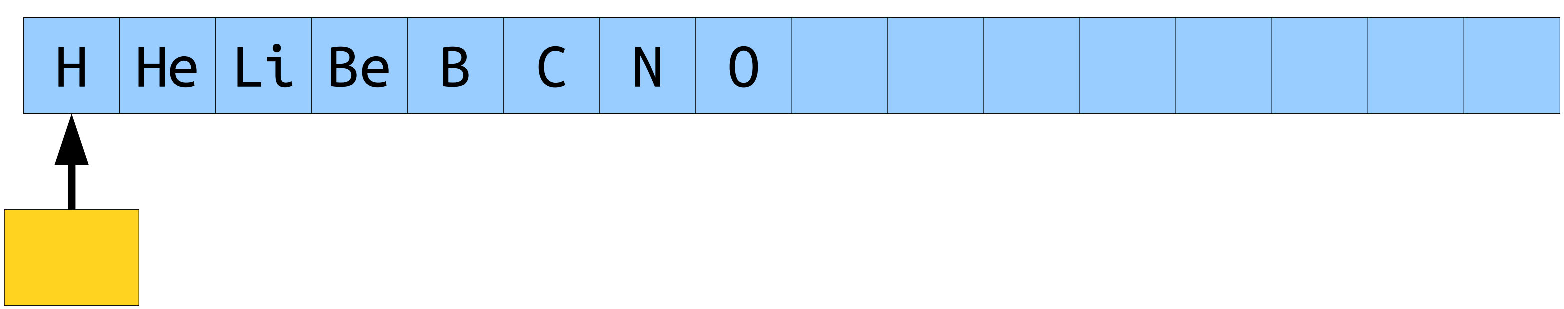
Analyzing Dynamic Arrays

- The amortized cost of an append is

$$\textit{amortized-cost} = \textit{real-cost} + k \cdot \Delta\Phi.$$

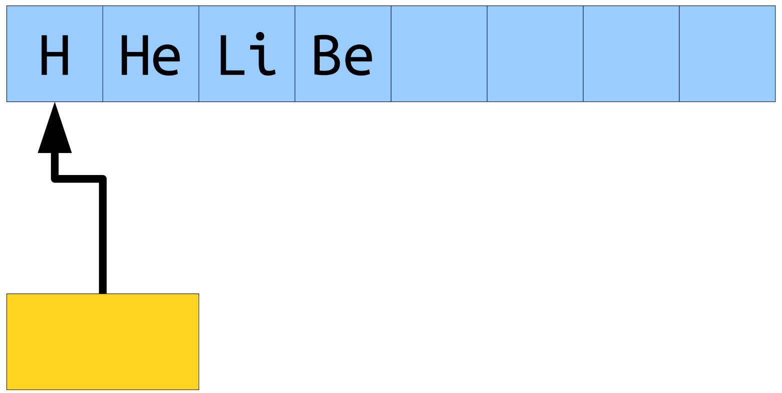
- When we double the array size, our real cost is $\Theta(n)$. We need $\Delta\Phi$ to be something like $-n$.
- ***Goal:*** Pick Φ so that
  - when there are no slots left, $\Phi \approx n$, and
  - right after we double the array size, $\Phi \approx 0$.
- With some trial and error, we can come up with

$$\Phi = \#\textit{elems} - \#\textit{free-slots}$$



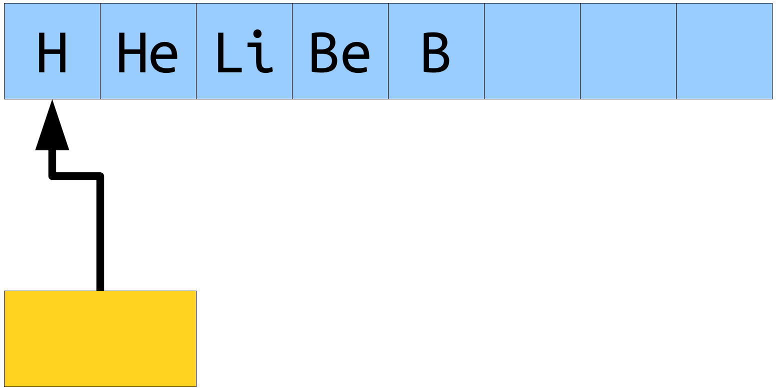
Dynamic Arrays

Φ = *#elems – #free-slots*



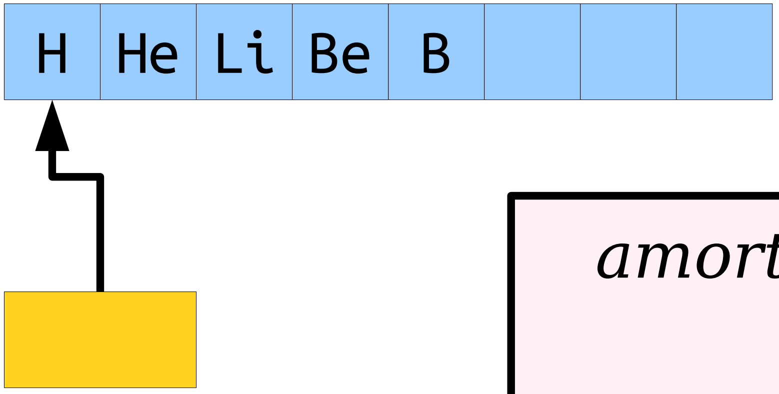
Dynamic Arrays

$$\Phi = \#elems - \#free\text{-}slots$$



Dynamic Arrays

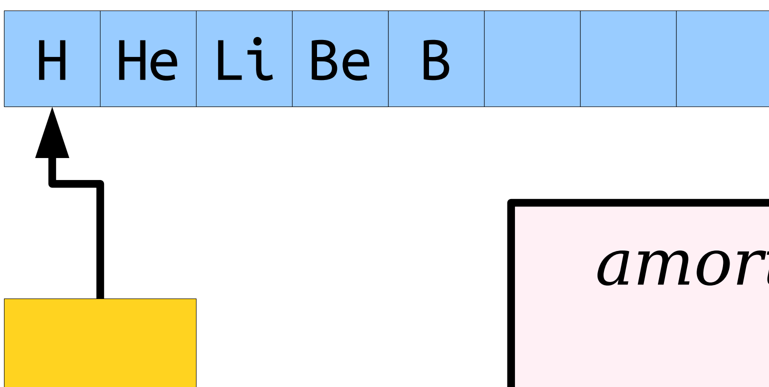
$\Phi = \#elems - \#free\text{-}slots$



$amortized\text{-}cost = real\text{-}cost + k \cdot \Delta\Phi$

Dynamic Arrays

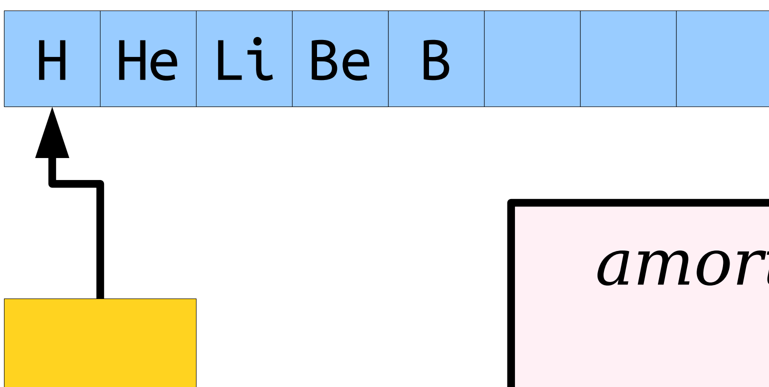
$\Phi = \#elems - \#free\text{-}slots$



$$\begin{aligned} amortized\text{-}cost &= real\text{-}cost + k \cdot \Delta\Phi \\ &= O(1) + k \cdot 2 \end{aligned}$$

Dynamic Arrays

$\Phi = \#elems - \#free\text{-}slots$



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amortized\text{-}cost &= real\text{-}cost + k \cdot \Delta\Phi \\
&= O(1) + k \cdot 2 \\
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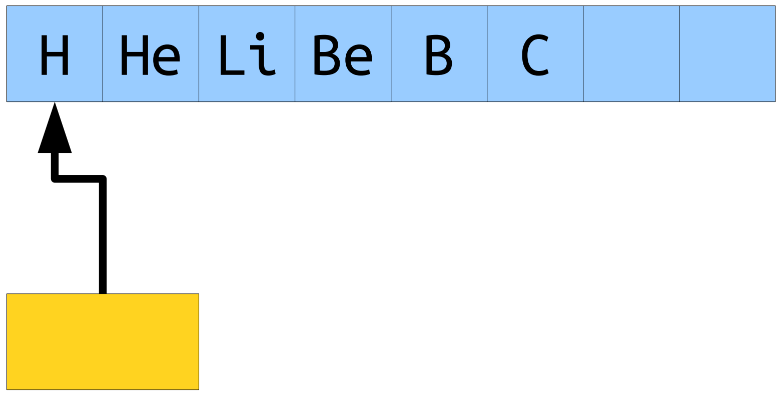
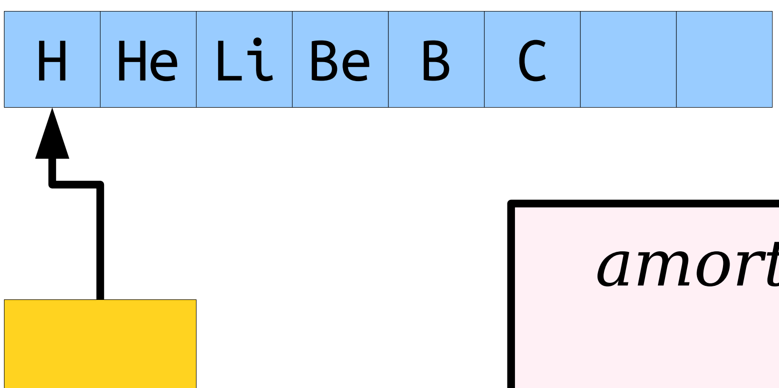
Dynamic Arrays

Φ = *#elems* – *#free-slots*



Dynamic Arrays

$$\Phi = \textit{\#elems} - \textit{\#free-slots}$$



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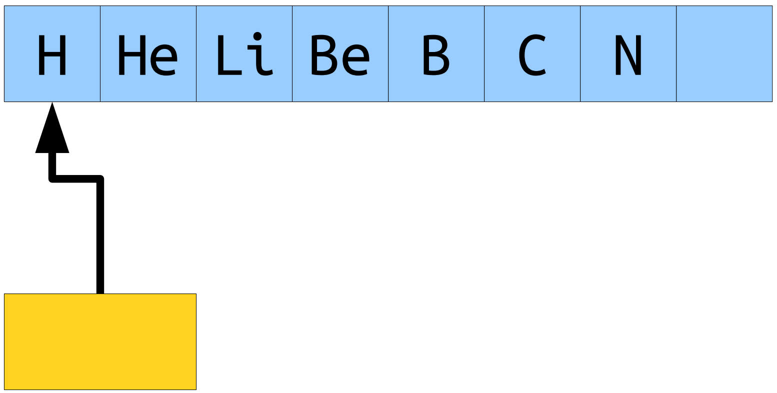
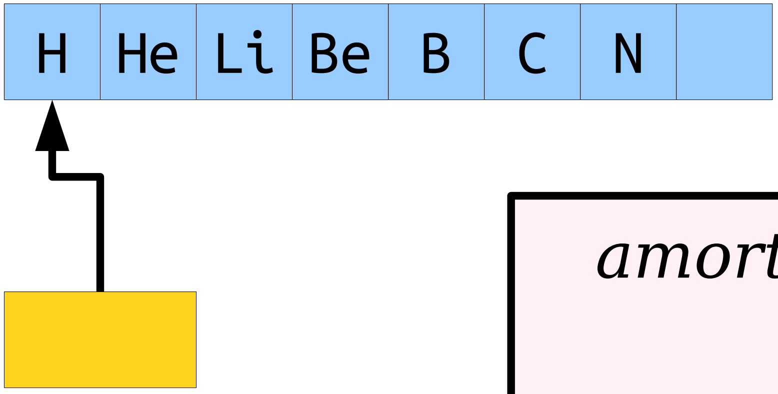
$\Phi = \#elems - \#free\text{-}slots$



Dynamic Arrays

$\Phi = \#elems - \#free\text{-}slots$



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Dynamic Arrays

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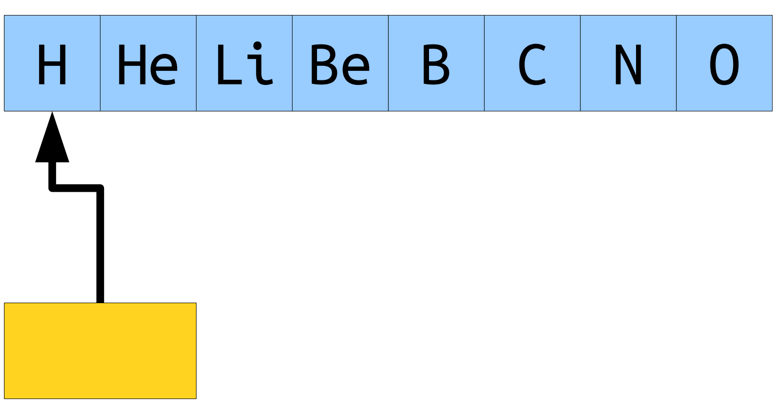

Dynamic Arrays

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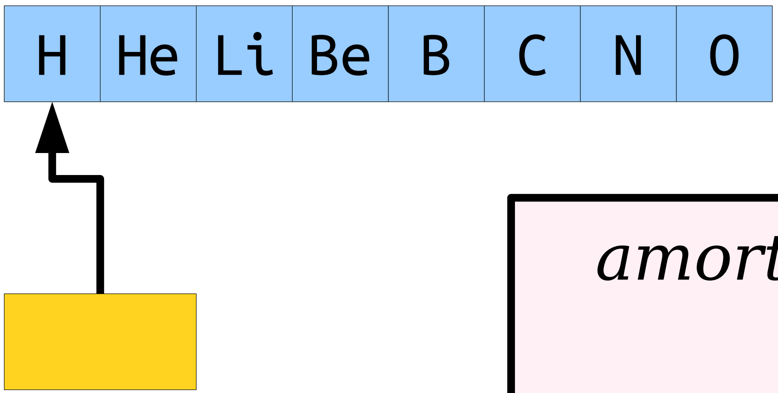

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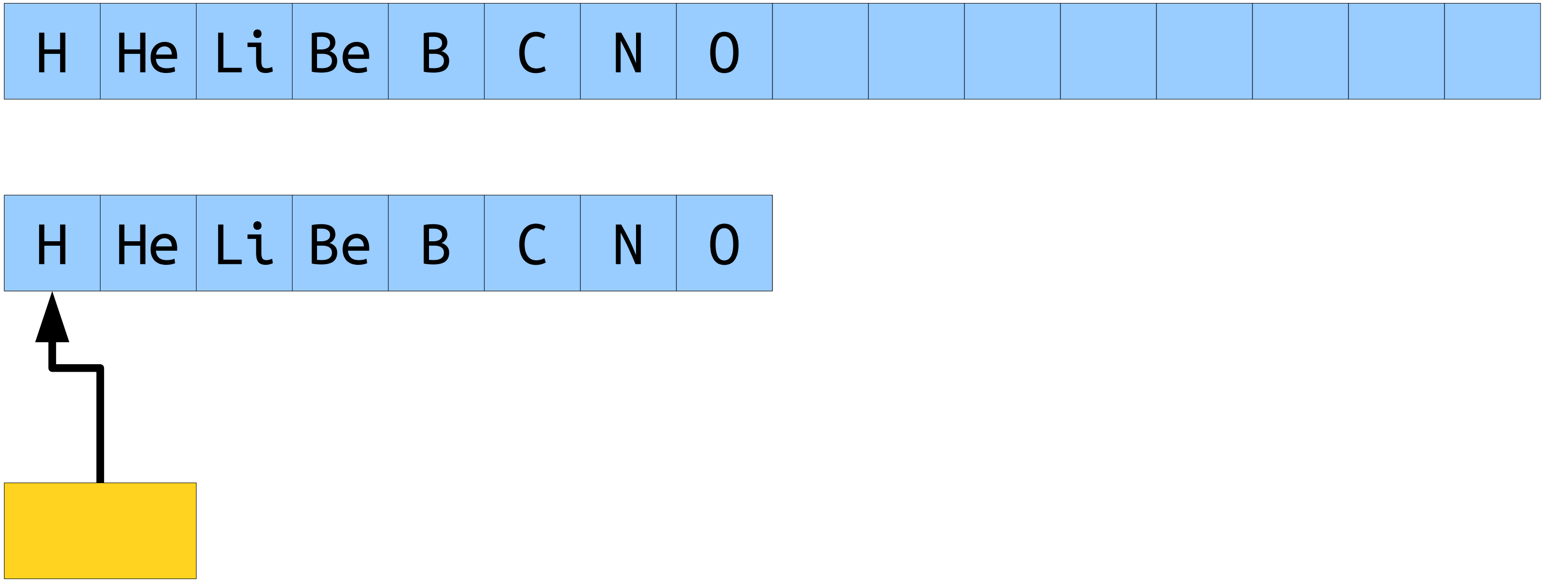
Dynamic Arrays

$\Phi = \#elems - \#free\text{-}slots$



Dynamic Arrays

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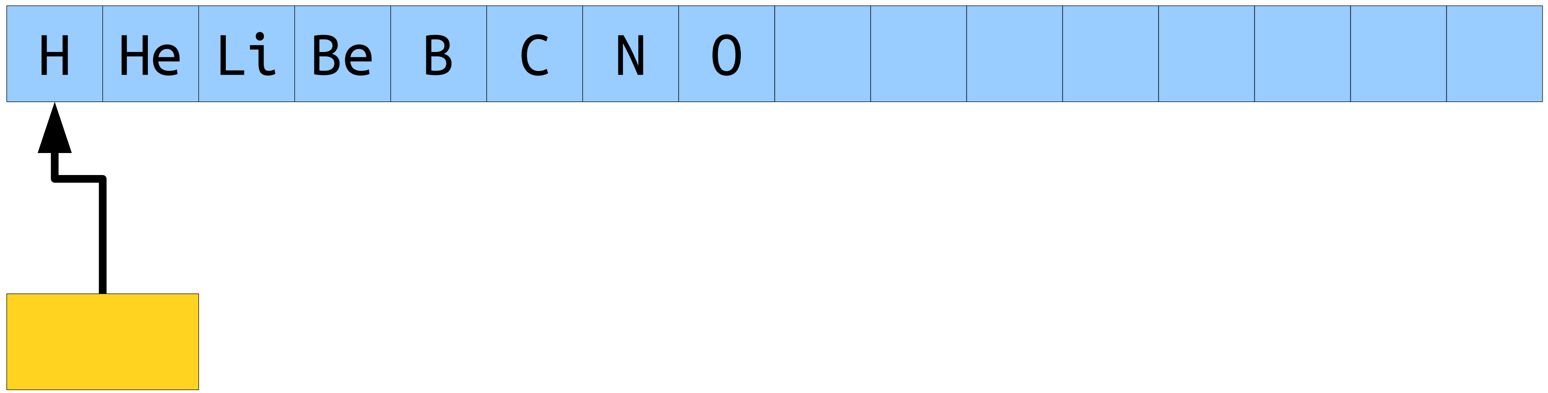
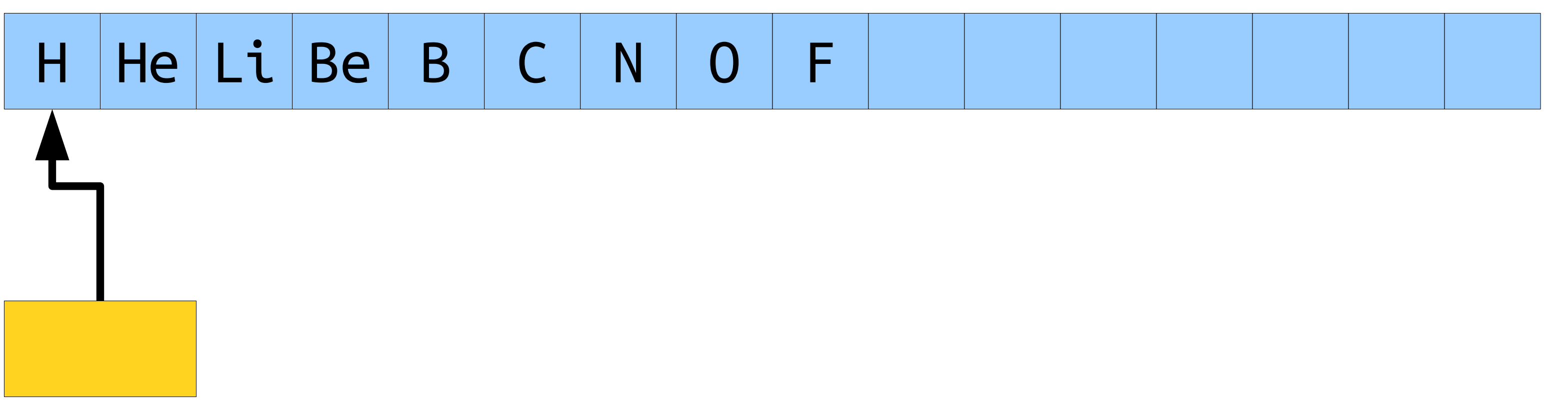


Dynamic Arrays

$$\Phi = \#\mathit{elems} - \#\mathit{free\text{-}slots}$$



Dynamic Arrays

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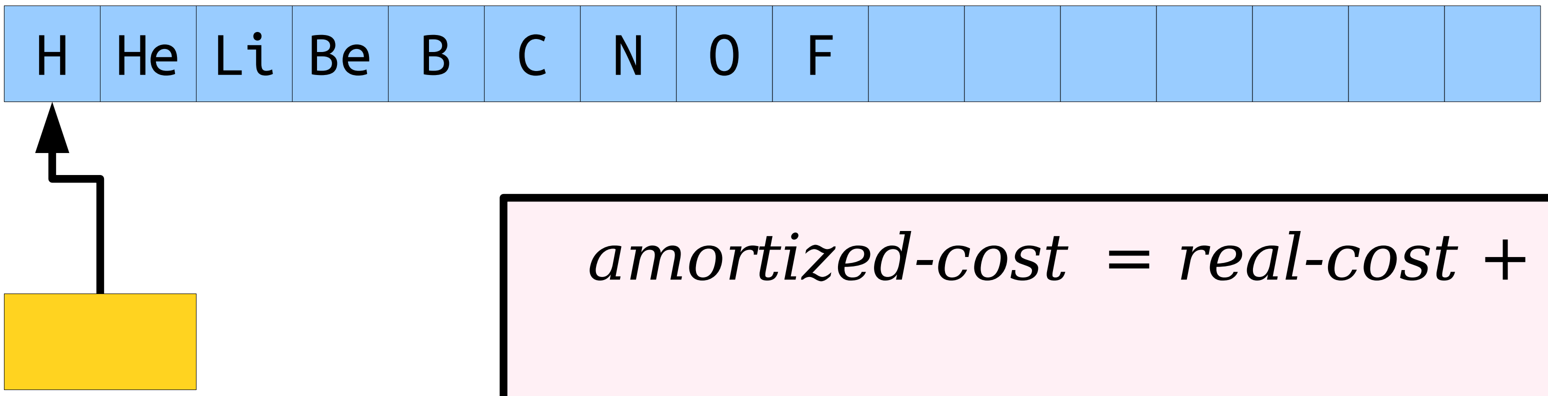
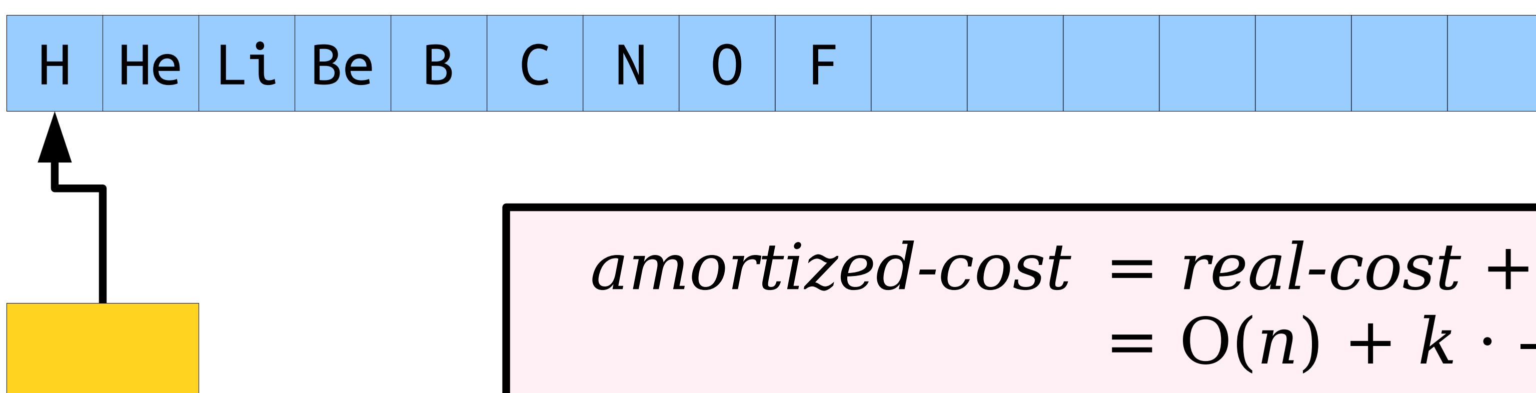


$amortized\text{-}cost = real\text{-}cost + k \cdot \Delta\Phi$

Dynamic Arrays

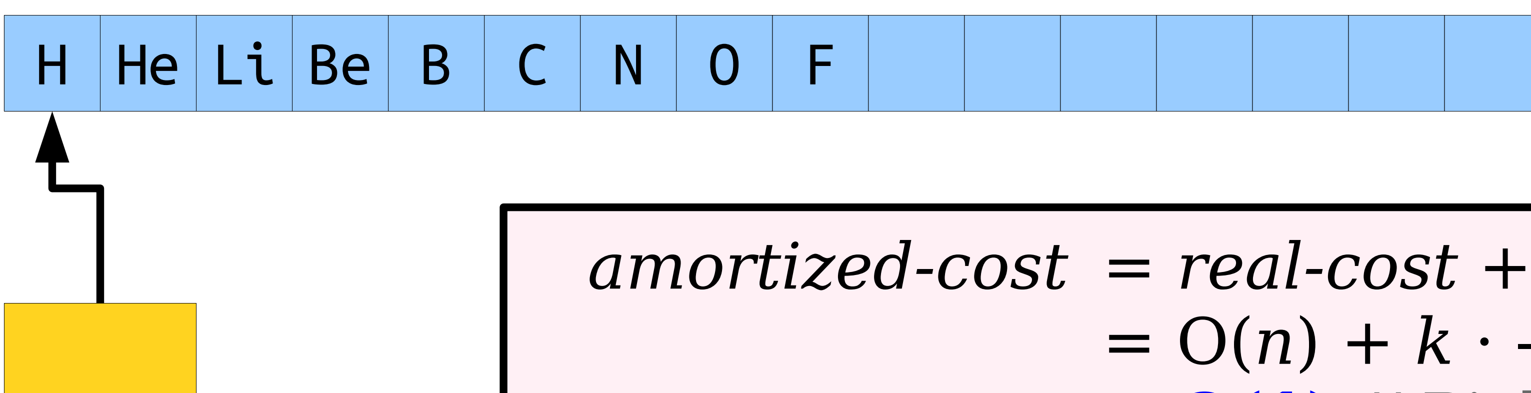
$\Phi = \#elems - \#free\text{-}slots$



$$\begin{aligned} amortized\text{-}cost &= real\text{-}cost + k \cdot \Delta\Phi \\ &= O(n) + k \cdot -\Theta(n) \end{aligned}$$

Dynamic Arrays

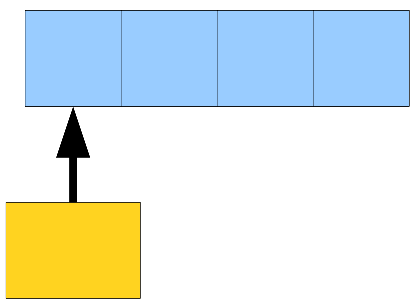
$\Phi = \#elems - \#free\text{-}slots$



$$\begin{aligned} amortized\text{-}cost &= real\text{-}cost + k \cdot \Delta\Phi \\ &= O(n) + k \cdot -\Theta(n) \\ &= \mathbf{O(1)} \quad \text{// Pick } k \text{ well} \end{aligned}$$

A Caveat

- We require that $\Phi_{start} = 0$ and that $\Phi \geq 0$.
- What happens when we have a newly-created dynamic array?



$\Phi = -4$

- ***Quick fix:*** This is an edge case, so set

$$\Phi = \max\{\ 0,\ \#elems\ -\ \#free\text{-}slots$$

Theorem: The amortized cost of an append to a dynamic array is O(1).

Proof: Suppose the dynamic array has initial capacity $2C = O(1)$. Then, define $\Phi = \max\{ 0, n - \#\textit{free-slots} \}$, where n is the number of elements stored in the dynamic array. Note that for $n < C$ that an append simply fills in a free slot and leaves $\Phi = 0$, so the amortized cost of such an append is O(1). Otherwise, we have $n > C$ and $\Phi = n - \#\textit{free-slots}$.

Consider any append. If the append does not trigger a resize, it does O(1) work, increases n by one, and decreases *#free-slots* by one, so the amortized cost is

$$O(1) + k \cdot \Delta\Phi = O(1) + k \cdot 2 = O(1).$$

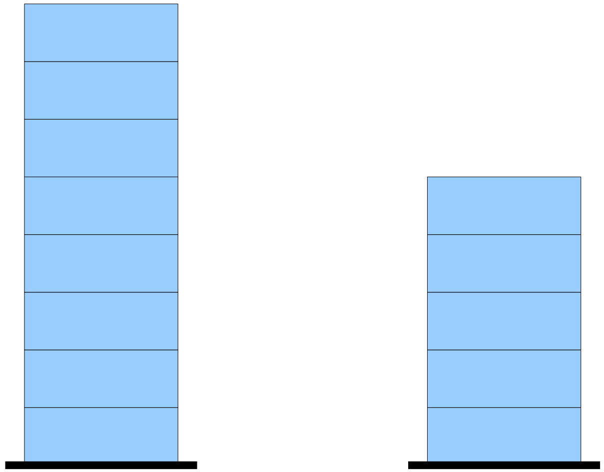
Otherwise, the operation copies n elements into a new array twice as large as before, increasing the number of free slots to n, then fills one of those slots. Just before the operation we had $\Phi = n$, and just after the operation we have $\Phi = 2$. Therefore, the amortized cost is

$$O(n) + k \cdot \Delta\Phi = O(n) + k \cdot (2 - n) = O(n) - nk + 2k,$$

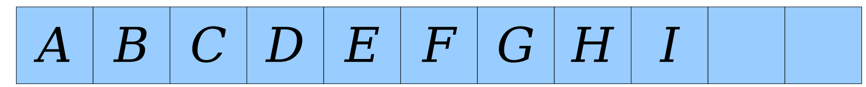
which can be made to equal O(1) by choosing the the k term to match the constant hidden in the $O(n)$ term. ■

Some Exercises

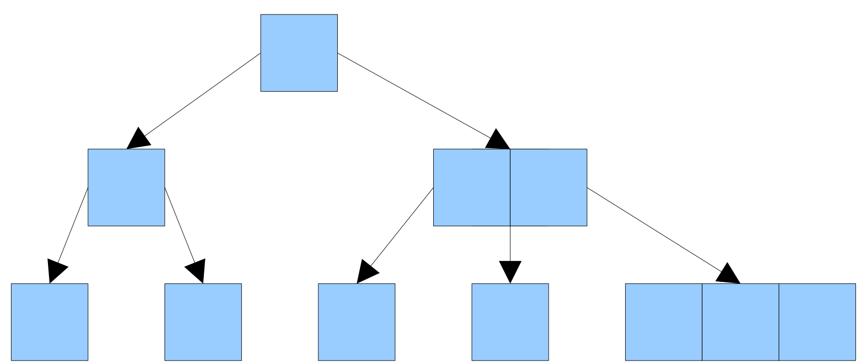
- Suppose we grow the array not by a factor of two, but by a fixed constant $\alpha > 1$. Find a choice of Φ so that the amortized cost of an append is O(1).
- Suppose we also allow elements to be removed from the array, and when it's ¼ full we shrink it by a factor of two. Find a choice of Φ so the amortized cost of appending or removing the last element is O(1).



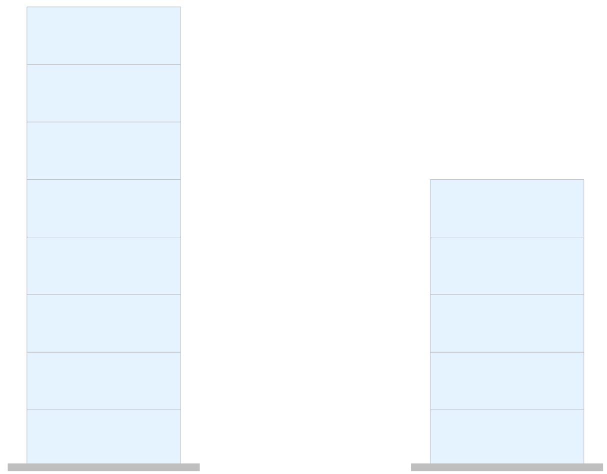
Two-Stack Queues



Dynamic Arrays

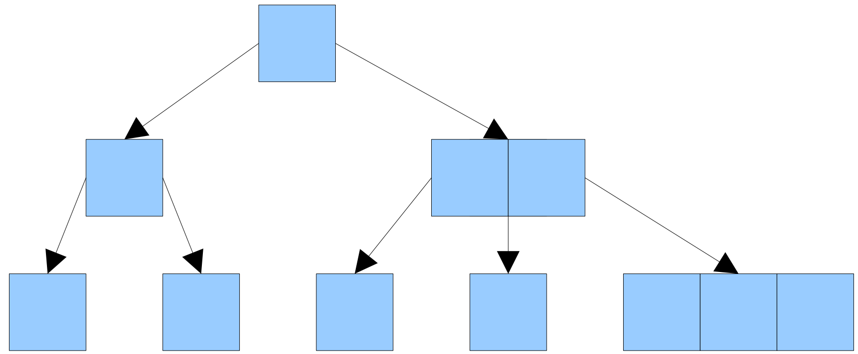


Building B-Trees



Two-Stack Queues

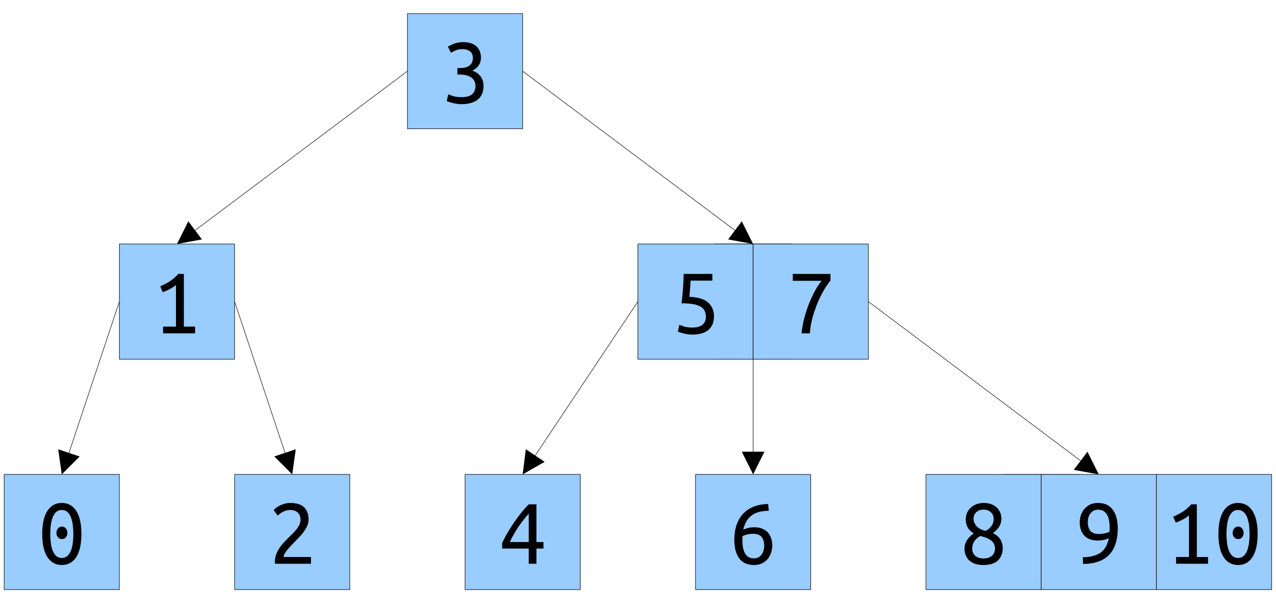
Dynamic Arrays



Building B-Trees

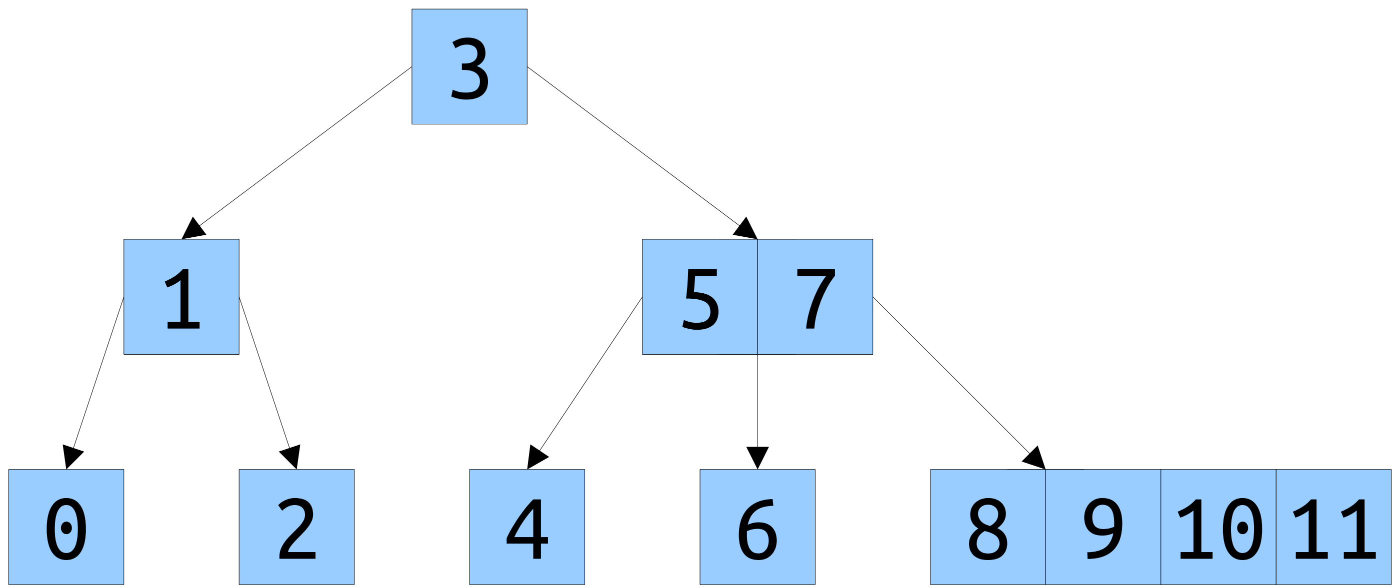
Building B-Trees

- ***Algorithm:*** Store a pointer to the rightmost leaf. To add an item, append it to the rightmost leaf, splitting and kicking the median key up if we are out of space.



Building B-Trees

- ***Algorithm:*** Store a pointer to the rightmost leaf. To add an item, append it to the rightmost leaf, splitting and kicking the median key up if we are out of space.



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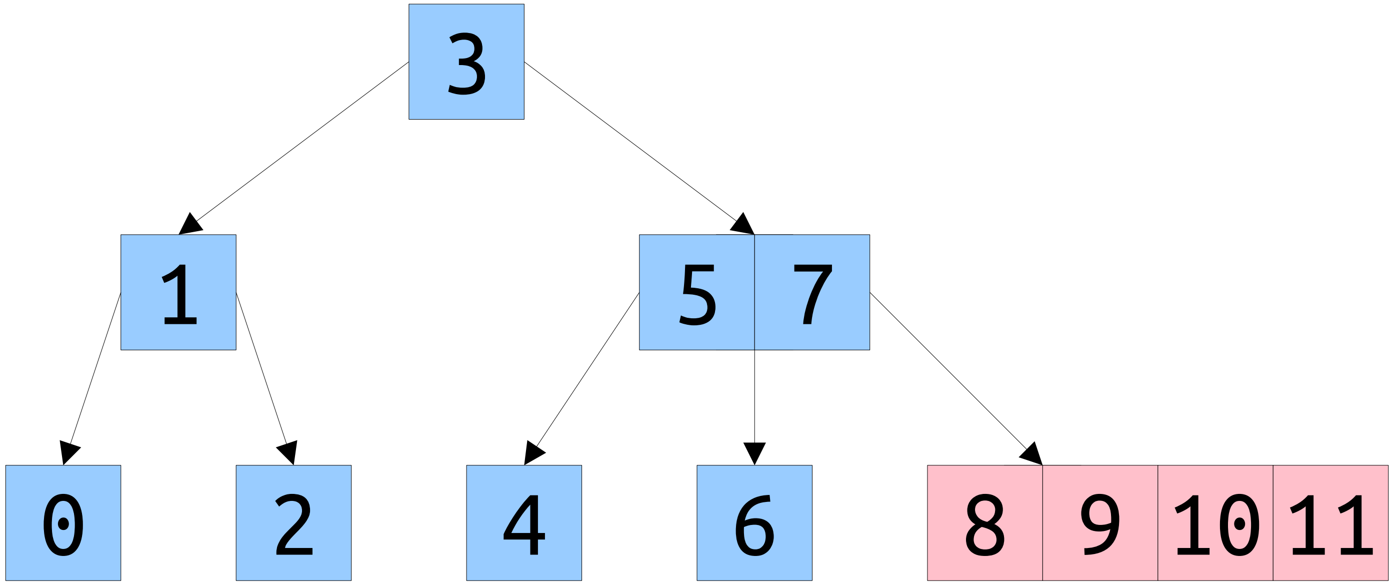

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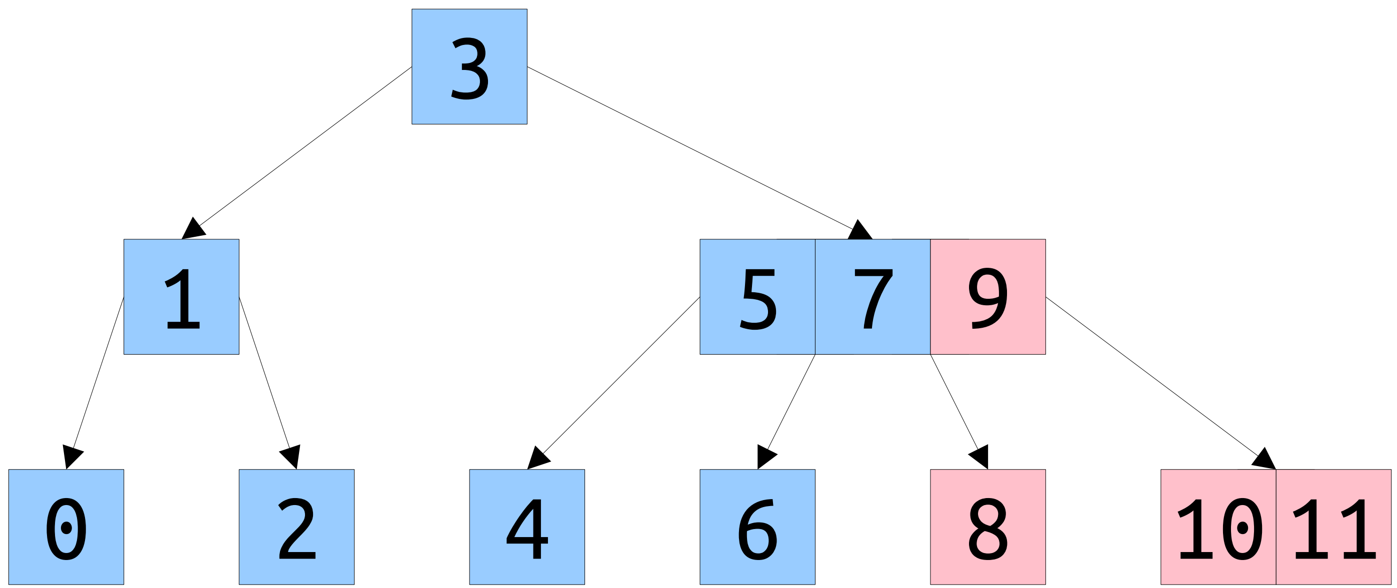

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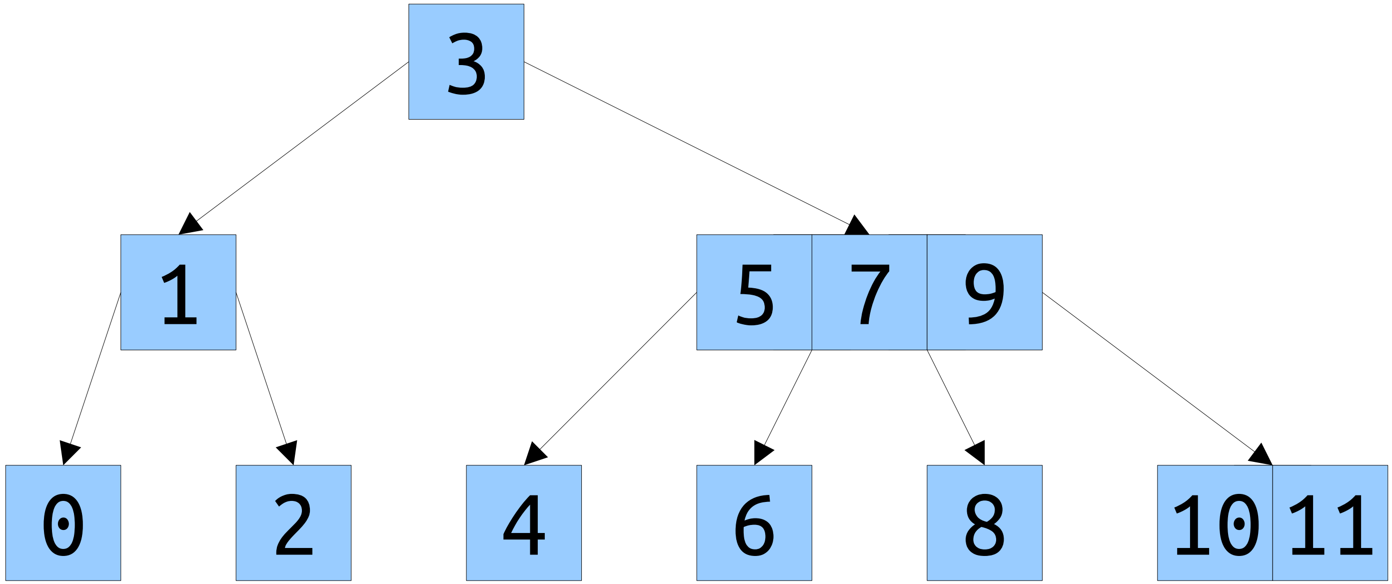

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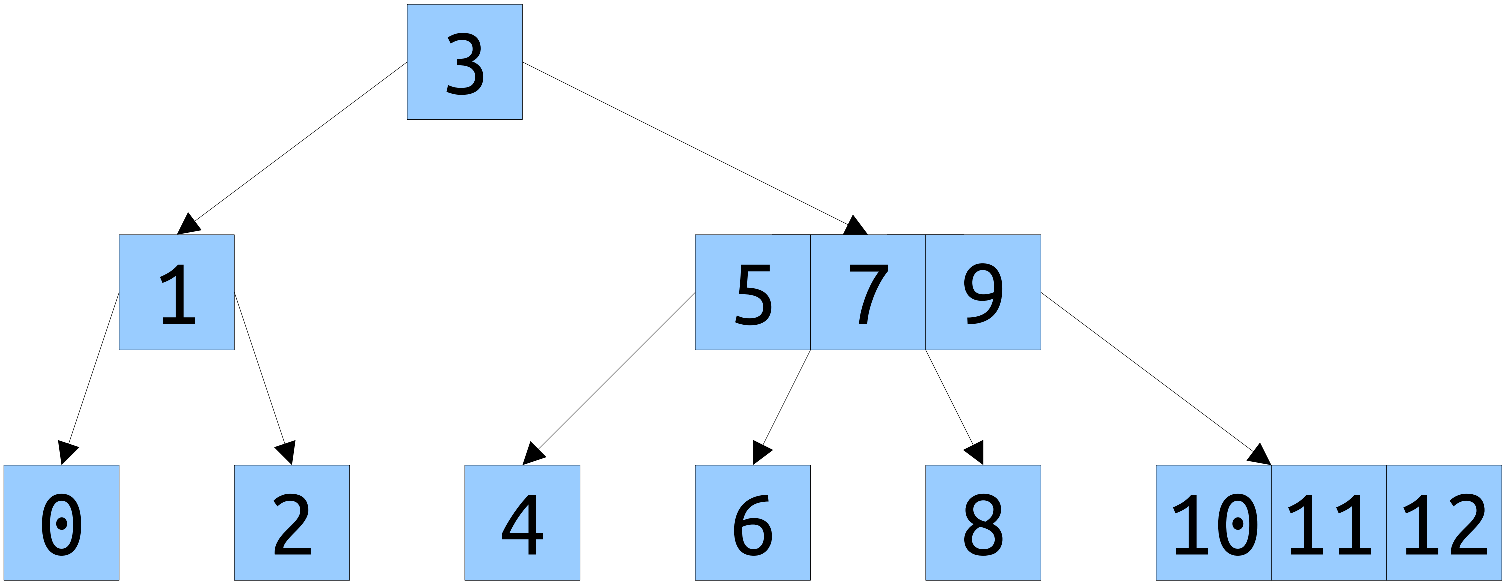

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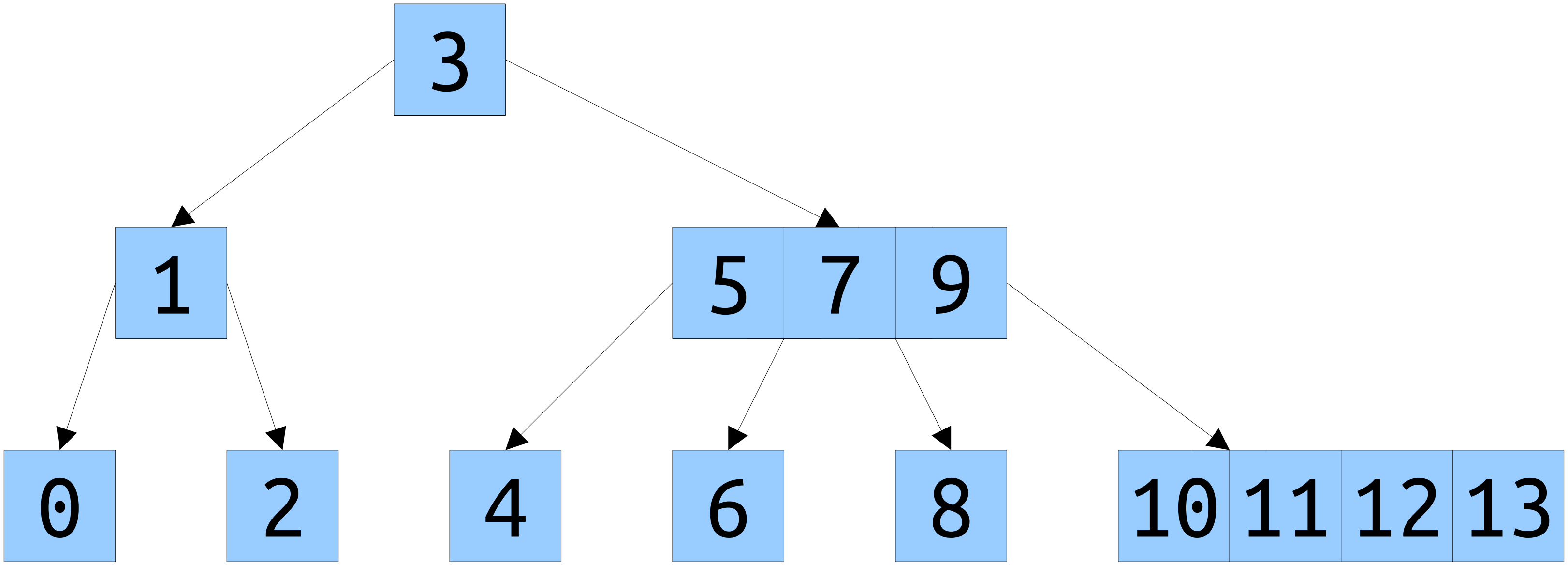

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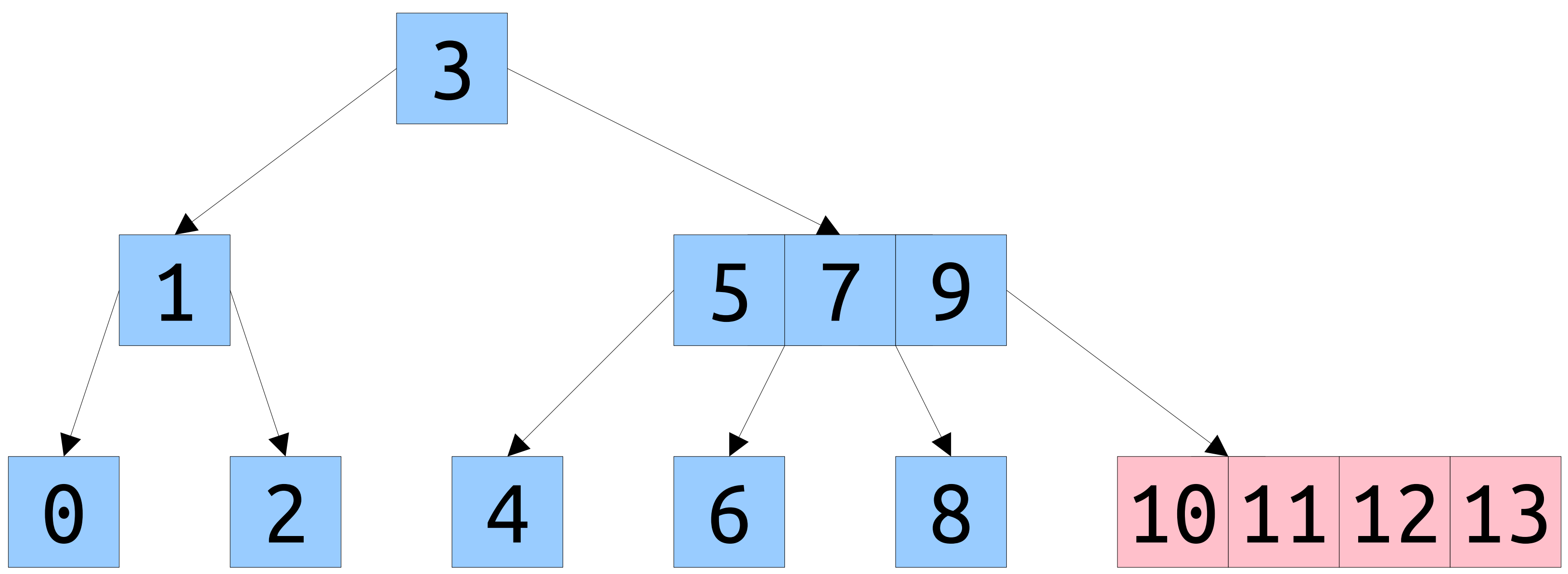

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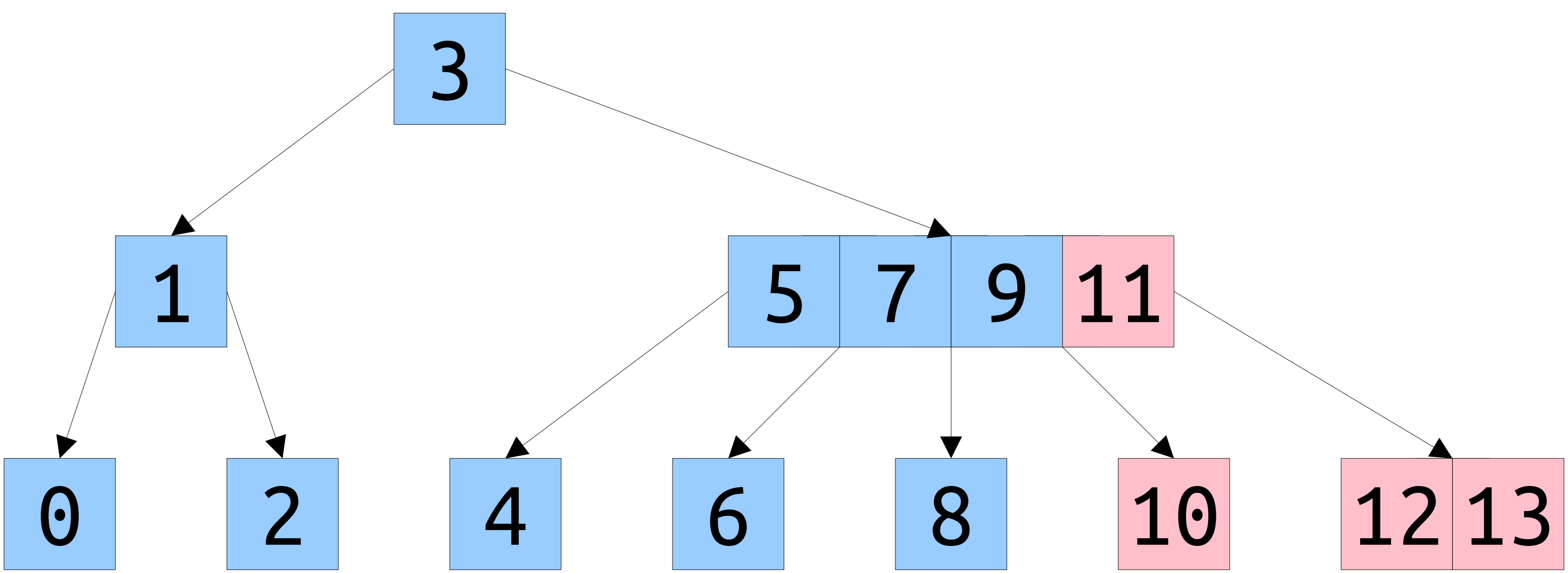

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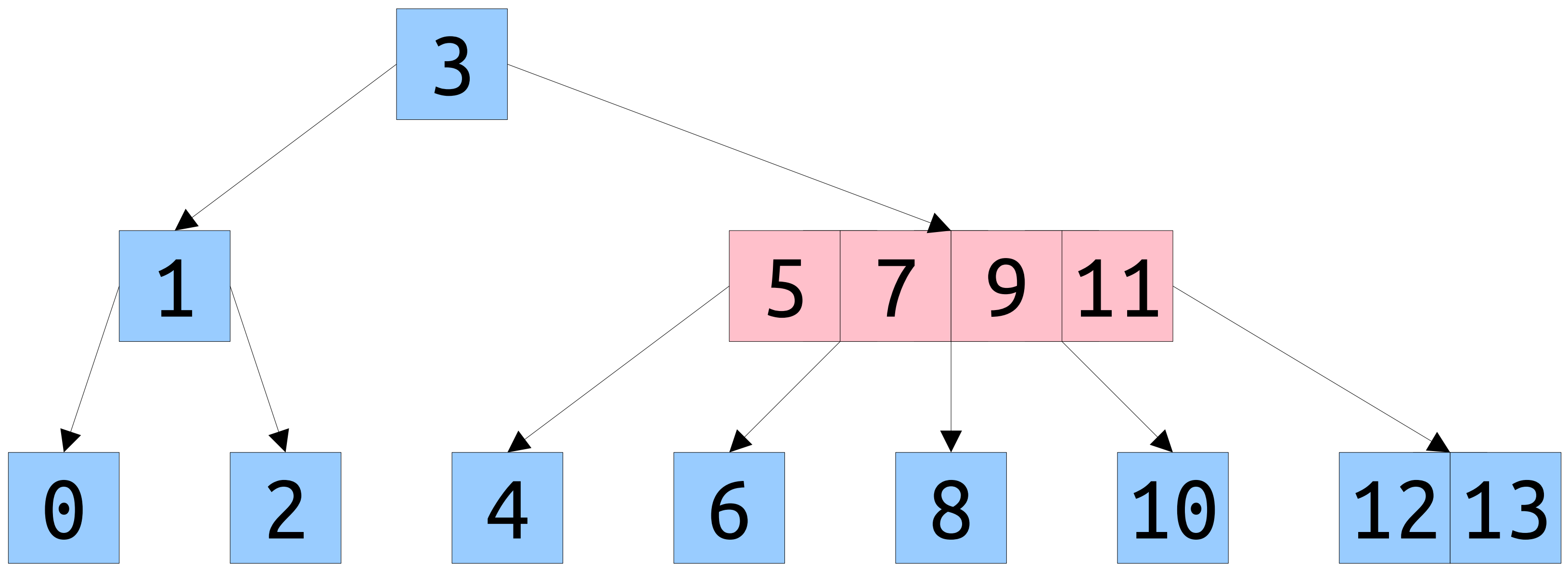

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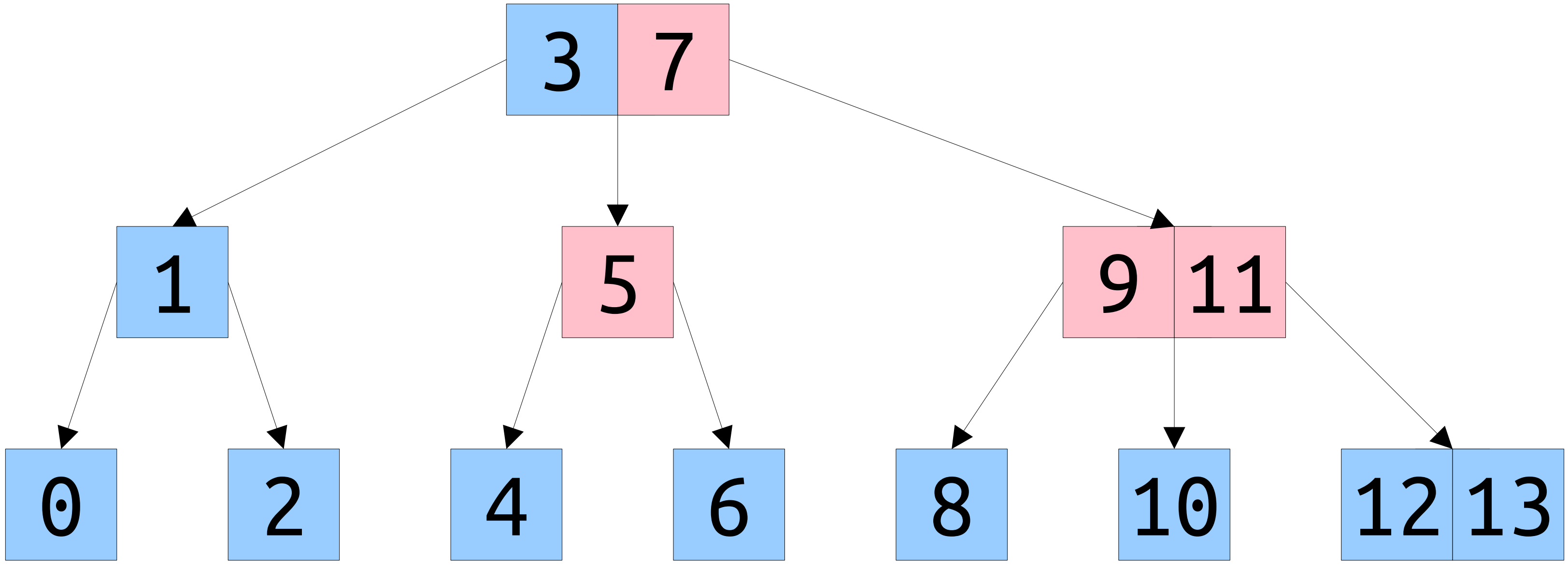

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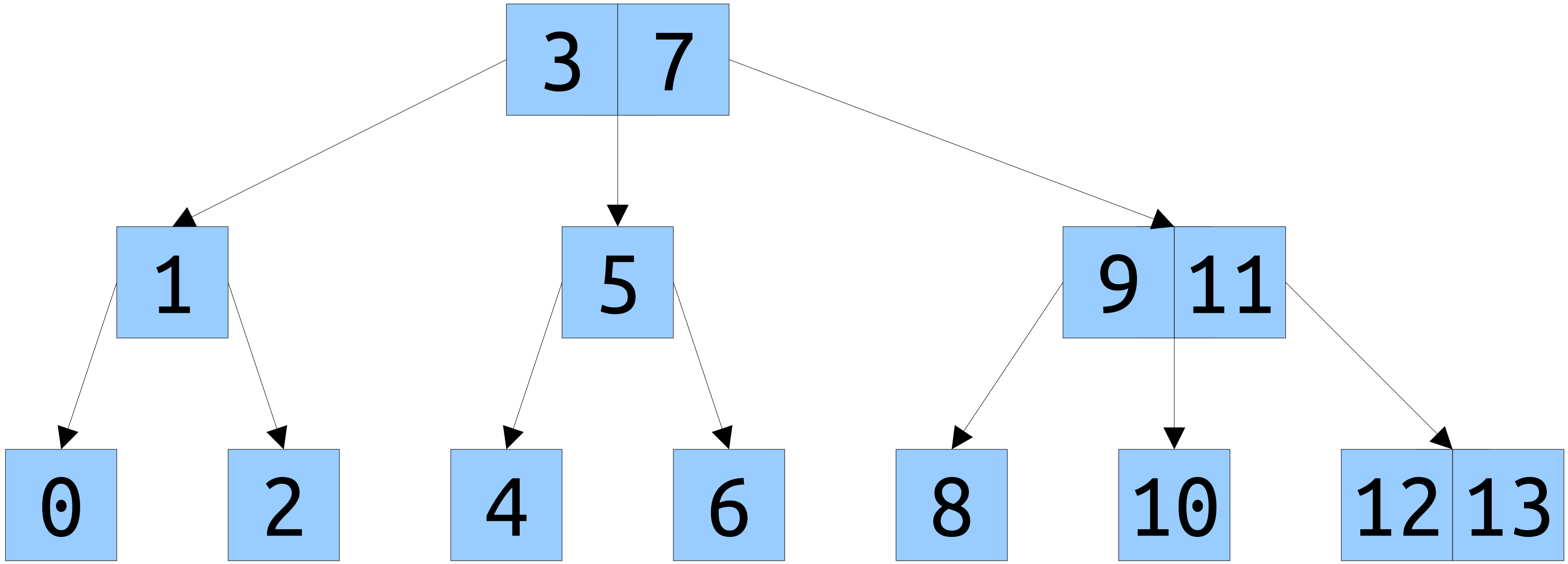
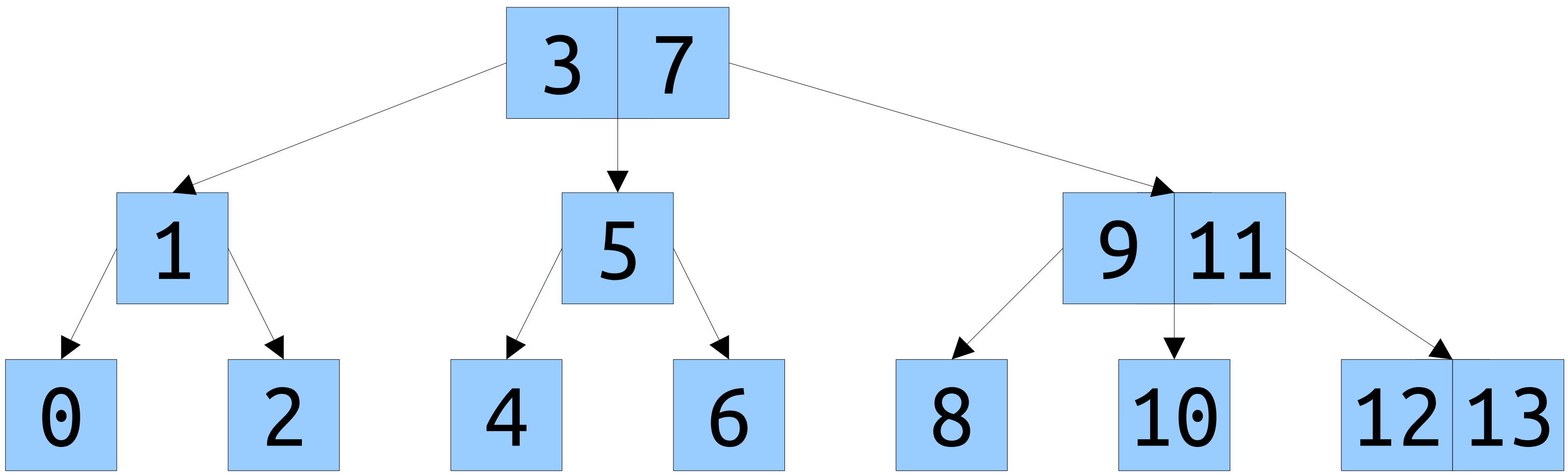


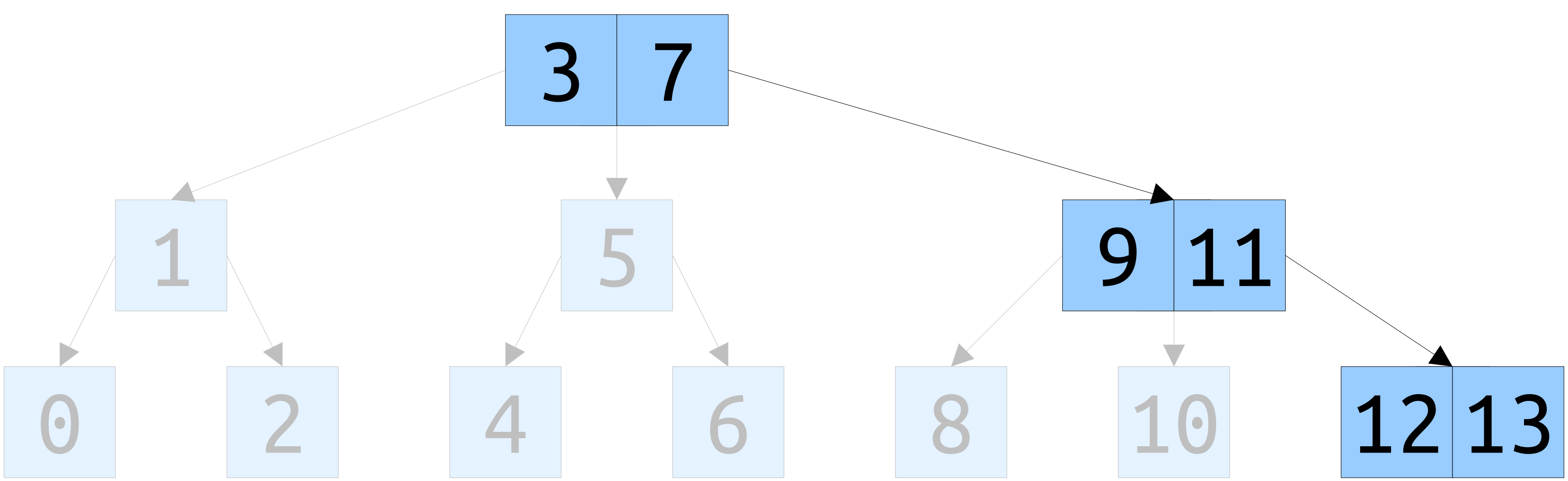
Building B-Trees

- What is the actual cost of appending an element?
  - Suppose that we perform splits at L layers in the tree.
  - Each split takes time $\Theta(b)$ to copy and move keys around.
  - Total cost: **$\Theta(bL)$**.
- ***Goal:*** Pick a potential function Φ so that we can offset this cost and make each append cost amortized O(1).



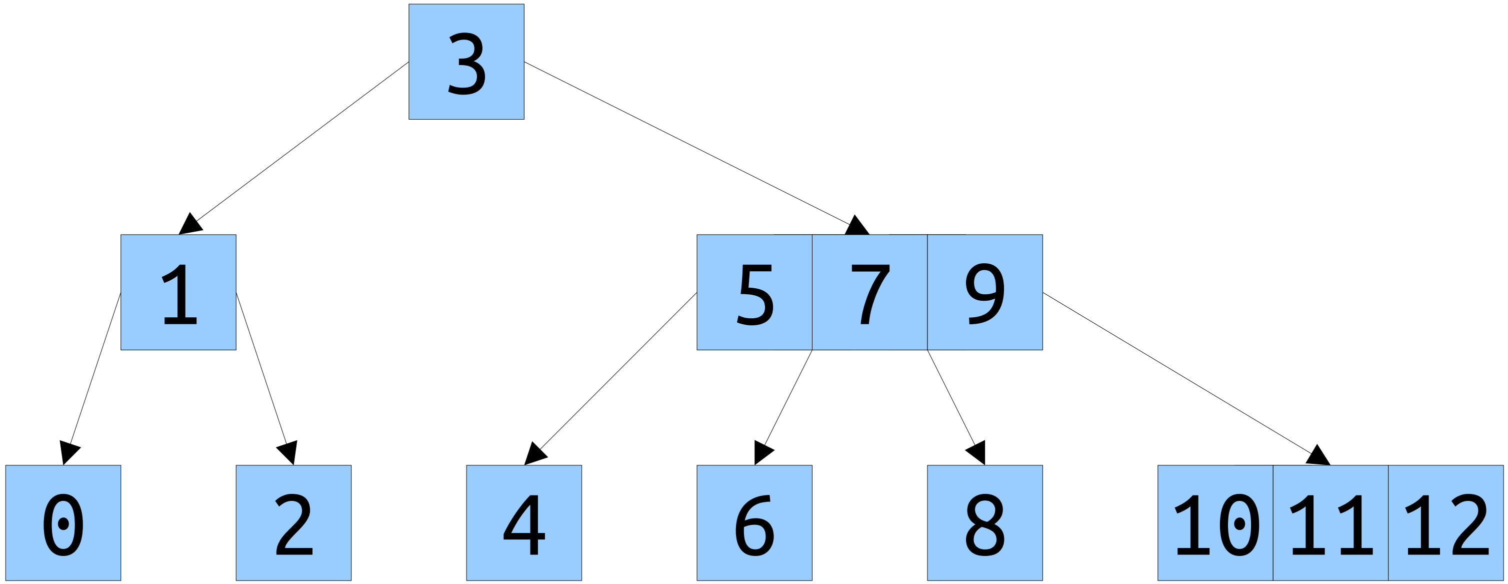
Building B-Trees

- Our potential function should, intuitively, quantify how "messy" our data structure is.
- Some observations:
  - We only care about nodes in the right spine of the tree.
  - Nodes in the right spine slowly have keys added to them. When they split, they lose (about) half of their keys.
- ***Idea:*** Set Φ to be the number of keys in the right spine of the tree.



Building B-Trees

- Let Φ be the number of keys on the right spine.
- Each split moves (roughly) half the keys from the split node into a node off the right spine.



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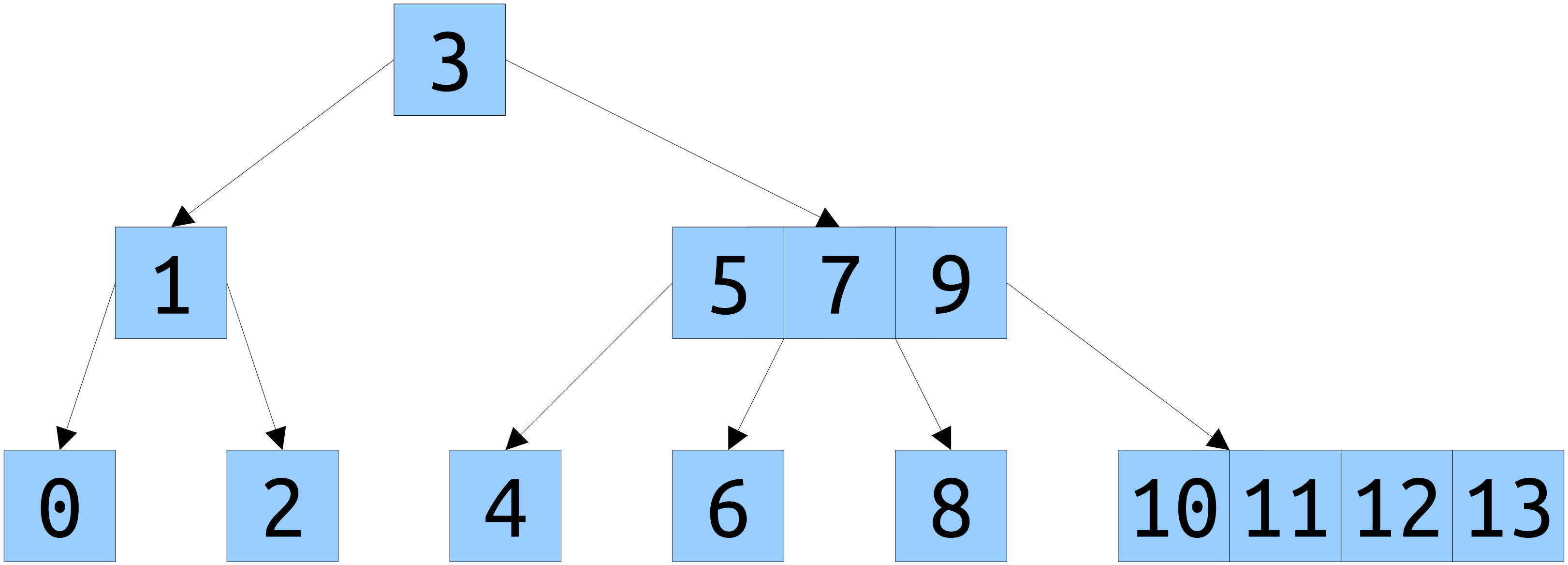

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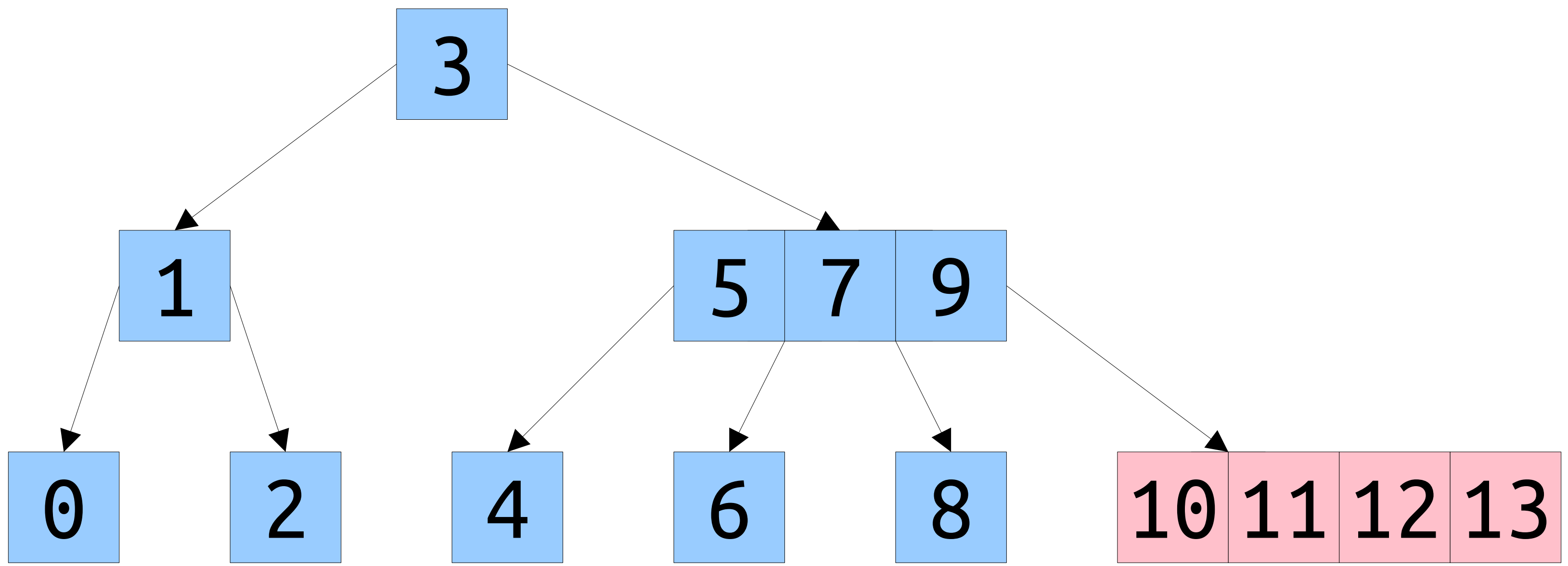

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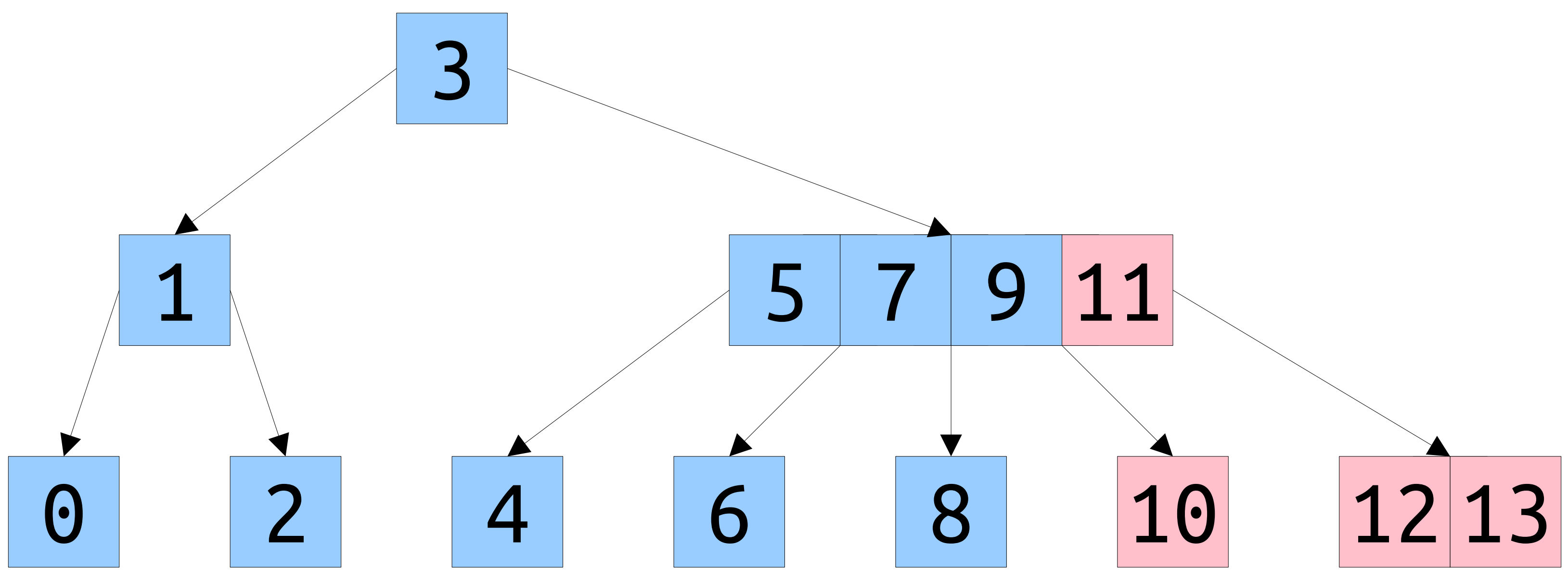

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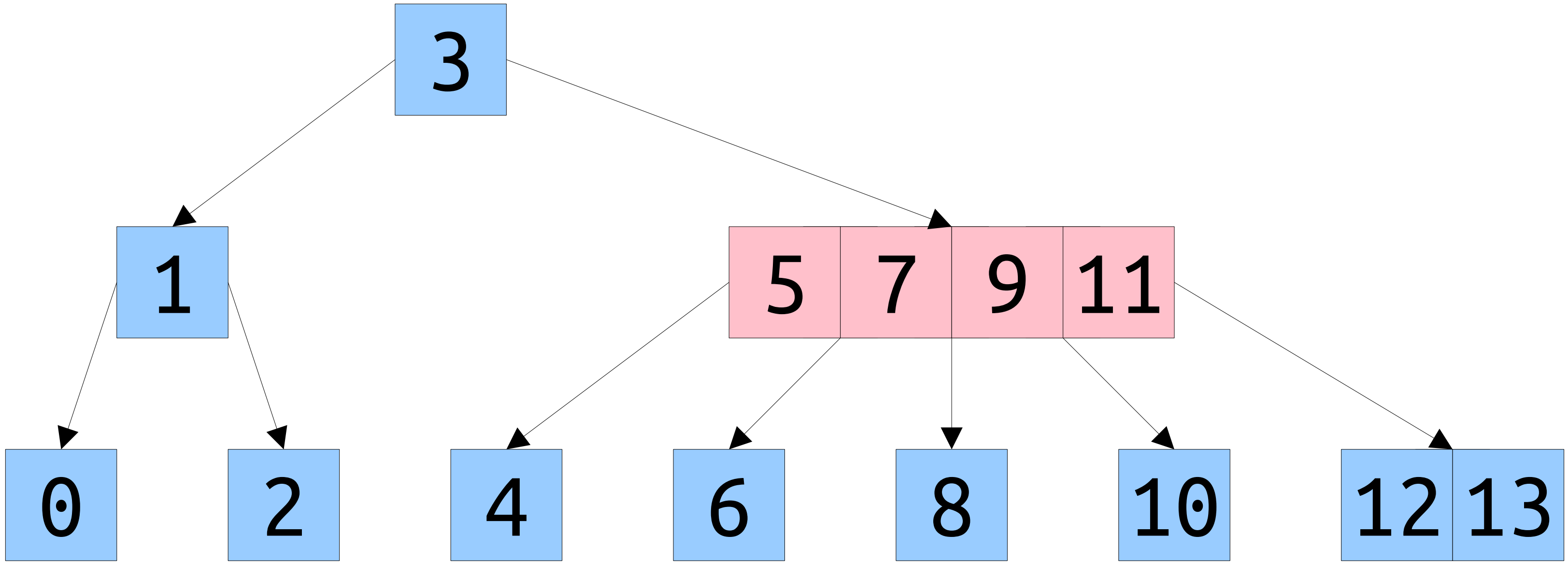

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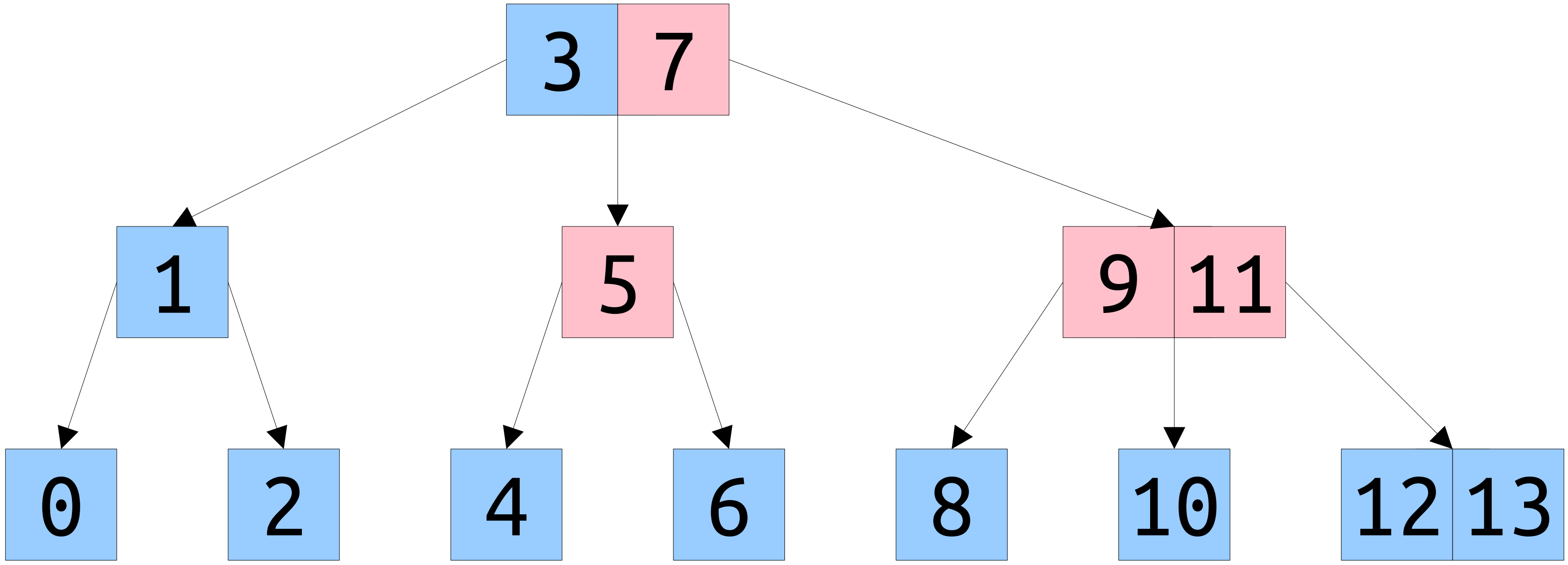
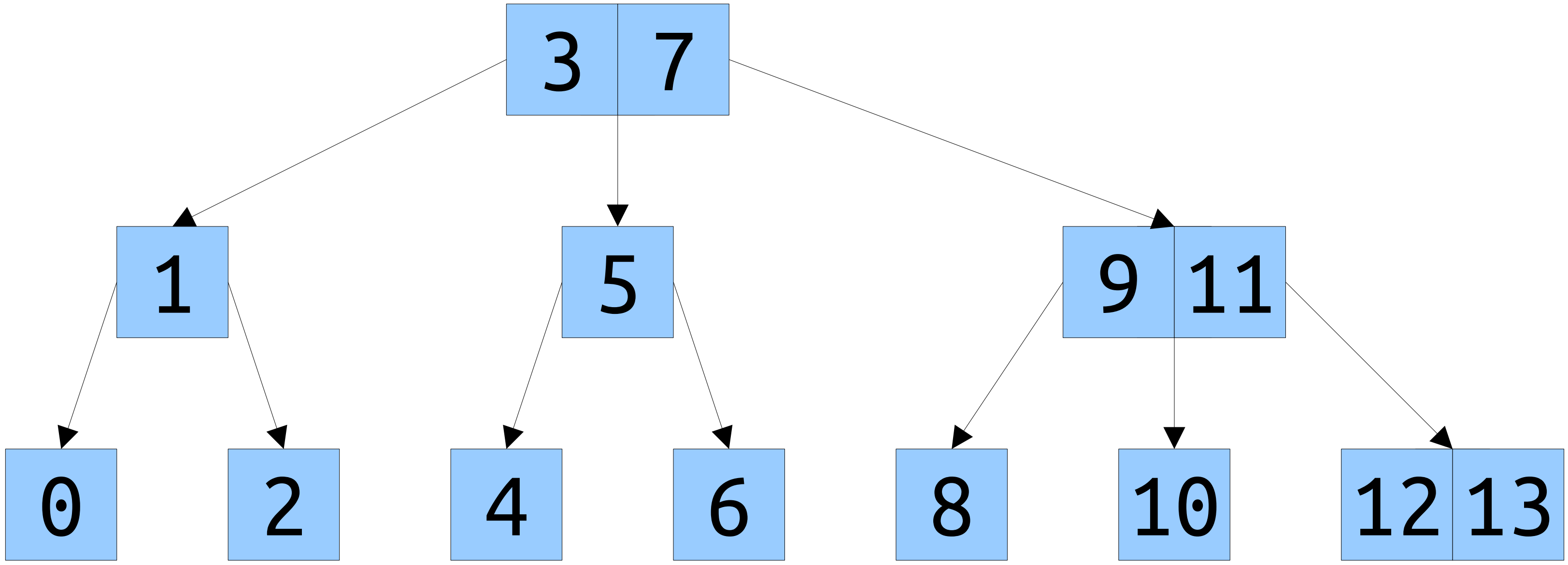


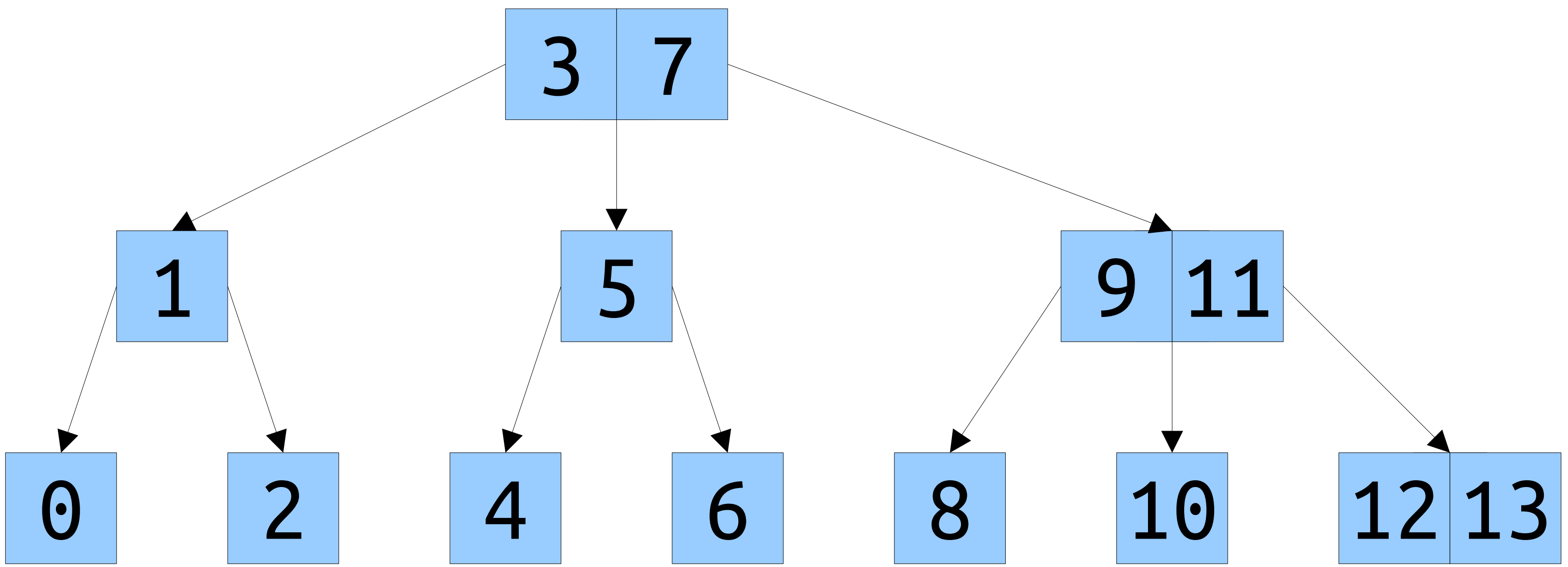
Building B-Trees

- Let Φ be the number of keys on the right spine.
- Each split moves (roughly) half the keys from the split node into a node off the right spine.
- Change in potential per split: $-\Theta(b)$.
- Net $\Delta\Phi$: **$-\Theta(bL)$**.



Building B-Trees

- Actual cost of an append that does L splits: $O(bL)$.
- $\Delta\Phi$ for that operation: $-\Theta(bL)$.
- Amortized cost: **O(1)**.



Theorem: The amortized cost of appending to a B-tree by inserting it into the rightmost leaf node and applying fixup rules is O(1).

Proof: Assume we are working with a B-tree of order b. Let Φ be the number of nodes on the right spine of the B-tree.

Suppose we insert a value into the tree using the algorithm described above. Suppose this causes L nodes to be split. Each of those splits requires $\Theta(b)$ work for a net total of $\Theta(bL)$ work.

Each of those L splits moves $\Theta(b)$ keys off of the right spine of the tree, decreasing Φ by $\Theta(b)$ for a net drop in potential of $-\Theta(bL)$. In the layer just above the last split, we add one more key into a node, increasing Φ by one. Therefore, $\Delta\Phi = -\Theta(bL)$.

Overall, this tells us that the amortized cost of inserting a key this way is

$$\Theta(bL) + k \cdot \Delta\Phi = \Theta(bL) - k \cdot \Theta(bL),$$

which can be made to be O(1) by choosing k to equate the constants hidden in the O and Θ terms. ■

More to Explore

- You can implement a ***deque*** (a doubly-ended queue) using a B-tree with pointers to the first and last leaves.
  - This is sometimes called a ***finger tree***.
  - Finger trees are used extensively in purely functional programming languages.
  - By extending the analysis from here, you can show the amortized cost of appending or removing from each end of the finger tree is O(1).
- Red/black trees are modeled on 2-3-4 trees. You can build a red/black tree from n sorted keys in time O(n) this way.
  - ***Great exercise:*** Explore how to do this, and work out what choice of Φ to make.

To Summarize

Amortized Analysis

- Some data structures accumulate messes slowly, then clean up those messes in single, large steps.
- We can assign ***amortized*** costs to operations. These are fake costs such that summing up the amortized costs never underestimates the sum of the real costs.
- To do so, we define a potential function Φ that, intuitively, measures how “messy” the data structure is. We then set

$$\textbf{\textit{amortized-cost}} = \textbf{\textit{real-cost}} + k \cdot \Delta\Phi.$$

- For simplicity, we assume that Φ is nonnegative and that Φ for an empty data structure is zero.

Next Time

- ***Binomial Heaps***
  - A very clever way to build a priority queue.
- ***Lazy Binomial Heaps***
  - Designing for amortization.