Discrete Probability

- **Sample space S: Set of possible outcomes of an experiment.**
  - » **Example:**
    - – **Experiment: roll two dice**
    - – **36 possible outcomes**
      - **S= {(1,1) (1,2) (1,3) …. (2,1) (2,2)…. (6,1)…(6,6)}**

- **Events: Subsets of sample space**
  - » **Example:**
    - – **Roll 3 with 2 dice : E= {(1,2) (2,1)}**
    - – **At least one of the dice shows up 1 :**

    E= {(1,1) (1,2), …. (1,6) (2,1) (3,1) … (6,1)}

Discrete Probability

- **Probability: Values between 0 to 1 assigned to all possible subsets of the sample space (events)**

$$P : 2^S \rightarrow [0\ 1]$$

- **Must satisfy following properties**

$$P(\phi) = 0, P(S) = 1$$

$$P(A \cup B) = P(A) + P(B) \ \ if \ \ A \cap B = \phi$$

- **For any event E, we have the property:**

$$P(E) = \sum_{s \in S} P(s)$$

Examples

Uniform distribution on outcome of rolling two Dice

P(s) = 1 / |S|

What is the probability, when two dice are rolled, that the sum is 12 ?

Subset: A={(6,6)}

P(A) = 1 / 36

What is the probability, when two dice are rolled, that the sum is 7 ?

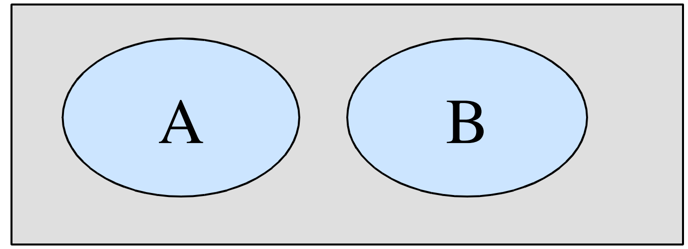
Subset: A = {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}

P(A) = P((1,6)) + P((2,5)) + P((3,4)) + P((4,3)) + P((5,2)) + P((6,1)) = 6* (1/36) = 1/6

Disjoint unions and Complement

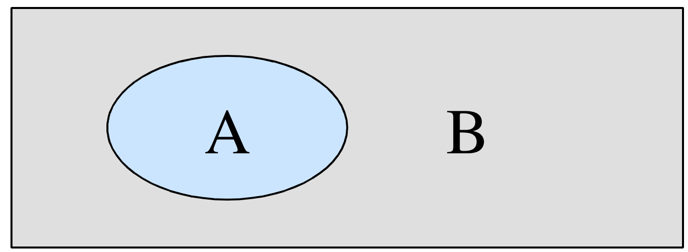
- **If A and B are disjoint (mutually exclusive)**

$$P(A \cup B) = P(A) + P(B)$$



- **Complement** $B = \overline{A}$

$$1 = P(S) = P(A) + P(B)$$



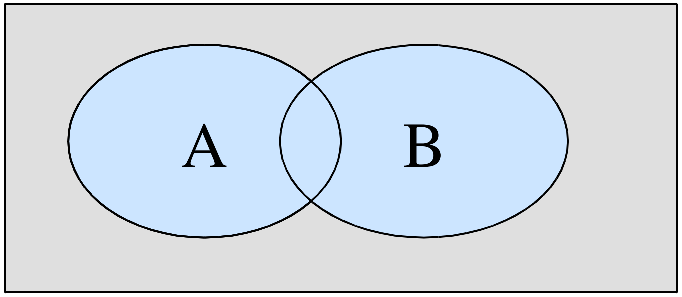
What is the probability that two dice never come up with a sum of 7 ?

P(A) = P(sum=7) = 1 / 6

P($\overline{A}$) = 1 - P(A) = 1 - 1 / 6 = 5 / 6

Union

- **Inclusion-Exclusion Principle:**
  Let A and B be two events in the same space S.
  Then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



- **Example: probability of having 1 on the first die or 2 on the second when die is thrown twice?**

$$P(A) = 1 \,.\, 6 / 36 = 6 / 36$$

$$P(B) = 6 \,.\, 1 / 36 = 6 / 36$$

$$P(A \cup B) = 6/36 + 6/36 - 1/36 = 11 / 36$$

Independence

- **Two events A and B are said to be independent if and only if**

$$P(A \cap B) = P(A) . P(B)$$

- **Different from mutually exclusive!**
- **Example**

A = dice 1 shows "2" = 1/6

B = dice 2 shows "3" = 1/6

$P(A \cap B) = \{(2,3)\} = 1/36$

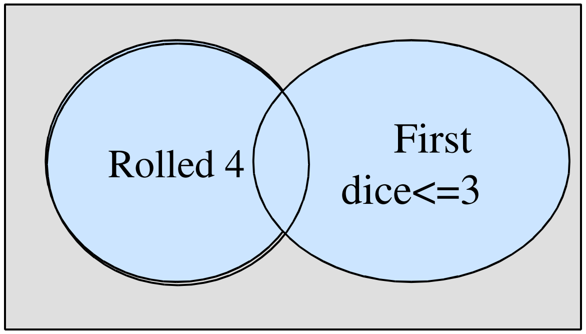
We remark that A and B are independent because $P(A \cap B) = P(A) . P(B)$

Conditional probability

- **Let A and B be events with P(B) > 0. The conditional probability of A given B, denoted P(A/B), is defined as:**

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

  - *(A new probability measure)*



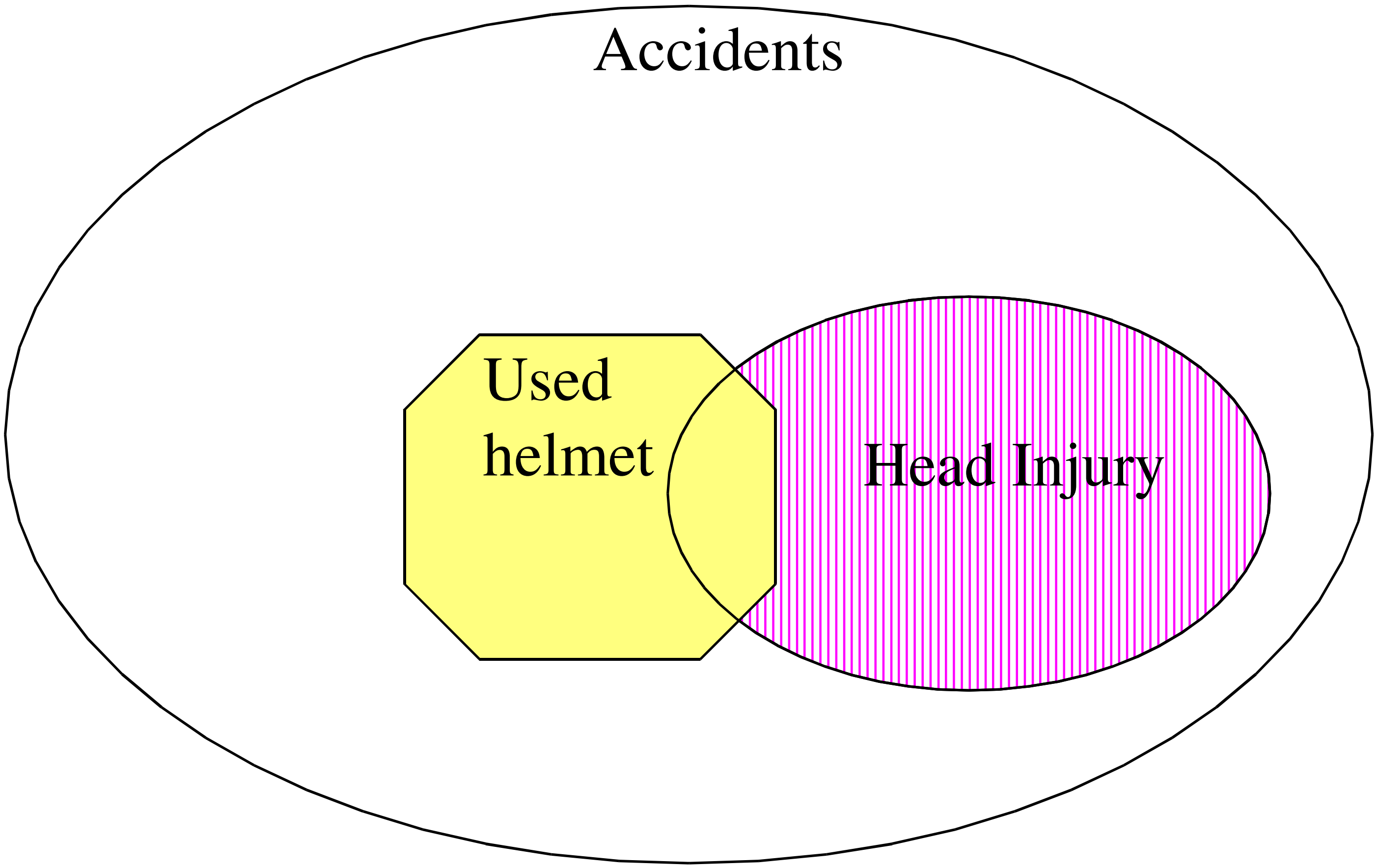
- **What is probability of rolling 4 given that first dice rolled something less than or equal to 3?**

Pr((3,1) (2,2) (1,3)) = 3/36 = 1/12

scale it to make a probability measure:

Conditional probability (1/12)/(1/2) = 1/6

Conditional probability



Two-colored divided by yellow:

Probability of head injury given that a helmet was used

Random Variables

- **A random variable is a function from the sample space of an experiment to the set of real numbers.**

$$X : S \rightarrow R$$

- **It assigns a real number to each outcome.**
- **Example: X(s) is the sum of outcomes of each dice**

$$X((1,2)) = 3$$

$$X((2,4)) = 6$$

Expected Value

- **The expected value of random variable X(s) on the sample space S is equal to**

$$E[X(s)] = \sum_{s} X(s)P(s)$$

- **Example:**

SUM	Pr x 36	SUM x Pr x 36
1	0	0
2	1	2
3	2	6
4	3	12
⋮	⋮	⋮
12	1	12

		252

E[X]=252/36=7

Linearity of Expectation

- **If X and Y are random variables on same space S, then**

$$E[X] = E[X] + E[Y]$$

- **If Xi, i=1,2,…,n with n a positive integer, are random variables on S, then**

$$E[X] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

- **Moreover, if a and b are real numbers then**

$$E[aX + b] = aE[X] + b$$

Independent Random Variables

- **The random variables X and Y on a sample space S are independent:**

$$P(X(s)=r_1 \; and \; Y(s)=r_2)=P(X(s)=r_1)\cdot P(Y(s)=r_2)$$
$$for \; all \; r_1, r_2$$

- **Example: a die is rolled twice. X is the number of spots on first roll, and Y is the number of spots on second roll.**

$$P(X=i, Y=j)=P((i,j))=\frac{1}{36}$$
$$P(X=i)=\frac{1}{6}, \qquad P(Y=j)=\frac{1}{6}$$

Thus the variables are independent.

- **If X, Y are independent, then**

$$E[XY]=E[X]E[Y]$$

Conditional Expectation

- **Conditional Expectation of random variable X given random variable Y with respect to conditional probability measure P(X|Y)**
- **Z=E[X|Y] is a random variable**

$$Z(j) = E[X \mid Y = j] = \sum_i iP(X = i \mid Y = j)$$

- **Example**
  - » Consider 1-dice toss.
  - » Let X be result of the toss, and Y be the event that the result is above 2.
    (Y=1 if above 2, Y=0 otherwise.)
    Note that Pr[Y=0]=2/6, Pr[Y=1]=4/6.
  - » Conditional expectation of X given Y.

$$Z(0) = E[X \mid Y = 0] = \sum_i iP(X = i \mid Y = 0) = (1+2)\frac{1/6}{2/6} = \frac{3}{2}$$

$$Z(1) = E[X \mid Y = 1] = \sum_i iP(X = i \mid Y = 1) = (3+4+5+6)\frac{1/6}{4/6} = \frac{9}{2}$$

Law of iterated expectations

- **The expected value of the conditional expected value of X given Y is the same as the expected value of X.**

$$E[E[X \mid Y]] = E[X]$$

- **Proof:**

$$\begin{aligned}
E_y\left[E_x[X \mid Y]\right] &= \sum_i \Pr[Y=i] E_x[X \mid Y=i] \\
&= \sum_i \Pr[Y=i] \sum_j j \Pr[X=j \mid Y=i] \\
&= \sum_i \sum_j j \Pr[X=j \mid Y=i] \Pr[Y=i] \\
&= \sum_j \sum_i j \Pr[X=j, Y=i]] \\
&= \sum_j j \Pr[\bigcup_i (X=j, Y=i)] \\
&= \sum_j j \Pr[X=j] \\
&= E[X]
\end{aligned}$$

Conditional Expectation example

- **Consider 1-dice toss.**
- **Let X be result of the toss, and Y be the event that the result is above 2. (Y=1 if above 2, Y=0 otherwise.)**
- **Condition on Y.**
  Note that Pr[Y=0]=2/6, Pr[Y=1]=4/6.

$$Z(0) = \frac{3}{2}, \quad Z(1) = \frac{9}{2}$$

$$E[E[X \mid Y]] = E[Z] = \frac{3}{2} \cdot \frac{2}{6} + \frac{9}{2} \cdot \frac{4}{6} = \frac{42}{12} = \frac{7}{2}$$

$$E[X] = (1+2+3+4+5+6) \cdot \frac{1}{6} = \frac{7}{2}$$