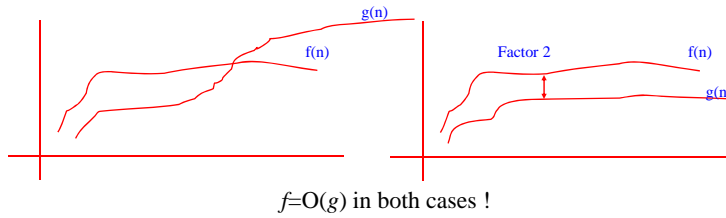


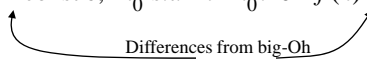
Asymptotics



Asymptotics

- **Small-oh notation:**

$$f(n) = o(g(n)) \Leftrightarrow \forall \text{const } c, \exists n_0 \text{ s.t. } \forall n \geq n_0: 0 \leq f(n) < cg(n)$$



Prove that $n = o(n^2)$:

Given c , let's take $n_0 = 2/c$

$$\Rightarrow \text{for } n \geq n_0, n^2 \geq \frac{2}{c}n \Rightarrow cn^2 \geq c\left(\frac{2}{c}n\right) = 2n > n \quad \text{QED}$$

Omega notation

- **Big-Omega:**

$$f(n) = \Omega(g(n)) \Leftrightarrow \exists \text{const } c, n_0 \text{ s.t. } \forall n \geq n_0: 0 \leq cg(n) \leq f(n)$$

- **Small-omega:**

$$f(n) = \omega(g(n)) \Leftrightarrow \forall \text{const } c, \exists n_0 \text{ s.t. } \forall n \geq n_0: 0 \leq cg(n) < f(n)$$

- **Intuition (works most of the time):**

$$\begin{array}{ll} O: \leq & o: < \\ \Omega: \geq & \omega: > \end{array}$$

17

Transitivity etc.

- **Most rules apply:**
Example: transitivity

$$a \leq b, b \leq c \Rightarrow a \leq c$$

$$f = O(g), g = O(h) \Rightarrow f = O(h)$$

Proof:

$$f = O(g) \Rightarrow \exists \text{const } c_1, n_1 \text{ s.t. } \forall n \geq n_1: 0 \leq f(n) \leq c_1 g(n)$$

$$g = O(h) \Rightarrow \exists \text{const } c_2, n_2 \text{ s.t. } \forall n \geq n_2: 0 \leq g(n) \leq c_2 h(n)$$

$$\text{Take } n_3 = \max(n_1, n_2), c_3 = c_1 c_2$$

$$\text{Then: } \forall n \geq n_3: 0 \leq f(n) \leq c_1 g(n) \leq c_1 c_2 h(n) = c_3 h(n)$$

$$\Rightarrow f(n) = O(h(n)) \quad \text{QED}$$

- **Not all rules apply!**

$$\exists f, g \text{ s.t. } f \neq O(g) \text{ and } g \neq O(f)$$

$$\text{example: } f = n, g = n^{1+\sin n}$$

18

Theta notation

- **Theta:**

$$f(n) = \Theta(g(n)) \Leftrightarrow \exists \text{const } c_1, c_2, n_0 \text{ s.t. } \forall n \geq n_0: 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

- Often confused with **Big-Oh** notation!

- **Example:** $n^2/2 - 2n = \Theta(n^2)$

Proof:

take $n_0 = 8$, then for $n \geq n_0$:

$$n^2/2 - 2n = n^2/4 + n^2/4 - 2n \geq n^2/4 + 8n/4 - 2n = n^2/4$$

On the other hand, we have: $n^2/2 - 2n < n^2/2$

Thus: $n^2/4 \leq n^2/2 - 2n \leq n^2/2$ i.e. $c_1 = 1/4, c_2 = 1/2$.

- **Claim:** Low order terms do not matter. Needs a proof! (HW?)

19

Simple Theorem

- **Claim:**
 $f(n) = O(g(n))$ and $g(n) = O(f(n)) \Rightarrow f(n) = \Theta(g(n))$

Proof:

$$\exists n_1, c_1 \text{ s.t. } \forall n \geq n_1: 0 \leq f(n) \leq c_1 g(n)$$

$$\exists n_2, c_2 \text{ s.t. } \forall n \geq n_2: 0 \leq g(n) \leq c_2 f(n)$$

$$\Rightarrow \forall n \geq \max(n_1, n_2): 0 \leq \frac{1}{c_2} g(n) \leq f(n) \leq c_1 g(n) \quad \text{QED}$$

20

Summary

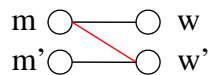
- Remember the definitions.
- Formally prove from definitions.
- Use intuition from the properties of " \leq ", " \geq ", etc.

21

Example of an algorithm Stable Marriage

- n man and n women
- Each woman ranks all men and each man ranks all women
- Find a way to match (marry) all men and women such that there are no two pairs (m,w) and (m',w') that are married and such that
 - » m prefers w' to w
 - » w' prefers m to m'

In other words, m will "steal" w' from m' and w' will agree



Red line shows "instability"

22

Discussion

- Bipartite graph
- Matching = legal set of marriages (no polygamy)
- Perfect matching = all men/women married
- Looking for perfect matching with “stability” constraint (similarity to shortest path, minimum spanning tree)
- Is there a perfect matching ?
- Is there a stable perfect matching ? Always or only for some input data ?
- Brute force approach ?

Gale-Shapley algorithm

- As long as there is a man that is not engaged
 - » Pick free man m
 - » m tries to propose to the next woman w on his list (going down in terms of preferences) that he did not propose to yet
 - » if w is free,
 - then m and w become engaged
 - else (w is engaged to m')
 - if w prefers m' to m ,
 - » then m remains free
 - » else m and w become engaged, m' becomes free (m “steals” w from m')

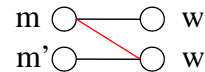
Some useful claims

- **Claim 1:** Once woman is engaged, she never becomes free.
Sequence of her partners improves (in terms of her preference list)
- **Claim 2:** The sequence of women a man m proposes to gets worse and worse (in terms of his preference list)
- **Claim 3:** If at some point m is free, then he has not yet proposed to all the women on his list
Proof:
 - » Assume m has proposed to all women already
 - » Since he is still free, all women are engaged (see Claim 1)
 - » But the number of men and women is the same - contradiction

25

Analysis

- **Termination:** follows from Claim 3
(At this point we have a **perfect matching**)
- **Correctness:**
Assume that the algorithm is incorrect,
i.e. there exist 2 engaged couples $(m,w), (m',w')$ such that
 - » m prefers w' to w
 - » w' prefers m to m'
 - » By construction, m last proposal was to w
 - » Did m propose to w' beforehand?
 - If not, then m prefers w to w' - contradiction using claim 2
 - If yes, then w' rejected him - contradiction using claim 1 (woman improves her partners)



26

Running time analysis

- Need to find “measure of progress”
- Consider number of possible proposal (m,w) pairs
 - » Proposal pairs never repeat
 - » Total n^2 possible proposal pairs
 - » n^2 iterations
 - » Each iteration $O(1)$ time
 - » Total running time $O(n^2)$