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19: Sampling and the Bootstrap

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[Lecture Discussion on Ed]

Sampling definitions

Motivating example

we really want to know the average happiness of the full distribution

You want to know the true mean and variance of happiness in Bhutan.

- But you can't ask everyone.
- You poll 200 random people.
- Your data looks like this:

Happiness = $\{\underbrace{72, 85, 79, 91, 68, \ldots, 71}_{\text{average}}\}$

- The mean of all these numbers is 83.

Is this the **true mean happiness** of all Bhutanese people?

Population

assume for simplicity that population is N=100,000

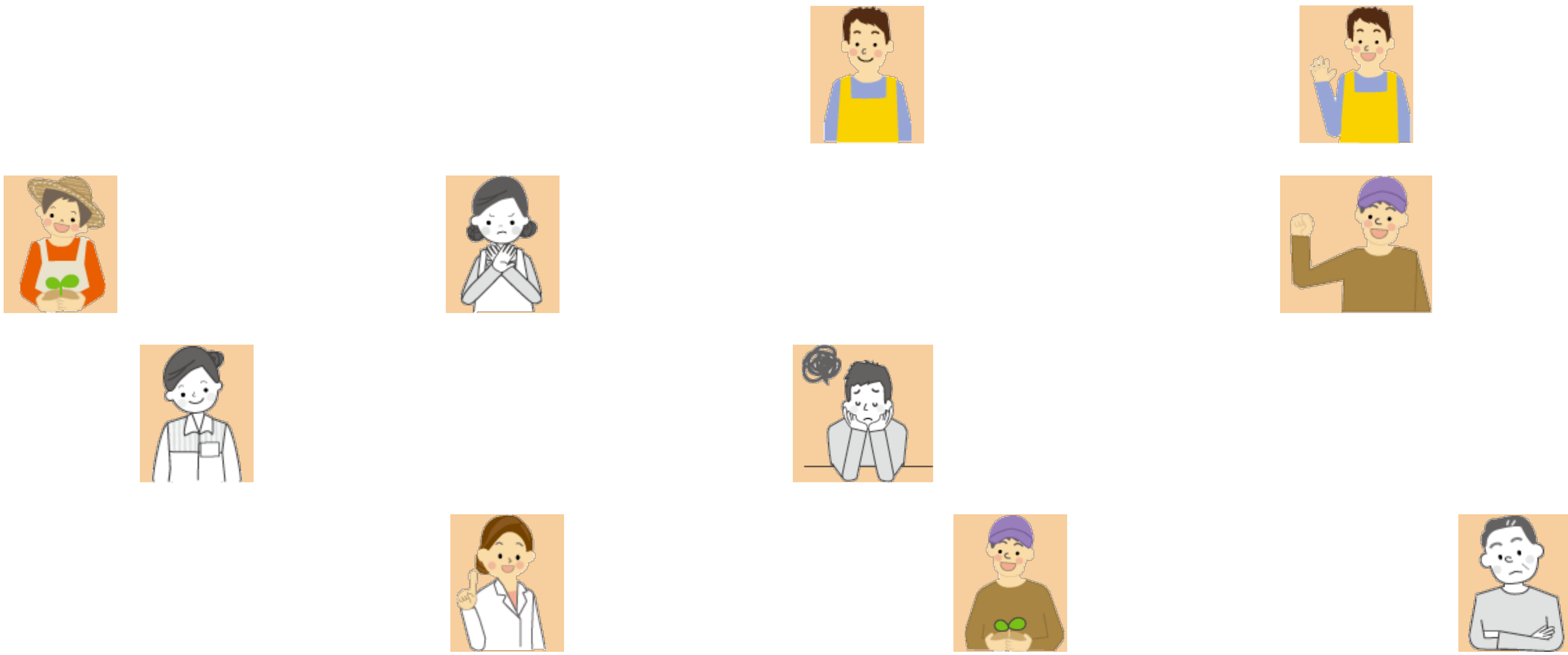
(true population is close to 800,000 now though)

This is a **population**.

Sample

A **sample** is selected from a population.

Sample



A **sample** is selected from a population.

Reasonable Questions Starting Out

e.g. interview, or inquire, or analyze, or otherwise observe, or

1. In situations where we can't obverse the entire population, what can we safely infer by polling a sample drawn from that population?

   safely here mean reliably, i.e. backed by data and/or scientific method.

2. How large does your sample need to be before your conclusions become trustworthy, and how do we express confidence in what we conclude.
3. Are there alternative ways to infer population statistics without polling entire populations?

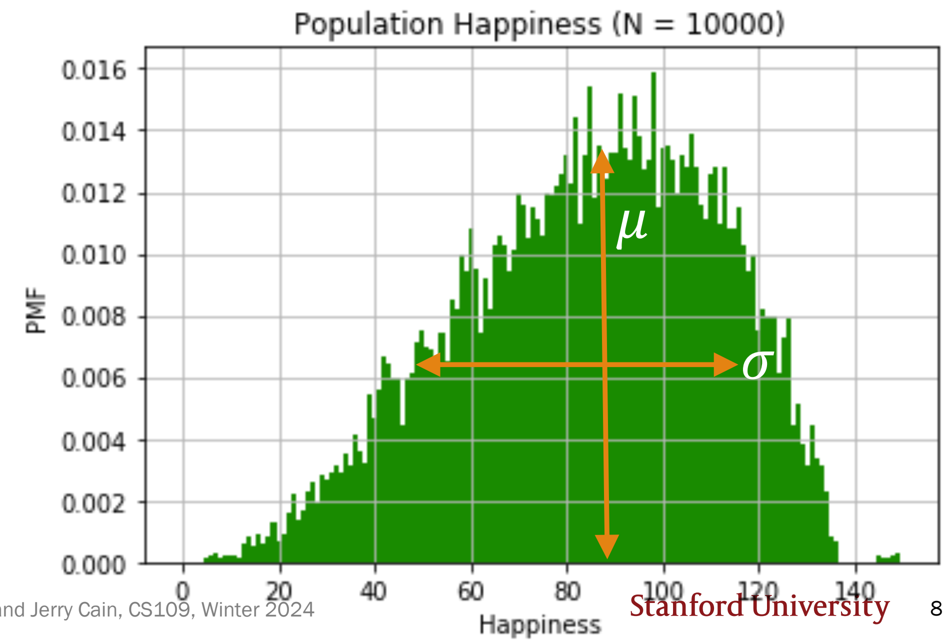
   yes. because here we are discussing some of them.

A sample, mathematically

Consider n random variables $X_1, X_2, \dots, X_n$.

The sequence $X_1, X_2, \dots, X_n$ is a **sample** from distribution F if:

- X_i are all independent and identically distributed (iid)
- X_i all have same distribution function F (the **underlying distribution**), where $E[X_i] = \mu$, $\text{Var}(X_i) = \sigma^2$



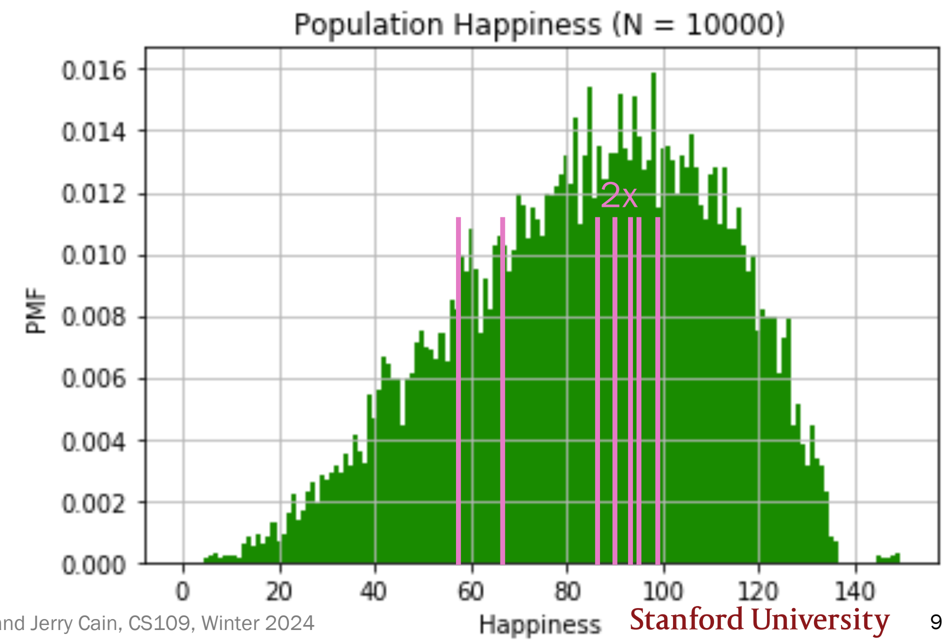
A sample, mathematically

A sample of size 8:

$$(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$$

The realization of a sample of size 8:

$$(59, 87, 94, 99, 87, 78, 69, 91)$$



A single sample

A happy
Bhutanese person

If we had a distribution F of our entire population, we could compute exact statistics about about happiness. the distribution from the prior slide pretends we do know the full population distribution F, though in general we won't have access to it.

But we only have 200 people—or rather, a sample.

Today: If we only have a single sample,

- How do we report **estimated** statistics?
  - We're careful to call them estimated mean and estimated variance, since they're based on samples (i.e., experiments)
- How do we report estimated errors on these estimates?
- How do we perform something called **hypothesis testing**? Oh, and what is it?

Unbiased estimators

A single sample

A happy
Bhutanese person

If we had a distribution F of our entire population, we could compute exact statistics about happiness. But again, we generally do not have access to it.

But we only have 200 people (a sample).

These population-level statistics are unknown:

- μ, the population mean
- σ^2, the population variance

A single sample

A happy
Bhutanese person

If we had a distribution F of our entire population, we could compute exact statistics about happiness.

But we only have 200 people (a sample).

- From these 200 people, what is our best estimate of the population mean and the population variance?
- How exactly do we define best estimate?

Estimating the population mean

1. What is our best estimate of μ, the **mean happiness** of Bhutanese people?

If we only have $(X_1, X_2, \dots, X_n)$:

The best estimate of μ is the **sample mean**: $$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

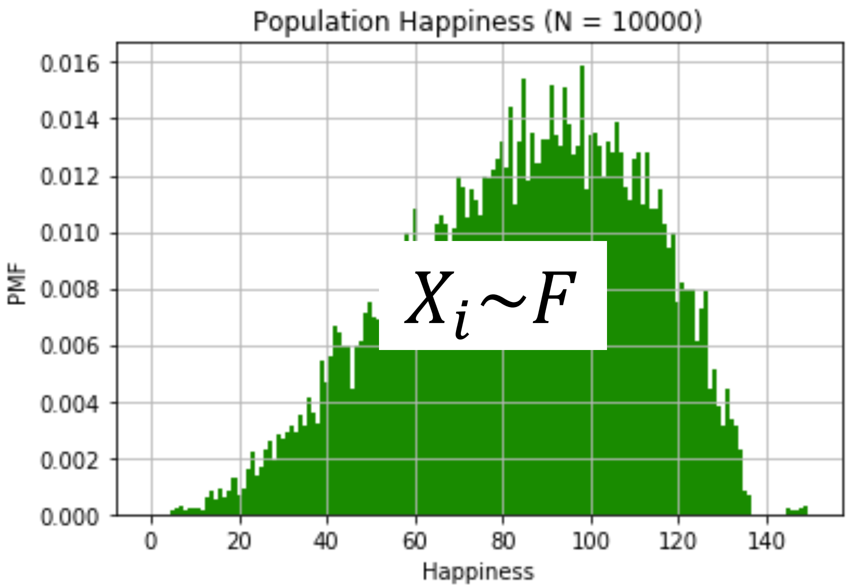
$\bar{X}$ is an **unbiased estimator** of the population mean μ. $E[\bar{X}] = \mu$

Intuition: By the CLT, $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$

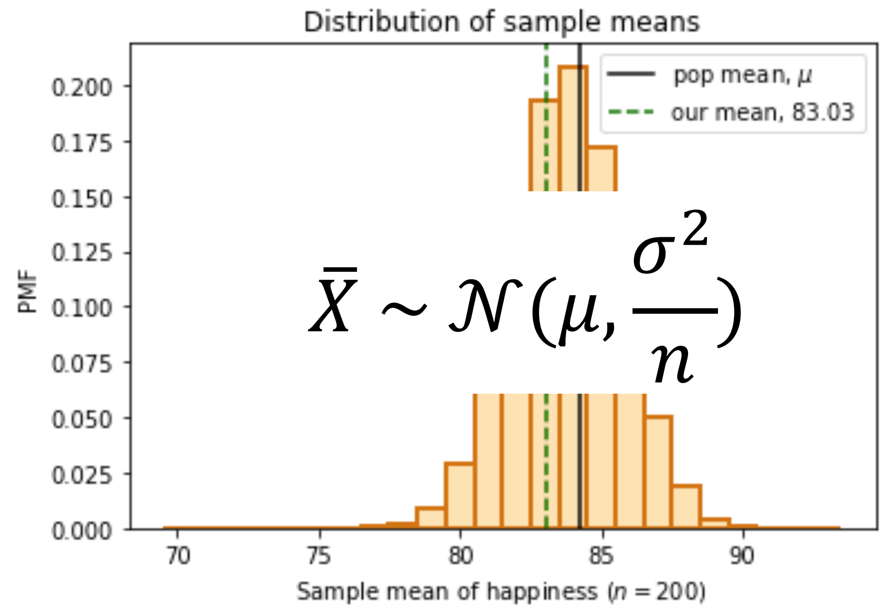
If we could take *multiple* samples of size n:

1. For each sample, compute sample mean
2. On average, we would get the population mean

Sample mean



$X_i \sim F$



$$\bar{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

Even if we can't report μ, we can report our sample mean 83.03, which is an unbiased estimate of μ.

Estimating the population variance

2. What is σ^2, the variance of happiness of Bhutanese people?

If we knew the entire population $(x_1, x_2, \dots, x_N)$:

population mean

population variance

$$\sigma^2 = E[(X-\mu)^2] = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$

If we only have one sample: $(X_1, X_2, \dots, X_n)$:

sample mean

sample variance

$$\boxed{S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

note that our estimated variance is defined in terms of another estimate — the sample mean.

reasonable question: why not n?

Intuition about the sample variance, S^2

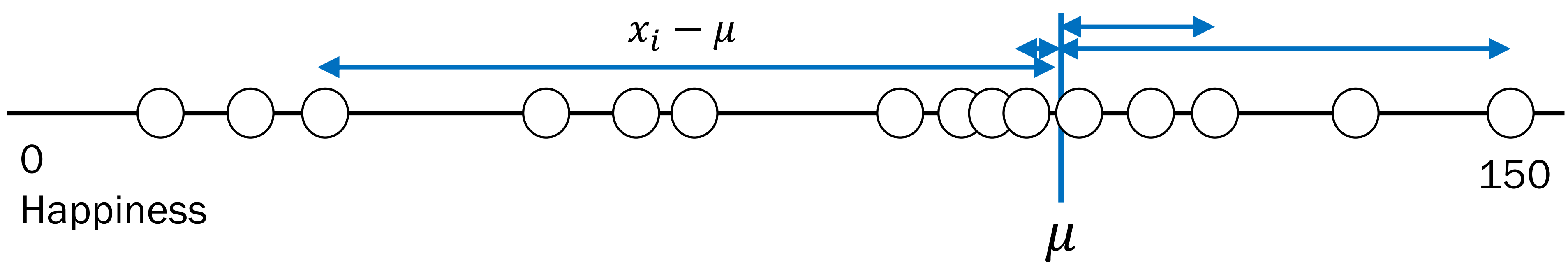
Actual, σ^2

population mean

population variance

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$

you have perfect information if you have access to the full population, because that's pretty much everything there is to know.



Population size, N

Calculating population statistics **exactly** requires us knowing all N datapoints.

Intuition about the sample variance, S^2

Actual, σ^2

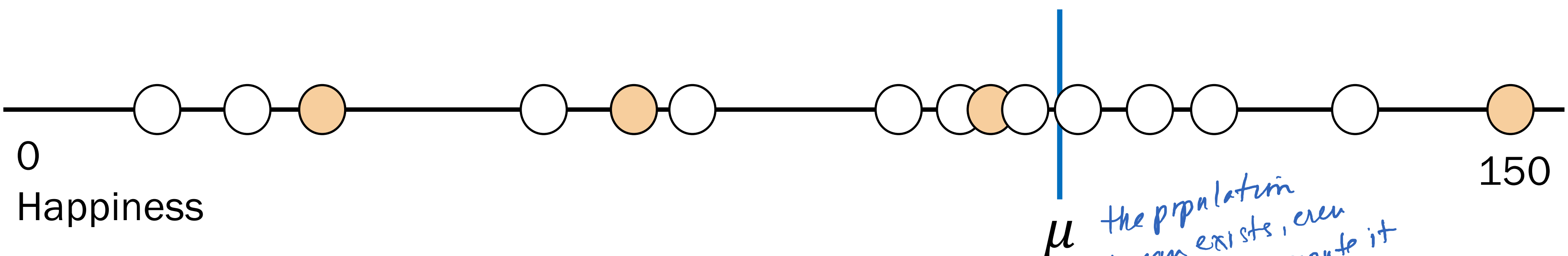
population mean

population variance $\qquad \sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$

Estimate, S^2

sample mean

sample variance $\qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$



the population mean exists, even if we can compute it exactly.

Population size, N

Intuition about the sample variance, S^2

Actual, σ^2

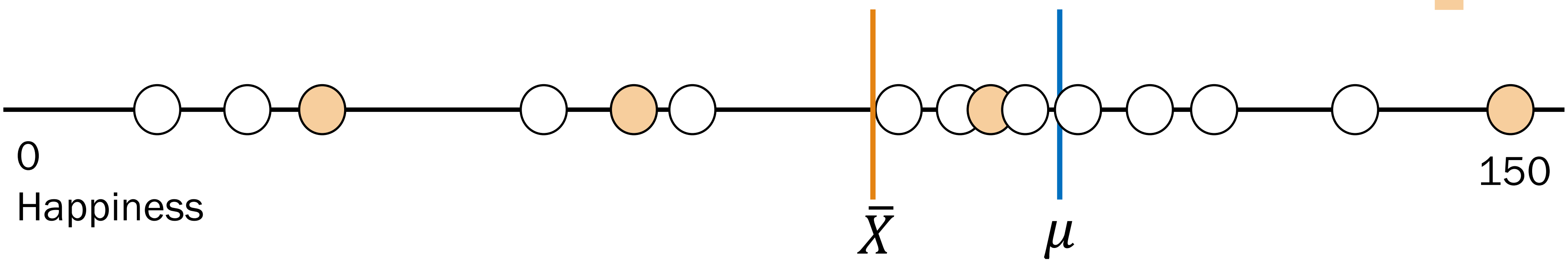
population mean

population variance $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$

Estimate, S^2

sample mean

sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$



Population size, N

Intuition about the sample variance, S^2

Actual, σ^2

population mean

population variance

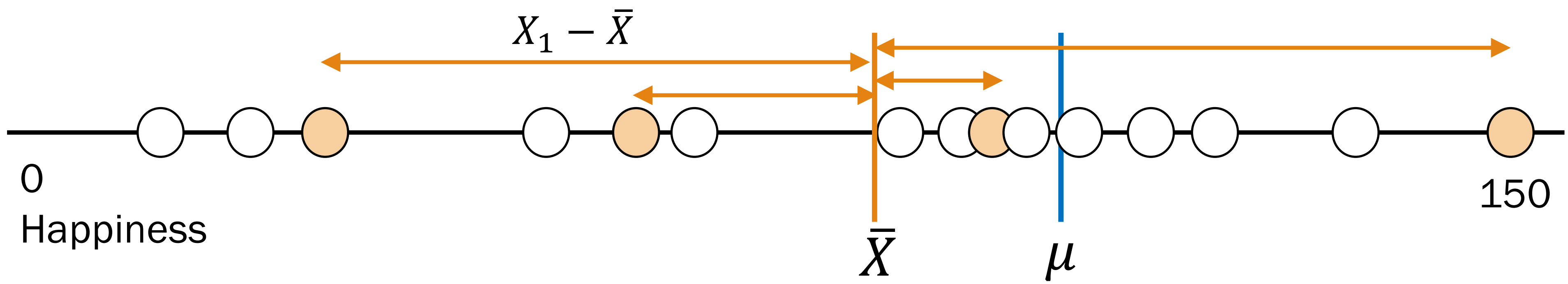
$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$

Estimate, S^2

sample mean

sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\underbrace{(X_i - \bar{X})}^2$$



Population size, N

Sample variance is an estimate using an estimate, so it requires additional scaling.

Estimating the population variance

2. What is σ^2, the **variance of happiness** of Bhutanese people?

If we only have a sample, $(X_1, X_2, \ldots, X_n)$:

The best estimate of σ^2 is the **sample variance:**

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})$ is also an estimator, but it's biased, because it's generally too low

S^2 is an **unbiased estimator** of the population variance, σ^2. $\quad E[S^2] = \sigma^2$

Proof that S^2 is unbiased (just for reference)

$E[S^2] = \sigma^2$

$$E[S^2] = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] \Rightarrow (n-1)E[S^2] = E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right]$$

$$(n-1)E[S^2] = E\left[\sum_{i=1}^{n}\left((X_i - \mu) + (\mu - \bar{X})\right)^2\right]$$

(introduce $\mu - \mu$)

$$= E\left[\sum_{i=1}^{n}(X_i - \mu)^2 + \sum_{i=1}^{n}(\mu - \bar{X})^2 + 2\sum_{i=1}^{n}(X_i - \mu)(\mu - \bar{X})\right]$$

$$2(\mu - \bar{X})\sum_{i=1}^{n}(X_i - \mu)$$

$$2(\mu - \bar{X})\left(\sum_{i=1}^{n}X_i - n\mu\right)$$

$$2(\mu - \bar{X})n(\bar{X} - \mu)$$

$$-2n(\mu - \bar{X})^2$$

$$= E\left[\sum_{i=1}^{n}(X_i - \mu)^2 + n(\mu - \bar{X})^2 - 2n(\mu - \bar{X})^2\right]$$

$$= E\left[\sum_{i=1}^{n}(X_i - \mu)^2 - n(\mu - \bar{X})^2\right] = \sum_{i=1}^{n}E[(X_i - \mu)]^2 - nE[(\bar{X} - \mu)^2]$$

$$= n\sigma^2 - n\mathrm{Var}(\bar{X}) = n\sigma^2 - n\frac{\sigma^2}{n} = n\sigma^2 - \sigma^2 = (n-1)\sigma^2$$

Therefore $E[S^2] = \sigma^2$

Standard error

Estimating population statistics

A particular outcome

1. Collect a sample, $X_1, X_2, \dots, X_n$.

$(72, 85, 79, 79, 91, 68, \dots, 71)$

$n = 200$

2. Compute **sample mean**, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$\bar{X} = 83$

85-83

72-83

79-83

3. Compute sample deviation, $X_i - \bar{X}$.

$(-11, 2, -4, -4, 8, -15, \dots, -12)$

4. Compute **sample variance**, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

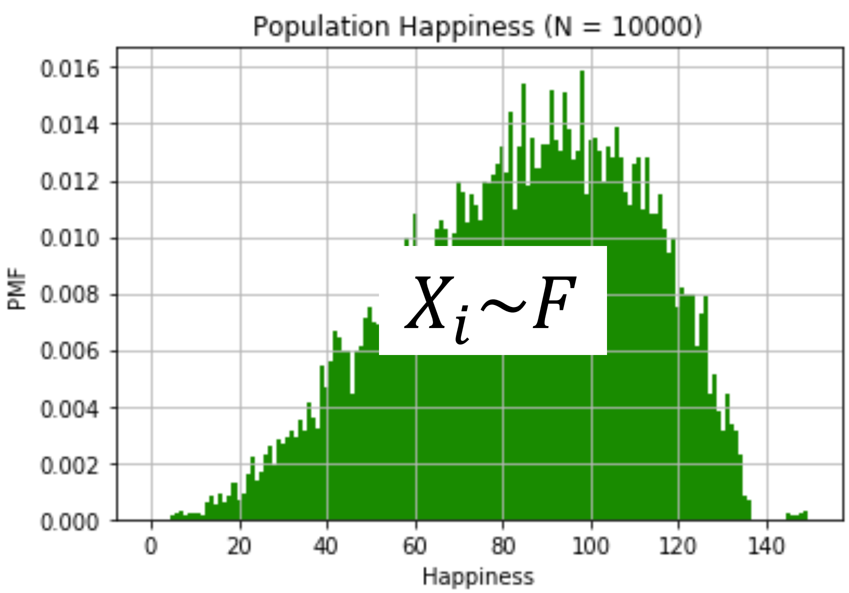
$S^2 = 793$

how close is $\bar{X}$ to μ?

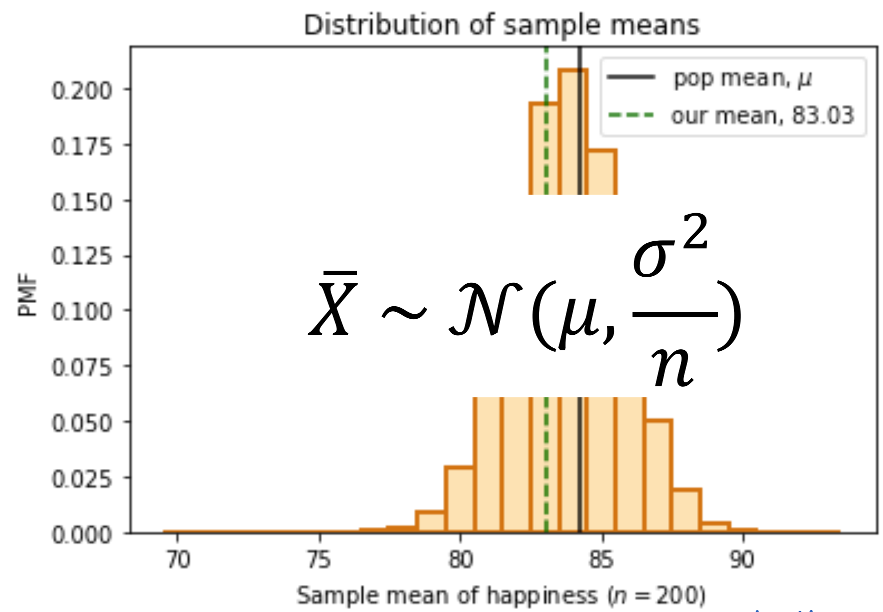
how close is S^2 to the true σ^2?

How close are our estimates $\bar{X}$ and S^2?

Sample mean



$$X_i \sim F$$



$$\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

if Var$(\bar{X})$ is large, we're not all that confident that $\bar{X}$ is close to μ.

conversely, if Var$(\bar{X})$ is small, then we're more confident it's close.

- Var$(\bar{X})$ is a measure of how close $\bar{X}$ is to μ.
- **How do we estimate Var$(\bar{X})$?**

How close is our estimate $\bar{X}$ to μ?

note, sadly, that we don't know what σ^2 is, as that's a population-level statistic.

$$E[\bar{X}] = \mu$$

$$\boxed{\text{Var}(\bar{X}) = \frac{\sigma^2}{n}}$$

We want to estimate this

$SD(\bar{X}) = \sqrt{Var(\bar{X})} = \sqrt{\frac{\sigma^2}{n}}$

we use S^2

<u>def</u> The **standard error** of the mean is an estimate of the standard deviation of $\bar{X}$.

$$SE = \sqrt{\frac{S^2}{n}}$$

$E[SE] < SD(\bar{X})$

less than because of the bias.

Intuition:

- S^2 is an unbiased estimate of σ^2
- S^2/n is an unbiased estimate of $\sigma^2/n = \text{Var}(\bar{X})$
- $\sqrt{S^2/n}$ can estimate $\sqrt{\text{Var}(\bar{X})}$

somewhat biased :(

but best we can do :(

More info on bias of standard error: wikipedia

Standard error of the mean

1. Mean happiness:

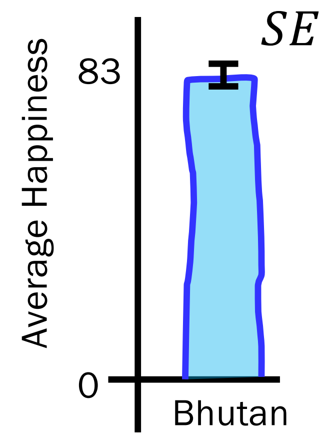
Claim: The average happiness of Bhutan is 83, with a standard error of 1.99.

this is our best estimate of μ

Closed form: $SE = \sqrt{\frac{S^2}{n}}$

this is our estimate of how close we are to μ

error bars



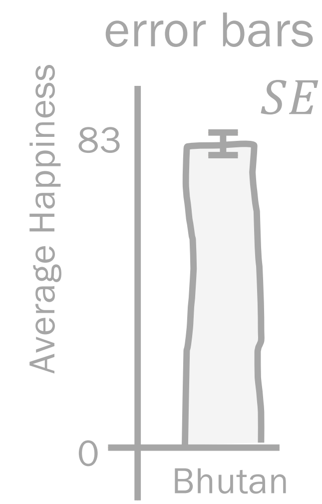
These 2 statistics give a sense of how $\bar{X}$—that is, the sample mean random variable—behaves.

Standard error of variance?

1. Mean happiness:

   Claim: The average happiness of Bhutan is 83,
   with a standard error of 1.99.

   Closed form: $SE = \sqrt{\frac{S^2}{n}}$



2. Variance of happiness:

   this is our best estimate of σ^2

   Claim: The variance of happiness of Bhutan is 793.

   Closed form: Not covered in CS109

   But how close are we?

Up next: Compute statistics with code!

Bootstrap: Sample mean

Bootstrap

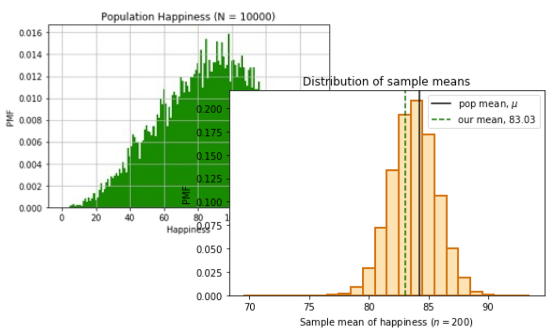
The Bootstrap:

Probability for Computer Scientists

Computing statistic of sample mean

What is the standard deviation of the sample mean $\bar{X}$? (sample size $n = 200$)

Population distribution
(we don't have this)



$$\frac{\sigma}{\sqrt{n}} = 1.886$$

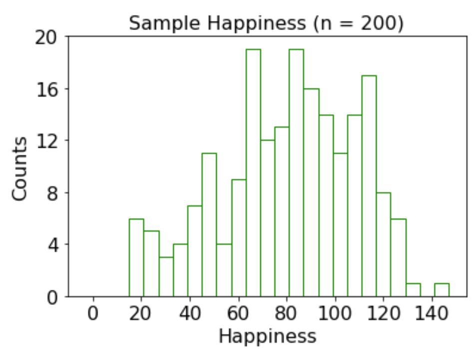
Exact statistic
(we don't have this)

these are the types of things we know with access to full population

1.869

Simulated statistic
(we don't have this)

Sample distribution
(we do have this)



$$SE = \frac{S}{\sqrt{n}} = 1.992$$

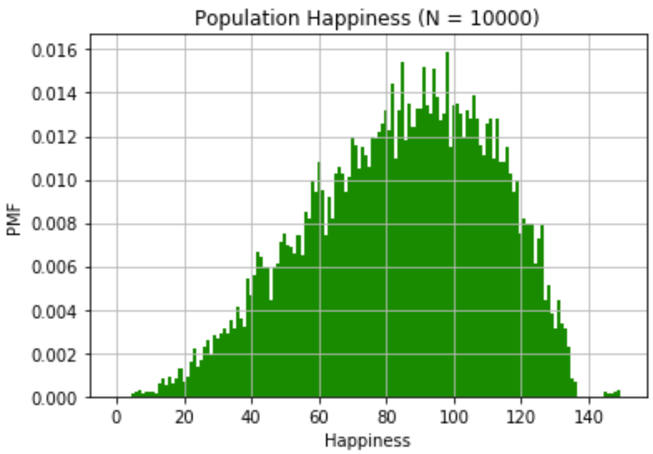
Estimated statistic, by formula, **standard error**

???

Simulated estimated statistic

Note: We don't have access to the population.
But Doris is sharing the exact statistic with you.

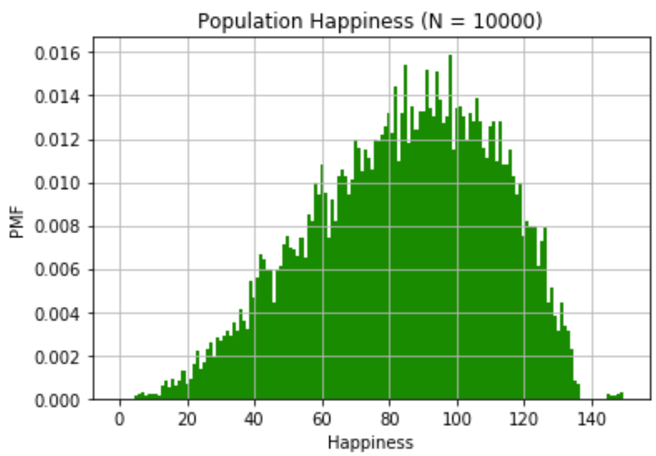
Bootstrap insight 1: Estimate the true distribution



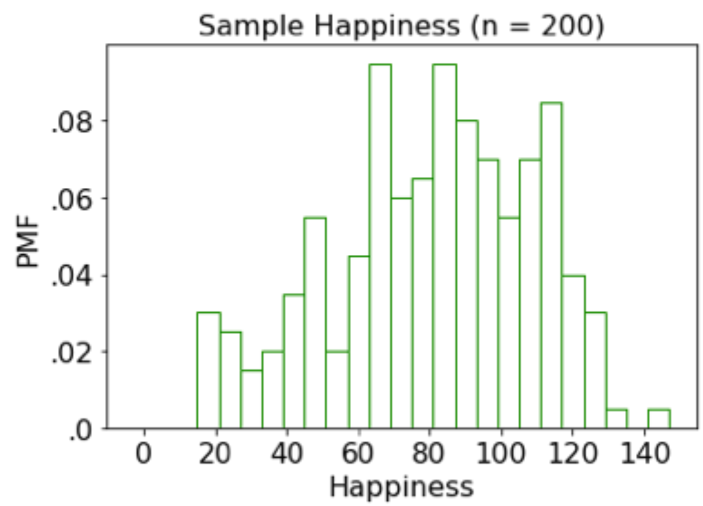
$\approx$

Bootstrap insight 1: Estimate the true distribution

You can estimate the PMF of the underlying distribution, using your sample.*



$\approx$



The underlying distribution

$$F \approx \hat{F}$$

the sample distribution (aka the histogram of your data)

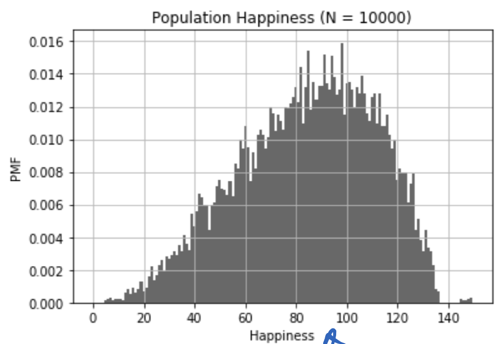
normalized to function as a PMF

*This is just a histogram of your data!

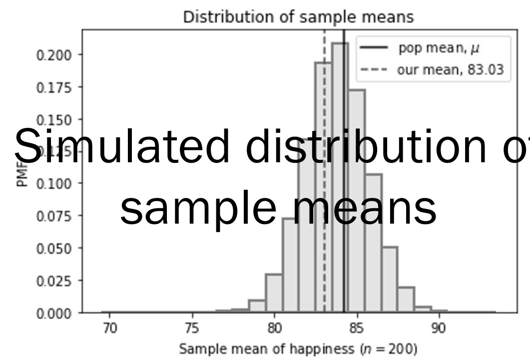
Bootstrap insight 2: Simulate a distribution

Approximate the procedure of simulating a distribution of a statistic, e.g., $\bar{X}$.

Population distribution
(we don't have this)



Distribution of $\bar{X}$

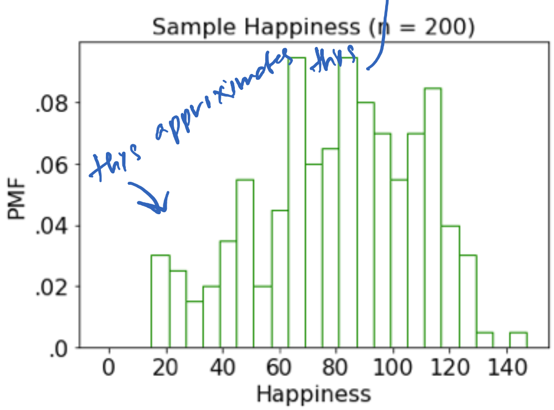


Simulated distribution of sample means

$\approx$

Sample distribution
(we do have this)

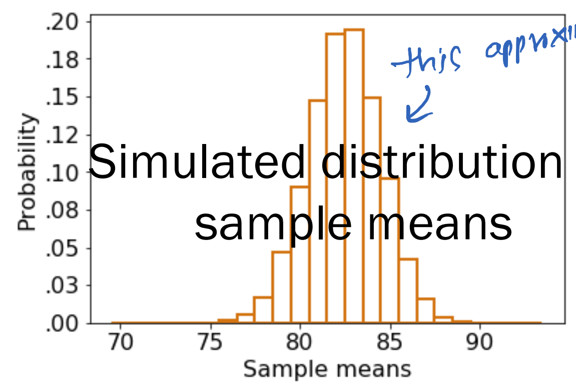
this approximates this



just about to learn this

Bootstrap means

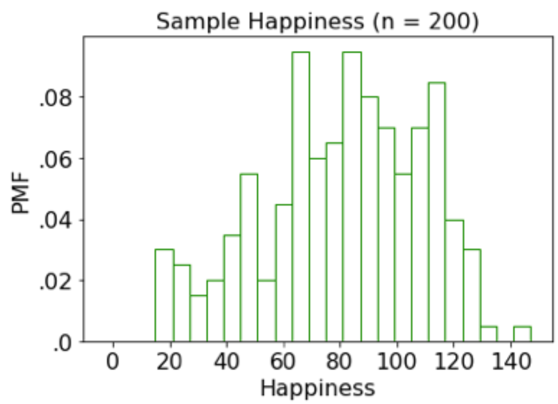
this approximates this

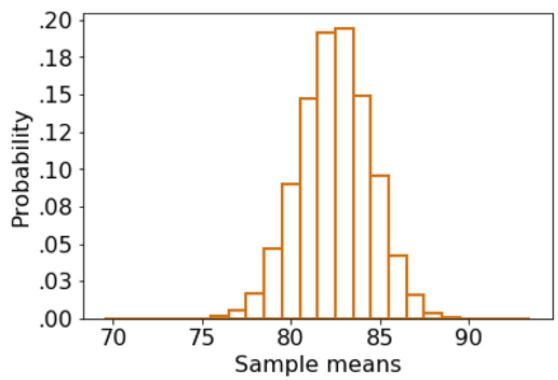


Simulated distribution of sample means

Bootstrapped sample means

```
means = [84.7, 83.9, 80.6, 79.8, 90.3, …, 85.2]
```






```
np.std(means)

2.003
```

Estimate the true PMF using our "PMF" (histogram) of our sample.

...generate a whole bunch of sample means of this estimated distribution...

from this

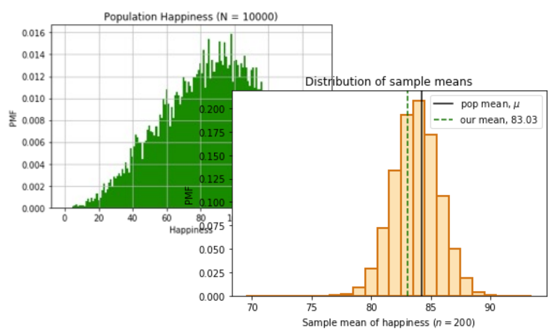
...and compute the standard deviation of this distribution.

by brute force computation.

Computing statistic of sample mean

What is the standard deviation of the sample mean $\bar{X}$? (sample size $n = 200$)

Population distribution
(we don't have this)



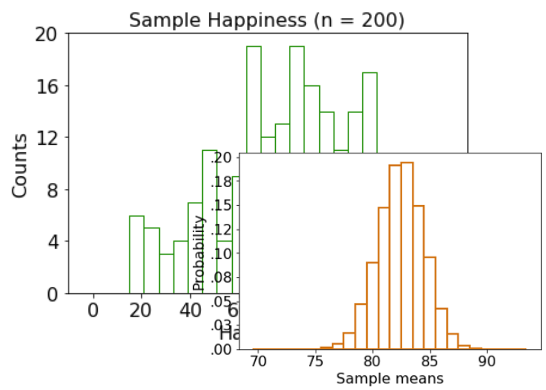
$$\frac{\sigma}{\sqrt{n}} = 1.886$$

Exact statistic
(we don't have this)

`1.869`

Simulated statistic
(we don't have this)

Sample distribution
(we do have this)



$$SE = \frac{S}{\sqrt{n}} = 1.992$$

Estimated statistic, by formula, **standard error**

`2.003`

Simulated estimated statistic, **bootstrapped standard error**

Bootstrap algorithm

Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Resample **sample.size**() from PMF
   b. Recalculate the **sample mean** on the resample
3. You now have a **distribution of your sample mean**

We sample with replacement, because the sample operates as a PMF (and if we didn't, we would just regenerate the original sample every time.)

What is the distribution of your sample mean?

We'll talk about this algorithm in detail with a demo!

Bootstrap algorithm

```
Bootstrap Algorithm (sample):
1. Estimate the PMF using the sample
2. Repeat 10,000 times:
     a. Resample sample.size() from PMF
     b. Recalculate the statistic on the resample
3. You now have a distribution of your statistic
```

What is the distribution of your statistic?

Bootstrapped sample variance

takeaway: this method — the bootstrap method — is generic and can be used to estimate all kinds of statistics.

Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Resample **sample.size**() from PMF
   b. Recalculate the sample variance on the resample
3. You now have a **distribution of your sample variance**

we are using bootstrapping to compute the standard error of the sample variance.

What is the distribution of your sample variance?

Even if we don't have a closed form equation,
we estimate statistics of sample variance with bootstrapping!

Bootstrap: Sample variance

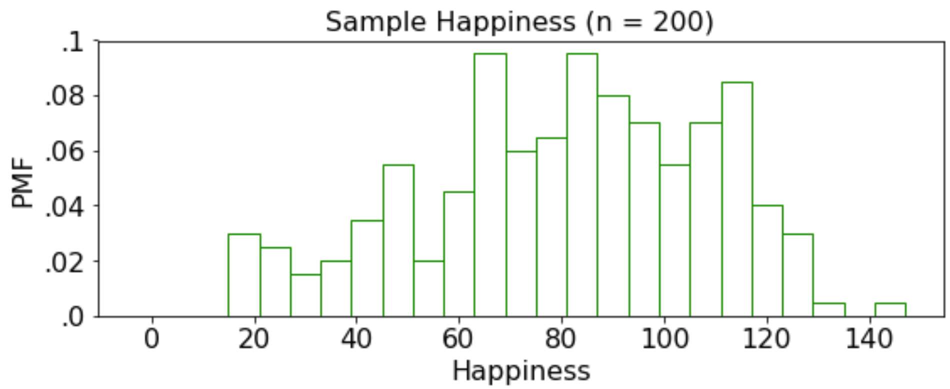
Bootstrapped sample variance

Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
    a. Resample **sample.size**() from PMF
    b. Recalculate the sample variance on the resample
3. You now have a **distribution of your sample variance**

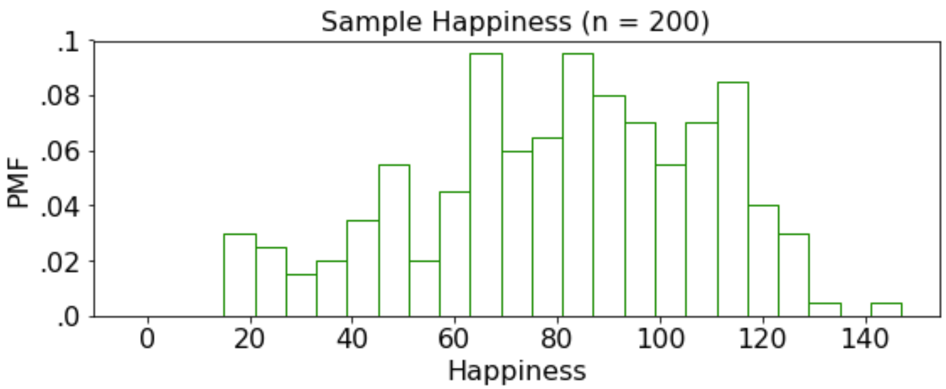
Goal What is the distribution of your sample variance?

Bootstrapped variance



1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
    a. Resample **sample.size**() from PMF
    b. Recalculate the **sample variance** on the resample
3. You now have a **distribution of your sample variance**

Bootstrapped variance



1. Estimate the **PMF** using the sample
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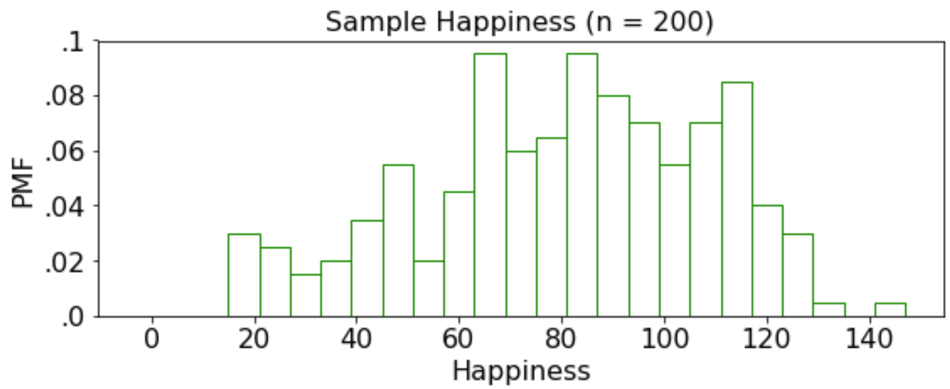
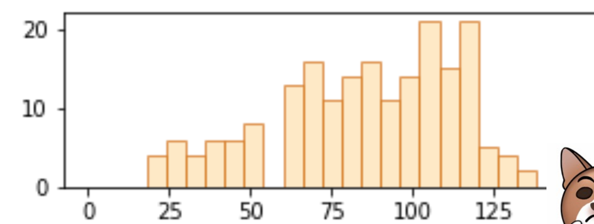
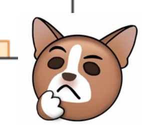
Bootstrapped variance



X_1 X_2 X_3 X_4 X_{200}

[52, 38, 98, 107, ..., 94]



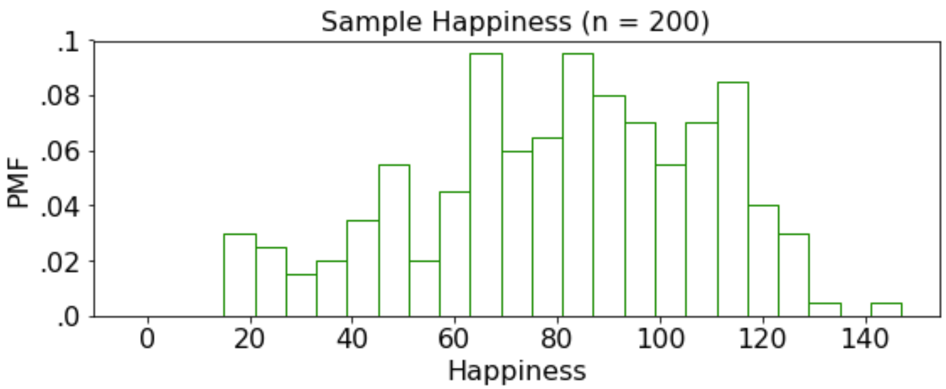


Why are these samples **different**?

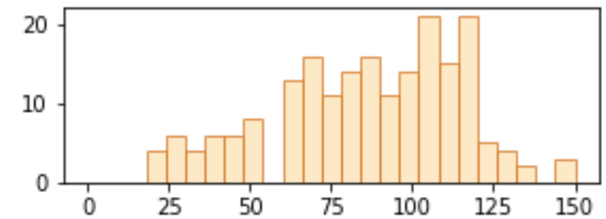
```
1. Estimate the PMF using the sample
2. Repeat 10,000 times:
   a. Resample sample.size() from PMF
   b. Recalculate the sample variance on the resample
3. You now have a distribution of your
```

This resampled sample is generated **with replacement.**

Bootstrapped variance



[52, 38, 98, 107, ..., 94]

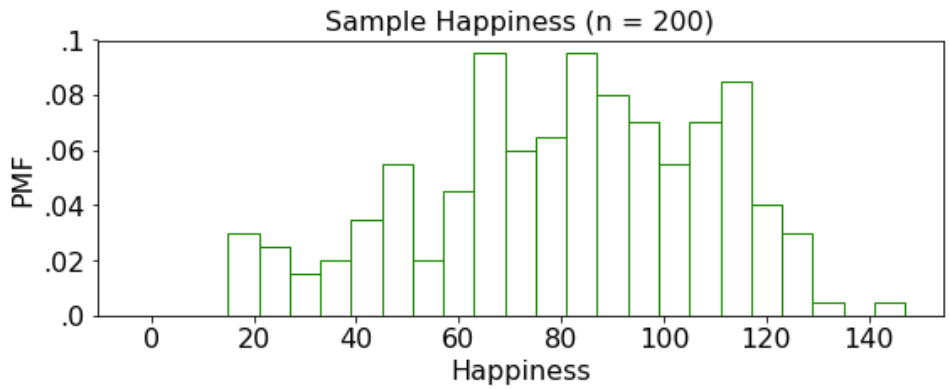


formula for s^2 is still $\frac{1}{n-1}\sum_{k=1}^{n}(x_i-\bar{x})^2$

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
    a. Resample **sample.size**() from PMF
    b. Recalculate the **sample variance** on the resample
3. You now have a **distribution of your sample variance**

variances = [827.4]

Bootstrapped variance



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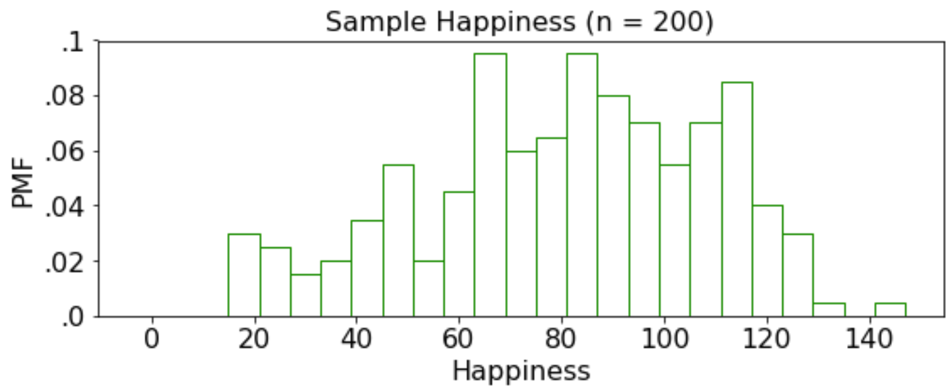
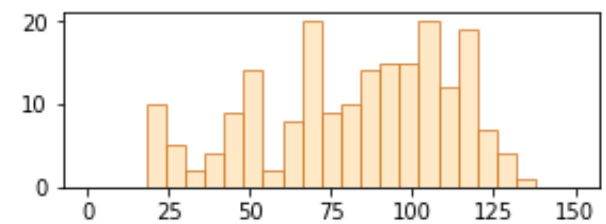
```
variances = [827.4]
```

Bootstrapped variance



[116, 76, 132, 85, ..., 78]



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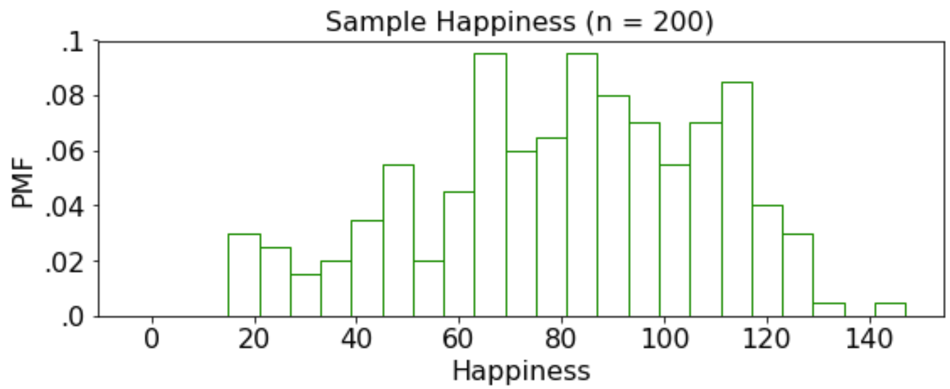
3. You now have a **distribution of your sample variance**

```
variances = [827.4]
```

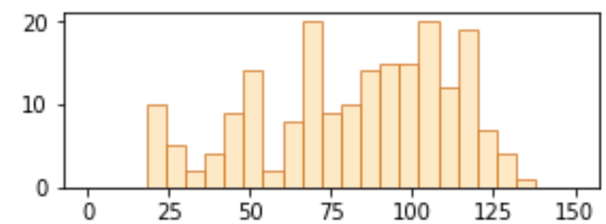
Bootstrapped variance

second iteration might generate this



x_1 x_2 x_3 x_4 x_{200}

[116, 76, 132, 85, ..., 78]



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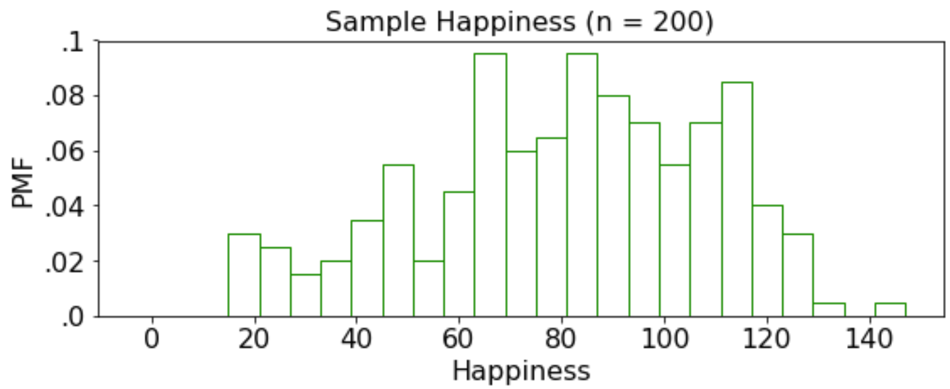
3. You now have a **distribution of your sample variance**

```
variances = [827.4, 846.1]
```

Bootstrapped variance



1. Estimate the **PMF** using the sample
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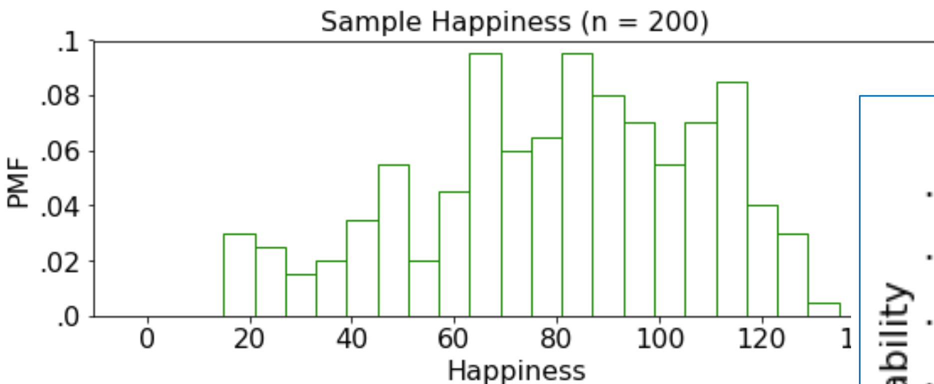
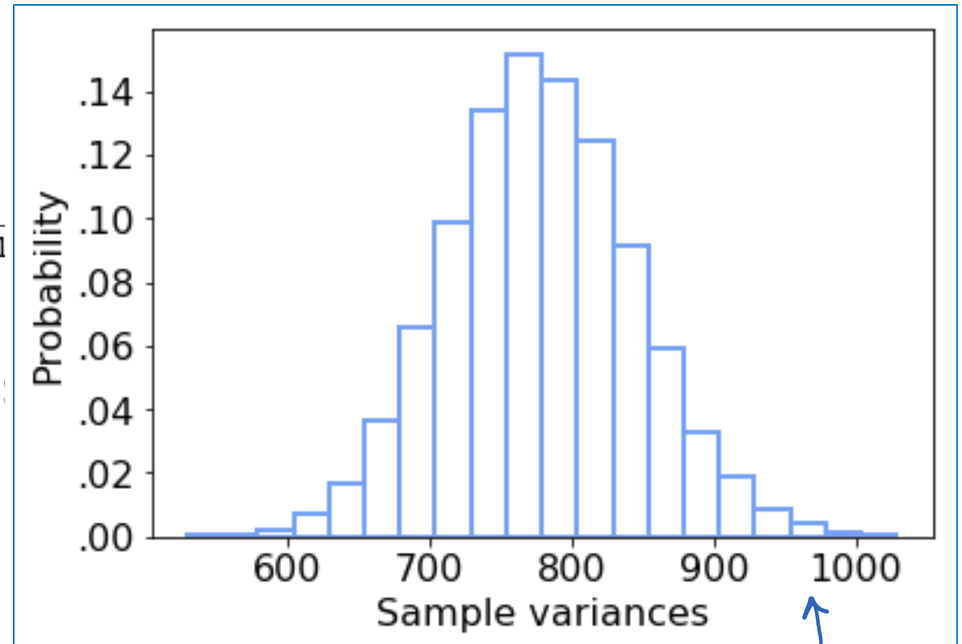
    a. Resample **sample.size**() from PMF
    b. Recalculate the **sample variance** on the resample
3. You now have a **distribution of your sample variance**

`variances = [827.4, 846.1]`

Bootstrapped variance





1. Estimate the **PMF** using the
2. Repeat **10,000** times:
    a. Resample **sample.size**()
    b. Recalculate the **sample**
3. You now have a **distribution of your sample variance**

data for this graph

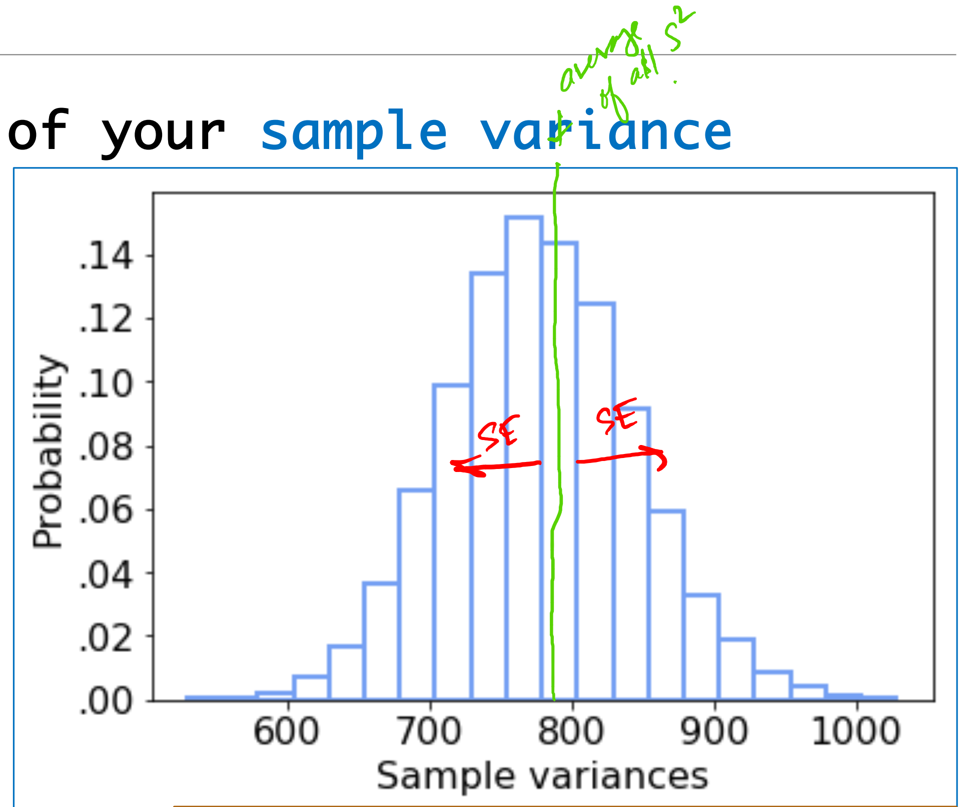
```
variances = [827.4, 846.1, 726.0, …, 860.7]
```

Bootstrapped variance

3. You now have a **distribution of your** sample variance

```
variances = [827.4,
     846.1, 726.0, …,
     860.7]
```

1000 of these



What is the bootstrapped standard error?

```
np.std(variances)
```

Bootstrapped standard error: 66.16

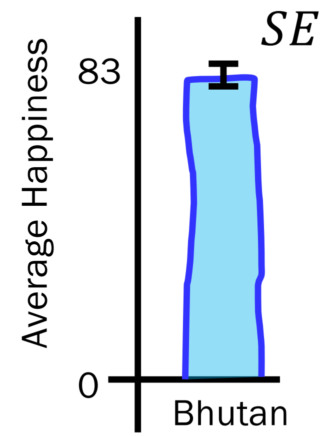
- Simulate a distribution of sample variances
- Compute standard deviation

Standard error

1. Mean happiness:

   Claim: The average happiness of Bhutan is 83, with a standard error of 1.99.

   Closed form: $SE = \sqrt{\frac{S^2}{n}}$



2. Variance of happiness:

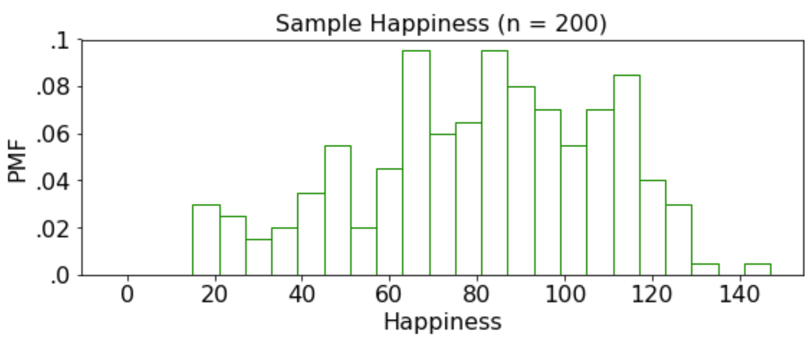
   Claim: The variance of happiness of Bhutan is 793, with a **bootstrapped standard error of 66.16**.

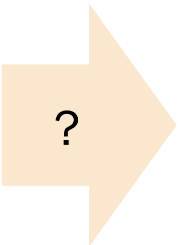
S^2 is our best estimate of σ^2

this is how close and confident we are, calculated by bootstrapping

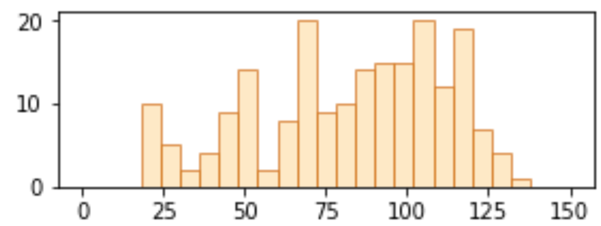
Algorithm in practice: Resampling

```
1. Estimate the PMF using the sample
2. Repeat 10,000 times:
   a. Resample sample.size() from PMF
   b. Recalculate the statistic on the resample
3. You now have a distribution of your statistic
```





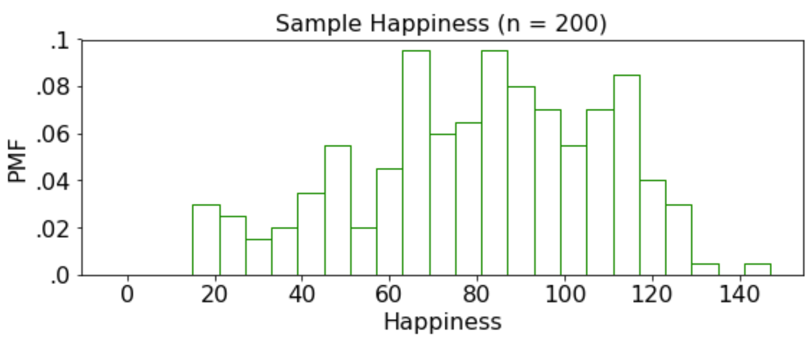
[116, 76, 132, 85, ..., 78]



$$P(X = k) = \frac{\text{\# values in sample equal to } k}{n}$$

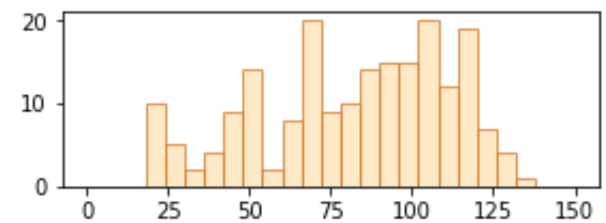
Algorithm in practice: Resampling

```python
def resample(sample, n):
    # estimate the PMF using the sample
    # draw n new samples from the PMF
    return np.random.choice(sample, n, replace=True)
```



?

[116, 76, 132, 85, ..., 78]



$$P(X = k) = \frac{\text{\# values in sample equal to } k}{n}$$

This resampled sample is generated **with replacement.**

To the code!

Bootstrap provides a way to calculate probabilities of statistics using code.

Bootstrapping works for any statistic*

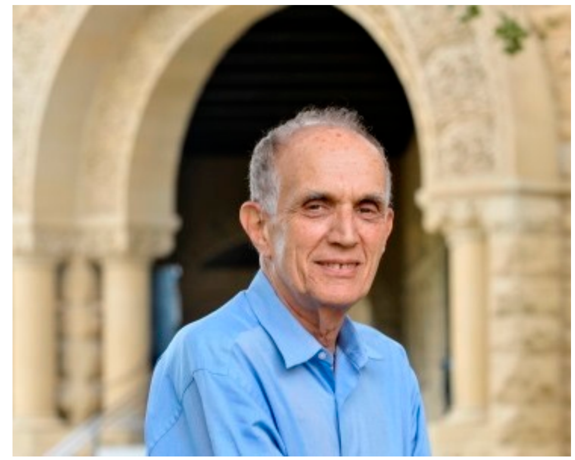
*as long as your sample is iid and the underlying distribution does not have a long tail

fancy way of saying that variance is finite

Google colab notebook link

Bradley Efron

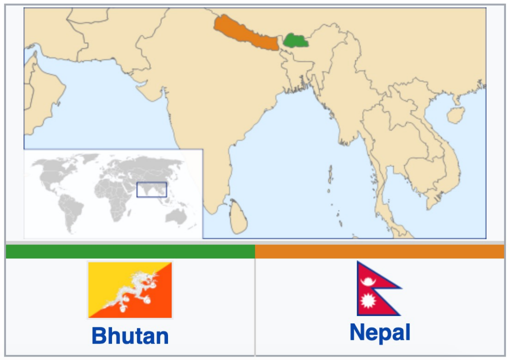
- Invented bootstrapping in 1979
- Still a professor at Stanford
- Won a National Science Medal



Inventor of Efron's dice: 4 dice A, B, C, D where:

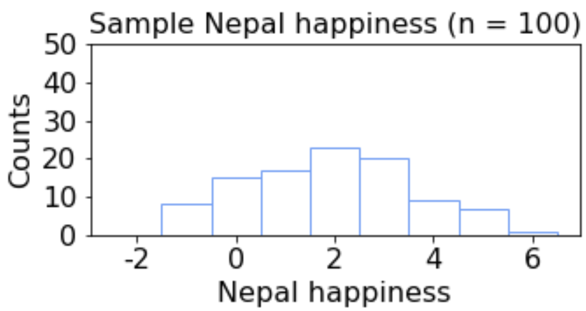
$$P(A > B) = P(B > C) = P(C > D) = P(D > A) = \frac{2}{3}$$

Bootstrap: p-value



Null hypothesis test

Nepal Happiness
4.45
2.45
6.37
2.07
…
1.63



Bhutan Happiness
0.91
0.34
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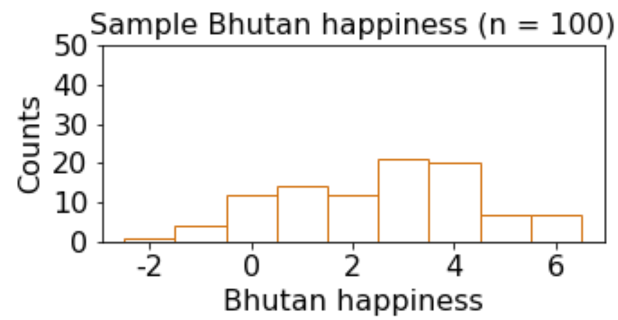
…
1.08



$\bar{X}_1 = 3.1$

$\bar{X}_2 = 2.4$

Claim: The difference in mean happiness between Nepal and Bhutan is 0.7 happiness points, and **this is statistically significant**.

Null hypothesis test

def **null hypothesis** – Even if there is no pattern (i.e., the two samples are from identical distributions), your claim might have arisen by chance.

def **p-value** – What is the probability that the observed difference occurs under the null hypothesis? that is, what is the chance you observed a significant difference by chance but it's not reproducible.

Example:

- Flip some coin 100 times.
- Flip the same coin another 150 times.

Null hypothesis assumes we use the same coin

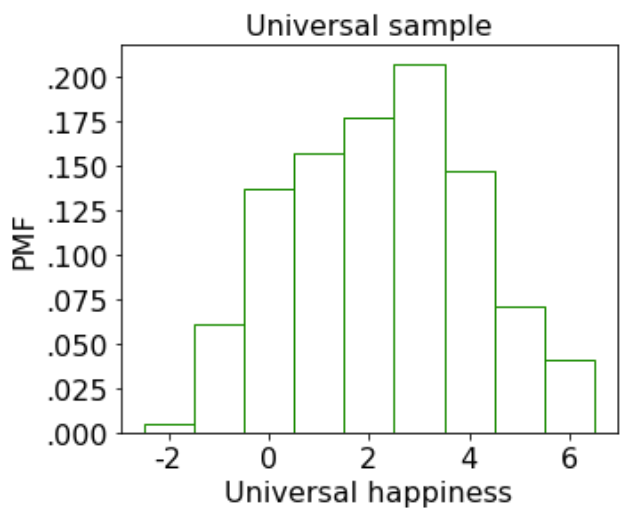
- Compute fraction of heads in both groups.
- There is a possibility we'll see the observed difference in these fractions even if we used the same coin

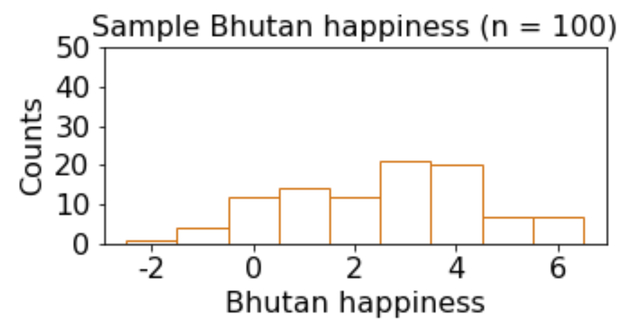
p-value

A **significant** p-value (< 0.05) means we reject the null hypothesis.

Universal sample

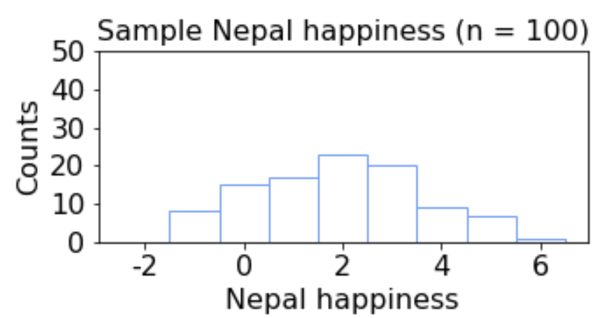
(this is what the null hypothesis assumes)





$\bar{X}_1 = 3.1$

how often do we see this from the combined populations?



$\bar{X}_2 = 2.4$

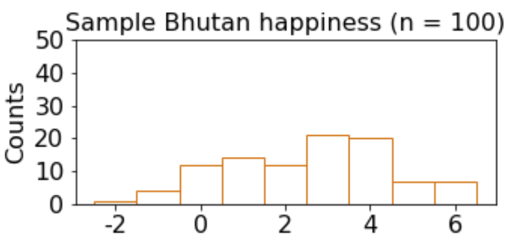
Want p-value: probability $|\bar{X}_1 - \bar{X}_2| = |3.1 - 2.4|$ happens under null hypothesis

Bootstrap for p-values

if two statistics are different by chance, then combining the populations shouldn't matter, right?

1. Create a **universal sample** using your two samples

i.e., recreate the null hypothesis



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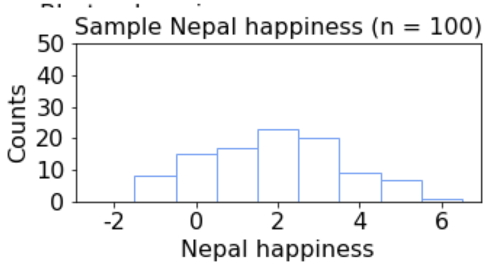

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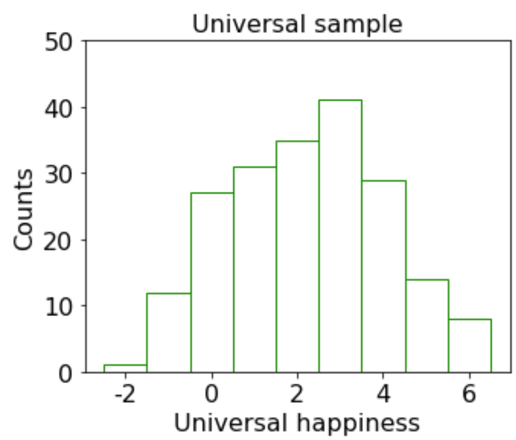

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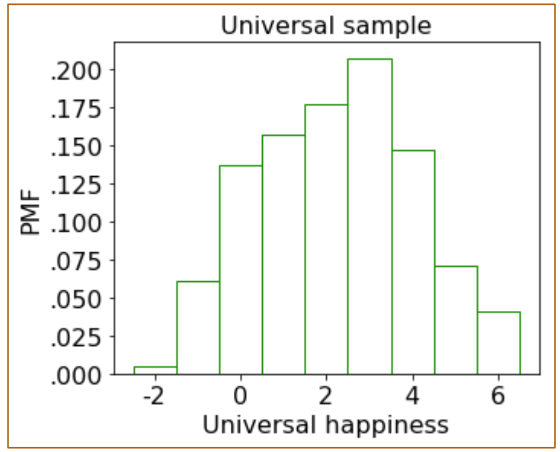
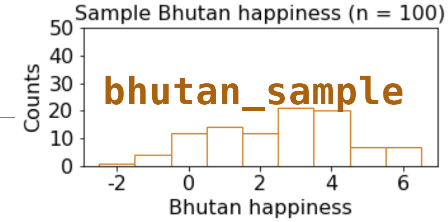
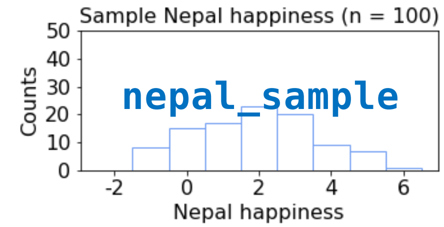


Bootstrap for p-values





1. Create a **universal sample** using your two samples
2. Repeat **10,000** times:
   a. Resample **both samples**
   b. Recalculate the **mean difference** between the resamples
3. **p-value** = $\dfrac{\text{\# (mean diffs >= observed diff)}}{n}$ $= 1000$

Probability that observed difference arose by chance

Bootstrap for p-values

```
def pvalue_boot(bhutan_sample, nepal_sample):
    N = size of the bhutan_sample
    M = size of the nepal_sample
    observed_diff = |mean of bhutan_sample – mean of nepal_sample|

    uni_sample = combine bhutan_sample and nepal_sample
    count = 0

    repeat 10,000 times:
        bhutan_resample = draw N resamples from the uni_sample
        nepal_resample = draw M resamples from the uni_sample
        muBhutan = sample mean of the bhutan_resample
        muNepal = sample mean of the nepal_resample
        diff = |muNepal – muBhutan|
        if diff >= observed_diff:
            count += 1

pValue = count / 10,000
```

Bootstrap for p-values

1. Create a universal sample using your two samples

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def pvalue_boot(bhutan_sample, nepal_sample):
    N = size of the bhutan_sample
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            count += 1

pValue = count / 10,000
```

this is the backbone of a null hypothesis.

Bootstrap for p-values

2. a. Resample **both samples**

```
def pvalue_boot(bhutan_sample, nepal_sample):
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        muNepal = sample mean of the nepal_resample
        diff = |muNepal – muBhutan|
        if diff >= observed_diff:
            count += 1

pValue = count / 10,000
```

it's important you go with the same sizes, otherwise you are using different random variables!

Bootstrap for p-values

2. b. Recalculate the **mean difference** b/t resamples

```
def pvalue_boot(bhutan_sample, nepal_sample):
    N = size of the bhutan_sample
    M = size of the nepal_sample
    observed_diff = |mean of bhutan_sample – mean of nepal_sample|

    uni_sample = combine bhutan_sample and nepal_sample
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        diff = |muNepal – muBhutan|
        if diff >= observed_diff:
            count += 1

pValue = count / 10,000
```

Bootstrap for p-values

3. **p-value** = $\frac{\text{\# (mean diffs > observed diff)}}{n}$

```
def pvalue_boot(bhutan_sample, nepal_sample):
    N = size of the bhutan_sample
    M = size of the nepal_sample
    observed_diff = |mean of bhutan_sample – mean of nepal_sample|

    uni_sample = combine bhutan_sample and nepal_sample
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        diff = |muNepal – muBhutan|
        if diff >= observed_diff:
            count += 1

    pValue = count / 10,000
```

Bootstrap for p-values

```
def pvalue_boot(bhutan_sample, nepal_sample):
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        muNepal = sample mean of the nepal_resample
        diff = |muNepal – muBhutan|
        if diff >= observed_diff:
            count += 1

pValue = count / 10,000
```

with replacement!

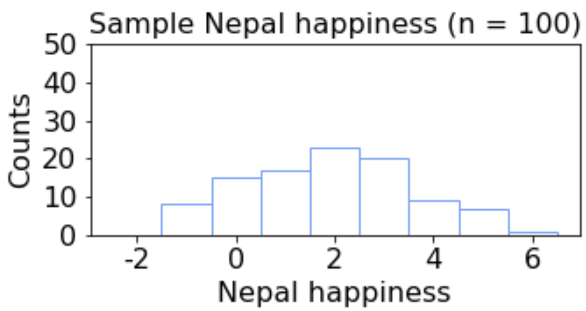
Bootstrap

Let's try it!

Google colab notebook link

Null hypothesis test

Nepal Happiness
4.45
2.45
6.37
2.07
…
1.63



Bhutan Happiness
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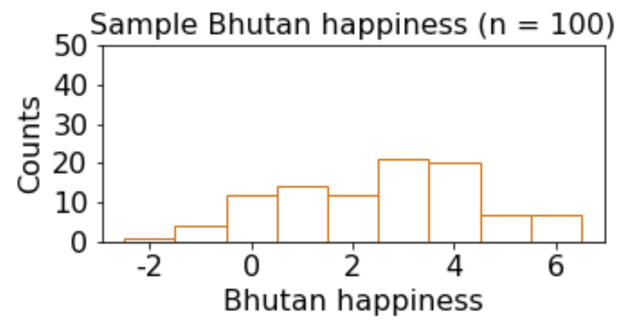

$\bar{X}_1 = 3.1$

$\bar{X}_2 = 2.4$

Claim: The happiness of Nepal and Bhutan have a 0.7 difference of means, and this is statistically significant ($p < 0.05$).