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## 16: Continuous Joint Distributions

Jerry Cain<br>© February 14, 2024 ©<br>Lecture Discussion on Ed

# Continuous Joint Distributions 

## Stanford logo with darts



The Stanford letterhead logo was created by throwing 500,000 darts according to a joint distribution.

If we throw another dart according to the same distribution, what is P (dart hits within $r$ pixels of center)?

Quick check: What is the probability that a dart hits at (456.2344132343, 532.1865739012)?

## CSiog logo with darts

P (dart hits within $r$ pixels of center)?



Possible dart counts (in $100 \times 100$ boxes)

## CSiog logo with darts

P (dart hits within $r$ pixels of center)?



Possible dart counts (in $50 x 50$ boxes)

## CS109 logo with darts

P (dart hits within $r$ pixels of center)?



Possible dart counts

## Continuous joint probability density functions

If two random variables $X$ and $Y$ are jointly continuous, then there exists a joint probability density function $f_{X, Y}$ defined over $-\infty<x, y<\infty$ such that:

$$
P\left(a_{1} \leq X \leq a_{2}, \quad b_{1} \leq Y \leq b_{2}\right)=\int_{a_{1}}^{a_{2}} \int_{b_{1}}^{b_{2}} f_{X, Y}(x, y) d y d x
$$

## From one continuous RV to jointly continuous RVs

Single continuous RV $X$

- PDF $f_{X}$ such that $\int_{-\infty}^{\infty} f_{X}(x) d x=1$
- Integrate to get probabilities


Jointly continuous RVs $X$ and $Y$

- PDF $f_{X, Y}$ such that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X, Y}(x, y) d y d x=1$
- Double integrate to get probabilities

Probability for jointly continuous RVs is volume under a surface.


## Double integrals without tears

Let $X$ and $Y$ be two continuous random variables.

- Support: $0 \leq X \leq 1,0 \leq Y \leq 2$.

Is $g(x, y)=x y$ a valid joint PDF over $X$ and $Y$ ?

Write down the definite double integral that
 must integrate to 1 :

## Double integrals without tears

Let $X$ and $Y$ be two continuous random variables.

- Support: $0 \leq X \leq 1,0 \leq Y \leq 2$.

Is $g(x, y)=x y$ a valid joint PDF over $X$ and $Y$ ?

Write down the definite double integral that
 must integrate to 1 :

$$
\underbrace{\int_{y=0}^{2} \int_{x=0}^{1} x y d x d y=1}_{\text {(used in next slide) }} \text { or } \int_{x=0}^{1} \int_{y=0}^{2} x y d y d x=1
$$



## Double integrals without tears

Let $X$ and $Y$ be two continuous random variables.

- Support: $0 \leq X \leq 1,0 \leq Y \leq 2$.

Is $g(x, y)=x y$ a valid joint PDF over $X$ and $Y$ ?
0 . Set up integral:
$1=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) d x d y=\int_{y=0}^{2} \int_{x=0}^{1} x y d x d y$


1. Evaluate inside integral by treating $y$ as a constant:

$$
\int_{y=0}^{2}\left(\int_{x=0}^{1} x y d x\right) d y=\int_{y=0}^{2} y\left(\int_{x=0}^{1} x d x\right) d y=\int_{y=0}^{2} y\left[\frac{x^{2}}{2}\right]_{0}^{1} d y=\int_{y=0}^{2} y \frac{1}{2} d y
$$

2. Evaluate remaining (single) integral:

$$
\int_{y=0}^{2} y \frac{1}{2} d y=\left[\frac{y^{2}}{4}\right]_{y=0}^{2}=1-0=1
$$

Yes, $g(x, y)$ is a valid joint PDF because it integrates to 1 .

## Marginal distributions

Suppose $X$ and $Y$ are continuous random variables with joint PDF:

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X, Y}(x, y) d y d x=1
$$



The marginal density functions-that is, the marginal PDFs-are therefore:

$$
f_{X}(a)=\int_{-\infty}^{\infty} f_{X, Y}(a, y) d y \quad f_{Y}(b)=\int_{-\infty}^{\infty} f_{X, Y}(x, b) d \mathrm{x}
$$

## Back to darts!



Match $X$ and $Y$ to their respective marginal PDFs:



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## Back to darts!



Match $X$ and $Y$ to their respective marginal PDFs:



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Joint CDFs

## CDFs and PDFs in one dimension

For a one-dimensional continuous random variable $X$ with $\operatorname{PDF} f$, the CDF (cumulative distribution function) is

$$
F(a)=P(X \leq a)=\int_{-\infty}^{a} f(x) d x
$$

The density $f$ is the derivative of the CDF, $F$ :

$$
f(x)=\frac{d}{d x} F(x)
$$

## Single variable CDF, graphically


$f_{X}(x)$


$\lim _{x \rightarrow-\infty} F_{X}(x)=0$

$$
F_{X}(x)=P(X \leq x)
$$

## Joint cumulative distribution function

For two random variables $X$ and $Y$, there can be a joint cumulative distribution function $F_{X, Y}$ :

$$
F_{X, Y}(a, b)=P(X \leq a, Y \leq b)
$$

For discrete $X$ and $Y$ :
For continuous $X$ and $Y$ :

$$
\begin{gathered}
F_{X, Y}(a, b)=\sum_{x \leq a} \sum_{y \leq b} p_{X, Y}(x, y) \quad F_{X, Y}(a, b)=\int_{-\infty}^{a} \int_{-\infty}^{b} f_{X, Y}(x, y) d y d x \\
f_{X, Y}(a, b)=\frac{\partial^{2}}{\partial a \partial b} F_{X, Y}(a, b)
\end{gathered}
$$

## Joint CDF, graphically



## Independent Continuous RVs

## Independent continuous RVs

Two continuous random variables $X$ and $Y$ are independent if:

$$
P(X \leq x, Y \leq y)=P(X \leq x) P(Y \leq y) \quad \forall x, y
$$

Equivalently:

$$
\begin{aligned}
F_{X, Y}(x, y) & =F_{X}(x) F_{Y}(y) \quad \forall x, y \\
f_{X, Y}(x, y) & =f_{X}(x) f_{Y}(y)
\end{aligned}
$$

Proof of PDF:

$$
\begin{aligned}
f_{X, Y}(x, y) & =\frac{\partial^{2}}{\partial x \partial y} F_{X, Y}(x, y)=\frac{\partial^{2}}{\partial x \partial y} F_{X}(x) F_{Y}(y) \\
& =\frac{\partial}{\partial x} \frac{\partial}{\partial y} F_{X}(x) F_{Y}(y)=\frac{\partial}{\partial x} F_{X}(x) \frac{\partial}{\partial y} F_{Y}(y) \\
& =f_{X}(x) f_{Y}(y)
\end{aligned}
$$

## Independent continuous RVs

## Two continuous random variables $X$ and $Y$ are independent if:

$$
P(X \leq x, Y \leq y)=P(X \leq x) P(Y \leq y) \quad \forall x, y
$$

Equivalently:

$$
\begin{aligned}
F_{X, Y}(x, y) & =F_{X}(x) F_{Y}(y) \quad \forall x, y \\
f_{X, Y}(x, y) & =f_{X}(x) f_{Y}(y)
\end{aligned}
$$

More generally, $X$ and $Y$ are independent if the joint PDF factors into two, single-variable marginal probability densities:

$$
f_{X, Y}(x, y)=g(x) h(y), \text { where }-\infty<x, y<\infty
$$

## Pop quiz! (just kidding)

```
f}\mp@subsup{f}{,Y}{}(x,y)=g(x)h(y)
    independent
    X and Y
```

Are $X$ and $Y$ independent in the following cases?

1. $f_{X, Y}(x, y)=6 e^{-3 x} e^{-2 y}$ where $0<x, y<\infty$
2. $f_{X, Y}(x, y)=4 x y$ where $0<x, y<1$
3. $f_{X, Y}(x, y)=24 x y$
where $0<x+y<1$

## Pop quiz! (just kidding)

```
f}\mp@subsup{f}{X,Y}{}(x,y)=g(x)h(y)
where - \infty<x,y<\infty

Are \(X\) and \(Y\) independent in the following cases?
1. \(f_{X, Y}(x, y)=6 e^{-3 x} e^{-2 y} \quad\) Separable functions: \(g(x)=3 e^{-3 x}\) where \(0<x, y<\infty\)
\[
h(y)=2 e^{-2 y}
\]
v 2. \(f_{X, Y}(x, y)=4 x y\) where \(0<x, y<1\)

Separable functions: \(g(x)=2 x\) \(h(y)=2 y\)

X 3. \(f_{X, Y}(x, y)=24 x y\) where \(0<x+y<1\)

Cannot capture constraint on \(x+y\) !

If you can factor densities over the entire support, you have independence.

\section*{More pop quiz! (more kidding)}
\(X\) and \(Y\) have the following joint PDF:
\[
\begin{gathered}
f_{X, Y}(x, y)=3 e^{-3 x} \\
\text { where } 0<x<\infty, 1<y<2
\end{gathered}
\]
1. Are \(X\) and \(Y\) independent?
2. What is the marginal PDF of \(X\) ? Of \(Y\) ?
3. What is \(E[X+Y]\) ?

\section*{More pop quiz! (more kidding)}
\(X\) and \(Y\) have the following joint PDF:
\[
f_{X, Y}(x, y)=3 e^{-3 x}
\]
\[
\text { where } 0<x<\infty, 1<y<2
\]
1. Are \(X\) and \(Y\) independent? \(\nabla\)
\[
\begin{array}{lll}
g(x)=3 C e^{-3 x}, & 0<x<\infty & C \text { is a } \\
h(y)=1 / C, & 1<y<2 & \text { constant }
\end{array}
\]
2. What is the marginal PDF of \(X\) ? Of \(Y\) ?
3. What is \(E[X+Y]\) ?



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\section*{More pop quiz! (more kidding)}
\(X\) and \(Y\) have the following joint PDF:
\[
\begin{gathered}
f_{X, Y}(x, y)=3 e^{-3 x} \\
\text { where } 0<x<\infty, 1<y<2
\end{gathered}
\]
1. Are \(X\) and \(Y\) independent? \(\nabla\)
\[
\begin{array}{ll}
g(x)=3 e^{-3 x} & 0<x<\infty \\
h(y)=1, & 1<y<2
\end{array}
\]
2. What is the marginal PDF of \(X\) ? Of \(Y\) ?
3. What is \(E[X+Y]\) ?

\section*{The joy of meetings}

Two people set up a meeting time. Each arrives independently at a time uniformly distributed between 12pm and 12:30pm.

Define \(\quad X=\#\) minutes past 12pm that person 1 arrives. \(X \sim \operatorname{Uni}(0,30)\)
\(Y=\#\) minutes past 12 pm that person 2 arrives. \(Y \sim \operatorname{Uni}(0,30)\)
What is the probability that the first to arrive waits \(>10\) mins for the other?
Compute: \(P(X+10<Y)+P(Y+10<X)=2 P(X+10<Y) \quad\) (by symmetry)
1. What is symmetry here?
2. How do we integrate to compute this probability?

\section*{Double integrals: A guide}

From last slide: \(\quad 2 P(X+10<Y)=2 \iint_{\substack{x+10<y, 0 \leq x, y, \leq 30}}(1 / 30)^{2} d x d y \quad \begin{array}{r}\text { (by symmetry, } \\ \text { independence) }\end{array}\)

\section*{Steps:}
1. Draw a picture.
2. Set bounds "from outside in".
- Outer integral bounds should be full range possible
- Inner integral can depend on integration variable of outer integral
\[
\begin{aligned}
& =\frac{2}{30^{2}} \int_{10}^{30} \int_{0}^{y-10} d x d y \\
& =\frac{2}{30^{2}} \int_{10}^{30}(y-10) d y \quad=\cdots=\frac{4}{9}
\end{aligned}
\]

Bivariate Normal
Distribution

\section*{Bivariate Normal Distribution}
\(X_{1}\) and \(X_{2}\) follow a bivariate normal distribution if their joint PDF \(f\) is
\[
f\left(x_{1}, x_{2}\right)=\frac{1}{2 \pi \sigma_{1} \sigma_{2} \sqrt{1-\rho^{2}}} e^{-\frac{1}{2\left(1-\rho^{2}\right)}\left(\frac{\left(x_{1}-\mu_{1}\right)^{2}}{\sigma_{1}^{2}}-\frac{2 \rho\left(x_{1}-\mu_{1}\right)\left(x_{2}-\mu_{2}\right)}{\sigma_{1} \sigma_{2}}+\frac{\left(x_{2}-\mu_{2}\right)^{2}}{\sigma_{2}^{2}}\right)}
\]

Can show that \(X_{1} \sim \mathcal{N}\left(\mu_{1}, \sigma_{1}^{2}\right), X_{2} \sim \mathcal{N}\left(\mu_{2}, \sigma_{2}^{2}\right)\)
(Ross chapter 6, example 5d)
Often written as:

\section*{\(\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})\)}
- Vector \(\boldsymbol{X}=\left(X_{1}, X_{2}\right)\)
- Mean vector \(\boldsymbol{\mu}=\left(\mu_{1}, \mu_{2}\right)\), Covariance matrix: \(\boldsymbol{\Sigma}=\left[\begin{array}{cc}\sigma_{1}^{2} & \rho \sigma_{1} \sigma_{2} \\ \rho \sigma_{1} \sigma_{2} & \sigma_{2}^{2}\end{array}\right]\)

Recall correlation: \(\rho=\frac{\operatorname{Cov}\left(X_{1}, X_{2}\right)}{\sigma_{1} \sigma_{2}}\)

We will focus on understanding the shape of a bivariate Normal RV.

\section*{Back to darts}

(side view)


Darts were thrown according to a bivariate normal distribution:
\[
\begin{aligned}
& \boldsymbol{\mu}=(450,600) \\
& \boldsymbol{\Sigma}=\left[\begin{array}{cc}
900^{2} / 4 & 0 \\
0 & 900^{2} / 25
\end{array}\right]
\end{aligned}
\]



\section*{A diagonal covariance matrix}

Let \(\boldsymbol{X}=\left(X_{1}, X_{2}\right)\) follow a bivariate normal distribution \(\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})\), where
\[
\boldsymbol{\mu}=\left(\mu_{1}, \mu_{2}\right), \quad \boldsymbol{\Sigma}=\left[\begin{array}{cc}
\sigma_{1}^{2} & 0 \\
0 & \sigma_{2}^{2}
\end{array}\right]
\]

Are \(X_{1}\) and \(X_{2}\) independent?
\[
\begin{aligned}
f\left(x_{1}, x_{2}\right) & =\frac{1}{2 \pi \sigma_{1} \sigma_{2} \sqrt{1-\rho^{2}}} e^{-\frac{1}{2\left(1-\rho^{2}\right)}\left(\frac{\left(x_{1}-\mu_{1}\right)^{2}}{\sigma_{1}^{2}}-\frac{2 \rho\left(x_{1}-\mu_{1}\right)\left(x_{2}-\mu_{2}\right)}{\sigma_{1} \sigma_{2}}+\frac{\left(x_{2}-\mu_{2}\right)^{2}}{\sigma_{2}^{2}}\right)} \\
& =\frac{1}{2 \pi \sigma_{1} \sigma_{2}} e^{-\frac{1}{2}\left(\frac{\left(x_{1}-\mu_{1}\right)^{2}}{\sigma_{1}^{2}}+\frac{\left(x_{2}-\mu_{2}\right)^{2}}{\sigma_{2}^{2}}\right)} \quad\left(\text { Note covariance: } \rho \sigma_{1} \sigma_{2}=0\right) \\
= & \frac{1}{\sigma_{1} \sqrt{2 \pi}} e^{-\left(x_{1}-\mu_{1}\right)^{2} / 2 \sigma_{1}^{2}} \frac{1}{\sigma_{2} \sqrt{2 \pi}} e^{-\left(x_{2}-\mu_{2}\right)^{2} / 2 \sigma_{2}^{2}}
\end{aligned} \begin{aligned}
& X_{1} \text { and } X_{2} \text { are independent } \\
& \text { with marginal distributions } \\
& X_{1} \sim \mathcal{N}\left(\mu_{1} \sigma_{1}^{2}\right), X_{2} \sim \mathcal{N}\left(\mu_{2} \sigma_{2}^{2}\right)
\end{aligned}
\]
\((X, Y)\) Matching (all have \(\boldsymbol{\mu}=(0,0)\) )

A. \(\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right] \quad\) B. \(\left[\begin{array}{ll}1 & 0 \\ 0 & 2\end{array}\right]\)
C. \(\left[\begin{array}{cc}1 & 0.5 \\ 0.5 & 1\end{array}\right]\) D. \(\left[\begin{array}{cc}1 & -0.5 \\ -0.5 & 1\end{array}\right]\)
1.

2.

3.


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\((X, Y)\) Matching (all have \(\boldsymbol{\mu}=(0,0)\) )
A. \(\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right] \quad\) B. \(\left[\begin{array}{ll}1 & 0 \\ 0 & 2\end{array}\right]\)
C. \(\left[\begin{array}{cc}1 & 0.5 \\ 0.5 & 1\end{array}\right]\) D. \(\left[\begin{array}{cc}1 & -0.5 \\ -0.5 & 1\end{array}\right]\)
1.

2.

3.




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\section*{Why are joint PDFs useful?}


Independence 2-D support Joint PDF Joint CDF
Marginal PDF Conditional PDF
- How 2 continuous RVs vary with each other
- How continuous RV is distributed given evidence (more on Friday)
- How a continuous RV can be decomposed into 2 RVs (or vice versa)
\[
P(X<Y)
\]
\[
\operatorname{Cov}(X, Y), \rho(X, Y)
\]

Given \(Y=y\), the distribution of \(X\)

Distribution of \(Z=X+Y\)
(which is a 1-D RV!)

\section*{Sum of Independent Gaussians}

\section*{Sum of independent Gaussians}
\[
\begin{gathered}
X \sim \mathcal{N}\left(\mu_{1}, \sigma_{1}^{2}\right), \\
Y \sim \mathcal{N}\left(\mu_{2}, \sigma_{2}^{2}\right) \\
X, Y \text { independent }
\end{gathered} \quad X+Y \sim \mathcal{N}\left(\mu_{1}+\mu_{2}, \sigma_{1}^{2}+\sigma_{2}^{2}\right)
\]

\author{
(proof left to Wikipedia)
}

Holds in general case:
\[
\begin{gathered}
X_{i} \sim \mathcal{N}\left(\mu_{i}, \sigma_{i}^{2}\right) \\
X_{i} \text { independent for } i=1, \ldots, n
\end{gathered} \quad \sum_{i=1}^{n} X_{i} \sim \mathcal{N}\left(\sum_{i=1}^{n} \mu_{i}, \sum_{i=1}^{n} \sigma_{i}^{2}\right)
\]

\section*{Back for another playoffs game}


What is the probability that the Warriors win? How do you model zero-sum games?
\[
P\left(A_{W}>A_{B}\right)
\]

This is a probability of an event involving two random variables!

We will compute:
\[
P(\underbrace{\left.A_{W}-A_{B}>0\right)}_{W}
\]

\section*{Motivating idea: Zero sum games}

Want: \(P\) (Warriors win) \(=P\left(A_{W}-A_{B}>0\right)\)
Assume \(A_{W}, A_{B}\) are independent.
Let \(D=A_{W}-A_{B}\).

What is the distribution of \(D\) ?
A. \(\quad D \sim \mathcal{N}\left(1657-1470,200^{2}-200^{2}\right)\)
B. \(D \sim \mathcal{N}\left(1657-1470,200^{2}+200^{2}\right)\)
C. \(D \sim \mathcal{N}\left(1657+1470,200^{2}+200^{2}\right)\)
D. \(D \sim \mathcal{N}\left(1657+1470,200^{2}\right)\)



\section*{Motivating idea: Zero sum games}

Want: \(P\) (Warriors win) \(=P\left(A_{W}-A_{B}>0\right)\)
Assume \(A_{W}, A_{B}\) are independent.
Let \(D=A_{W}-A_{B}\).

What is the distribution of \(D\) ?
\[
\begin{aligned}
& \text { A. } D \sim \mathcal{N}\left(1657-1470,200^{2}-200^{2}\right) \\
& \text { B. } D \sim \mathcal{N}\left(1657-1470,200^{2}+200^{2}\right) \\
& \text { C. } D \sim \mathcal{N}\left(1657+1470,200^{2}+200^{2}\right) \\
& \text { D. } D \sim \mathcal{N}\left(1657+1470,200^{2}\right)
\end{aligned}
\]


Opponents \(A_{B} \sim \mathcal{N}\left(S=1470,200^{2}\right)\)
\[
\begin{aligned}
& \text { If } X \sim \mathcal{N}\left(\mu_{1}, \sigma^{2}\right) \\
& \text { then }(-X) \sim \mathcal{N}\left(-\mu,(-1)^{2} \sigma^{2}=\sigma^{2}\right)
\end{aligned}
\]

\section*{Motivating idea: Zero sum games}

Want: \(P\) (Warriors win) \(=P\left(A_{W}-A_{B}>0\right)\)
Assume \(A_{W}, A_{B}\) are independent.
Let \(D=A_{W}-A_{B}\).
\[
\begin{aligned}
& D \sim \mathcal{N}\left(1657-1470, \quad 200^{2}+200^{2}\right) \\
& \sim \mathcal{N}\left(187,2 \cdot 200^{2}\right) \quad \sigma \approx 282.842 \\
& \\
& P(D>0)=1-F_{D}(0)=1-\Phi\left(\frac{0-187}{282.842}\right) \\
& \\
& \quad \approx 0.74574
\end{aligned}
\]

Compare with 0.7488 , calculated by sampling!

Warriors \(A_{W} \sim \mathcal{N}\left(S=1657,200^{2}\right)\)


>>> from scipy.stats import norm
>>> 1 - norm (187, 80000 ** 0.5).cdf(0) 0.7457402843526317
>>> \(1-\operatorname{norm}(0,1) . \operatorname{cdf}(-187 /(80000\) ** 0.5)) 0.7457402843526317

\section*{Virus infections}

Suppose you are working with the WHO to initiate a response to the onset of a virus. There are two exposed groups:
- G1: 20000 people, each independently infected with \(p_{1}=0.1\)
- G2: 10000 people, each independently infected with \(p_{2}=0.4\)

What is \(P\) (people infected \(\geq 6100\) )? An approximation is okay.
1. Define RVs
\& state goal
Let \(A=\#\) infected in G1.
\(A \sim \operatorname{Bin}(20000,0.1)\)
\(B=\#\) infected in G2.
\(B \sim \operatorname{Bin}(10000,0.4)\)

Strategy:
A. Sum of independent Binomials
B. Sum of independent Poissons
C. Sum of independent Gaussians
D. Sum of independent Exponentials

Want: \(P(A+B \geq 6100)\)

\section*{Virus infections}

Suppose you are working with the WHO to initiate a response to the onset of a virus. There are two exposed groups:
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What is \(P\) (people infected \(\geq 6100\) )? An approximation is okay.
1. Define RVs
\& state goal
Let \(A=\) \# infected in G1.
\(A \sim \operatorname{Bin}(20000,0.1)\)
\(B=\#\) infected in G2.
\(B \sim \operatorname{Bin}(10000,0.4)\)
Want: \(P(A+B \geq 6100)\)
2. Approximate as sum of Gaussians
\(A \approx X \sim \mathcal{N}(2000,1800) B \approx Y \sim \mathcal{N}(4000,2400)\) \(P(A+B \geq 6100) \approx P(X+Y \geq 6099.5)_{\substack{\text { contrection }}}^{\text {contity }}\)
3. Solve

\section*{Virus infections}

Suppose you are working with the WHO to initiate a response to the onset of a virus. There are two exposed groups:
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1. Define RVs
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Let \(A=\) \# infected in G1.
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\(B=\) \# infected in G2.
\(B \sim \operatorname{Bin}(10000,0.4)\)
Want: \(P(A+B \geq 6100)\)
2. Approximate as sum of Gaussians
\[
A \approx X \sim \mathcal{N}(2000,1800) B \approx Y \sim \mathcal{N}(4000,2400)
\]
\[
P(A+B \geq 6100) \approx P(X+Y \geq 6099.5)_{\text {correction }}^{\text {continuty }}
\]
3. Solve

Let \(W=X+Y \sim \mathcal{N}(6000,4200)\)
\(P(W \geq 6099.5)=1-\Phi\left(\frac{6099.5-6000}{\sqrt{4200}}\right)\)
\(\approx 1-\Phi(1.53531) \approx 0.06235\)

\section*{Sum of independent Gaussians}
\[
\begin{gathered}
X_{1} \sim \mathcal{N}\left(\mu_{1}, \sigma_{1}^{2}\right), \\
X_{2} \sim \mathcal{N}\left(\mu_{2}, \sigma_{2}^{2}\right) \\
X_{1}, X_{2} \text { independent }
\end{gathered} \quad X_{1}+X_{2} \sim \mathcal{N}\left(\mu_{1}+\mu_{2}, \sigma_{1}^{2}+\sigma_{2}^{2}\right)
\]

Is this related to linear transformations of Gaussians?
Recall:
\[
\text { If } Y=a X+b \text {, then } Y \sim \mathcal{N}\left(a \mu_{X}+b, a^{2} \sigma_{X}^{2}\right)
\]

\section*{Teaser: Linear transforms vs. independence}

Let \(X \sim \mathcal{N}\left(\mu, \sigma^{2}\right)\) and \(Y=X+X\). What is the distribution of \(Y\) ?
- Are both approaches valid?

\section*{Independent RVs approach}

Let \(X_{1} \sim \mathcal{N}\left(\mu_{1}, \sigma_{1}^{2}\right), X_{2} \sim \mathcal{N}\left(\mu_{2}, \sigma_{2}^{2}\right)\) be independent.
Then \(Y=X_{1}+X_{2} \sim \mathcal{N}\left(\mu_{1}+\mu_{2}, \sigma_{1}^{2}+\sigma_{2}^{2}\right)\)

\section*{Linear transform approach}
\[
\begin{gathered}
\text { Let } X \sim \mathcal{N}\left(\mu, \sigma^{2}\right) . \\
\text { If } Y=a X+b, \\
\text { then } Y \sim \mathcal{N}\left(a \mu+b, a^{2} \sigma^{2}\right) .
\end{gathered}
\]

\section*{Teaser: Linear transforms vs. independence}

Let \(X \sim \mathcal{N}\left(\mu, \sigma^{2}\right)\) and \(Y=X+X\). What is the distribution of \(Y\) ?
- Are both approaches valid?

\section*{Independent RVs approach}

Let \(X_{1} \sim \mathcal{N}\left(\mu_{1}, \sigma_{1}^{2}\right), X_{2} \sim \mathcal{N}\left(\mu_{2}, \sigma_{2}^{2}\right)\) be independent.
Then \(Y=X_{1}+X_{2} \sim \mathcal{N}\left(\mu_{1}+\mu_{2}, \sigma_{1}^{2}+\sigma_{2}^{2}\right)\)
\[
\begin{aligned}
& Y=X+X \\
& X+X \sim \mathcal{N}\left(\mu+\mu, \sigma^{2}+\sigma^{2}\right) ? \\
& Y \sim \mathcal{N}\left(2 \mu, 2 \sigma^{2}\right) ?
\end{aligned}
\]

\section*{Linear transform approach}
\[
\begin{gathered}
\text { Let } X \sim \mathcal{N}\left(\mu, \sigma^{2}\right) . \\
\text { If } Y=a X+b, \\
\text { then } Y \sim \mathcal{N}\left(a \mu+b, a^{2} \sigma^{2}\right) .
\end{gathered}
\]
\[
\begin{aligned}
& Y=2 X \\
& Y \sim \mathcal{N}\left(2 \mu, 4 \sigma^{2}\right)
\end{aligned}
\]

For independent \(X_{1} \sim \mathcal{N}\left(\mu_{1}, \sigma_{1}^{2}\right), X_{2} \sim \mathcal{N}\left(\mu_{2}, \sigma_{2}^{2}\right)\),
\[
a X_{1}+b X_{2}+c \sim \mathcal{N}\left(a \mu_{1}+b \mu_{2}+c, a^{2} \sigma_{1}^{2}+b^{2} \sigma_{2}^{2}\right)
\]```

