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13: Statistics on Multiple Random Variables

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[Lecture Discussion on Ed]

Coupon Collecting

Coupon collecting and server requests

The **coupon collector's problem** in probability theory:

- You buy boxes of cereal.
- There are k different types of coupons
- For each box you buy, you "collect" a coupon of type i.

Servers

requests

k servers

request to server i

1. How many coupons do you expect after buying n boxes of cereal?

What is the expected number of servers utilized after n requests?

* 52% of Amazon profits

** more profitable than Amazon's North America commerce operations

source

Computer cluster utilization

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Consider a computer cluster with k servers. We send n requests.

- Requests independently go to server i with probability p_i
- Let $X = \#$ servers that receive ≥ 1 request.

What is $E[X]$?

Computer cluster utilization

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Consider a computer cluster with k servers. We send n requests.

- Requests independently go to server i with probability p_i
- Let X = # servers that receive ≥ 1 request.

What is $E[X]$?

1. Define additional random variables.

Let: A_i = event that server i receives ≥ 1 request

X_i = indicator for A_i

$X_i = \begin{cases} 1 & \text{if } A_i \text{ holds} \\ 0 & \text{if } A_i^C \text{ holds instead} \end{cases}$

$P(A_i) = 1 - P(\text{no requests to } i)$

$= 1 - (1 - p_i)^n$

Note: A_i are dependent!

2. Solve.

$E[X_i] = P(A_i)$

$$E[X_i] = P(A_i) = 1 - (1 - p_i)^n$$

$$E[X] = E\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} E[X_i] = \sum_{i=1}^{k} \left(1 - (1 - p_i)^n\right)$$

$$= \sum_{i=1}^{k} 1 - \sum_{i=1}^{k} (1 - p_i)^n = k - \sum_{i=1}^{k} (1 - p_i)^n$$

does this result make sense for $n = 0$? how about $n = 1$?

Coupon collecting problems: Hash tables

The **coupon collector's problem** in probability theory:

- You buy boxes of cereal.
- There are k different types of coupons
- For each box you buy, you "collect" a coupon of type i.

Servers	Hash Tables
requests	strings
k servers	k buckets
request to server i	hashed to bucket i

1. How many coupons do you expect after buying n boxes of cereal? → What is the expected number of utilized servers after n requests?

2. How many boxes do you expect to buy until you have one of each coupon? → What is the expected number of strings to hash until each bucket has ≥ 1 string?

Hash Tables

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Consider a hash table with k buckets.

- Strings are equally likely to get hashed into any bucket (independently).
- Let $Y = \#$ strings to hash until each bucket ≥ 1 string.

What is $E[Y]$?

1. Define additional random variables.

   How should we define Y_i such that $Y = \sum_i Y_i$?

2. Solve.

Hash Tables

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

assume perfect hashing, so that $p_i = \frac{1}{k}$

Consider a hash table with k buckets.

- Strings are equally likely to get hashed into any bucket (independently).
- Let Y = # strings to hash until each bucket ≥ 1 string.

What is $E[Y]$?

Y_0 = # trials needed until first bucket gets a string
Y_1 = # trials beyond Y_0 until second buckets get a string
Y_2 = # trials beyond Y_1 until third bucket sees a string

1. Define additional random variables.

Let: Y_i = # of trials needed to get success after i-th success

- Success: hash string to previously empty bucket
- If i non-empty buckets: $P(\text{success}) = \frac{k-i}{k}$ ← # empty buckets

2. Solve.

$$P(Y_i = n) = \left(\frac{i}{k}\right)^{n-1} \left(\frac{k-i}{k}\right)$$

Equivalently, $Y_i \sim \text{Geo}\left(p = \frac{k-i}{k}\right)$ $\quad E[Y_i] = \frac{1}{p} = \frac{k}{k-i}$

Hash Tables

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Consider a hash table with k buckets.

- Strings are equally likely to get hashed into any bucket (independently).
- Let Y = # strings to hash until each bucket $\geq$ 1 string.

What is $E[Y]$?

1. Define additional random variables.

   Let: Y_i = # of trials to needed get success after i-th success

   $$Y_i \sim \text{Geo}\left(p = \frac{k-i}{k}\right), \qquad E[Y_i] = \frac{1}{p} = \frac{k}{k-i}$$

2. Solve.

$$Y = Y_0 + Y_1 + \cdots + Y_{k-1}$$

$$\sum_{m=1}^{k} \frac{1}{m} \approx \int_1^k \frac{1}{m}\, dm = \ln k$$

$$E[Y] = E[Y_0] + E[Y_k] + \cdots + E[Y_{k-1}]$$

$$= \frac{k}{k} + \frac{k}{k-1} + \frac{k}{k-2} + \cdots + \frac{k}{1} = k\left[\frac{1}{k} + \frac{1}{k-1} + \cdots + 1\right] = O(k \log k)$$

Covariance

Statistics of sums of RVs

For any random variables X and Y,

$$E[X+Y] = E[X] + E[Y]$$

$$\text{Var}(X+Y) = \quad ?$$

But first, a new statistic!

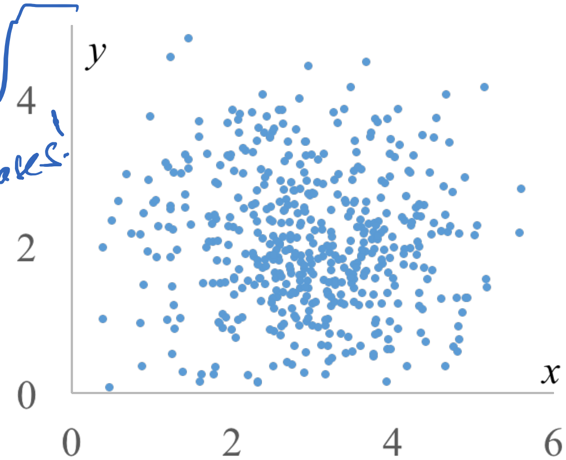
Spot the difference

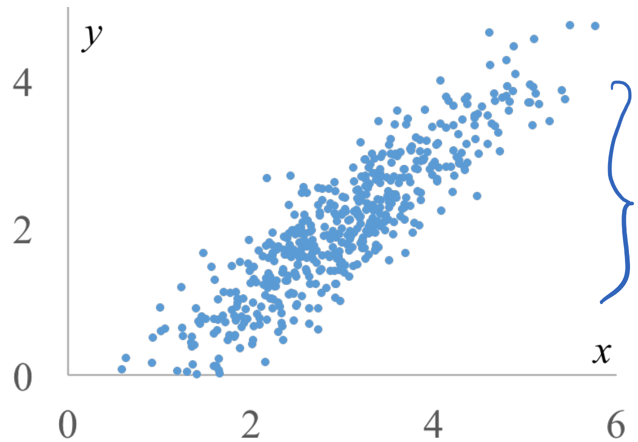
Compare/contrast the following two distributions:

Assume all points are equally likely.

$$P(X = x, Y = y) = \frac{1}{N}$$

can't predict how y will change as x increases!





can predict, at least roughly, how y will change as x increases!

these four statistics don't capture how x and y are coupled!

Both distributions have the same $E[X]$, $E[Y]$, Var(X), and Var(Y)

Difference: how the two variables vary with **each other**.

Covariance

The **covariance** of two variables X and Y is:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Proof of second part (rewriting $E[X]$, $E[Y]$ as μ_X, μ_Y to emphasize the fact they're each constants):

$$\begin{aligned}
\text{Cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] = E[(X - \mu_X)(Y - \mu_Y)] \\
&= E[XY - \mu_Y X - \mu_X Y + \mu_X \mu_Y] \\
&= E[XY] - E[\mu_Y X] - E[\mu_X Y] + E[\mu_X \mu_Y] && \text{(linearity of expectation)} \\
&= E[XY] - \mu_X \mu_Y - \mu_X \mu_Y + \mu_X \mu_Y && (\mu_X, \mu_Y \text{ are constants}) \\
&= E[XY] - \mu_X \mu_Y = E[XY] - E[X]E[Y]
\end{aligned}$$

Covariance

The covariance of two variables X and Y is:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

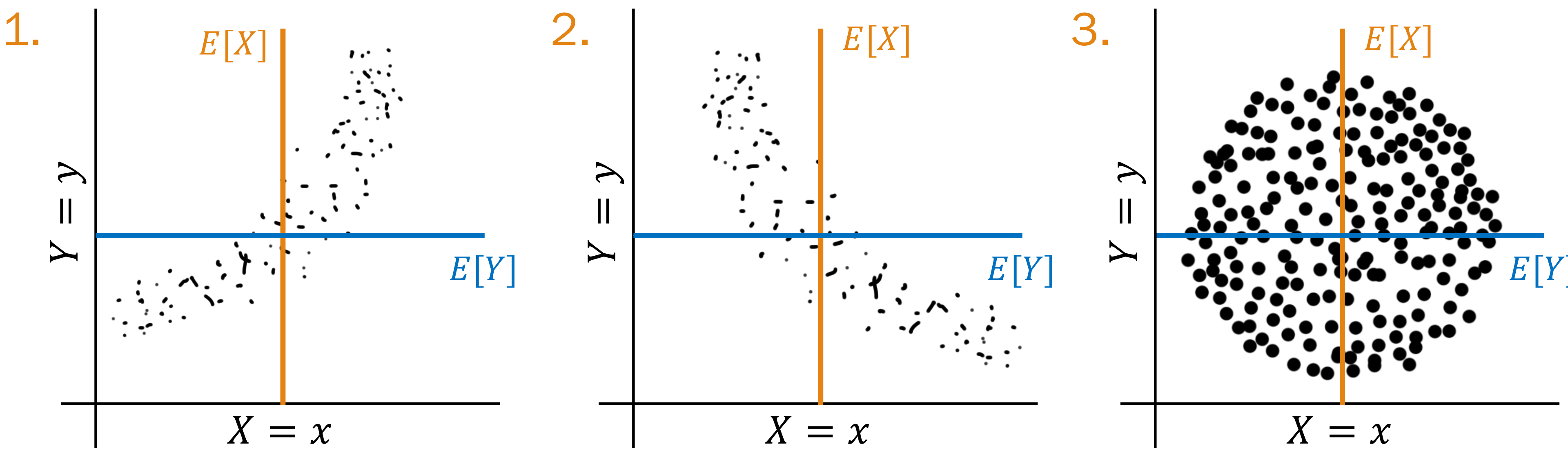
Covariance measures how one random variable varies with a second.

- Outside temperature and utility bills have a negative covariance.
- Handedness and musical ability have near zero covariance.
- Product demand and price have a positive covariance.

Feel the covariance

$$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Is the covariance positive, negative, or zero?



Feel the covariance

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$$

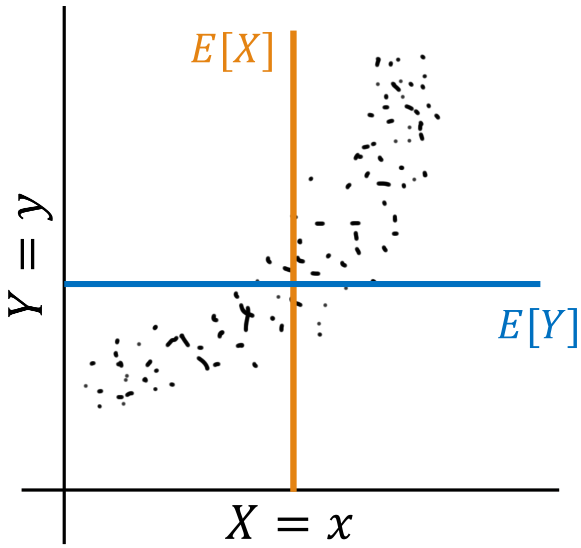
Is the covariance positive, negative, or zero?

as x increases, so does y ∴ positive covariance

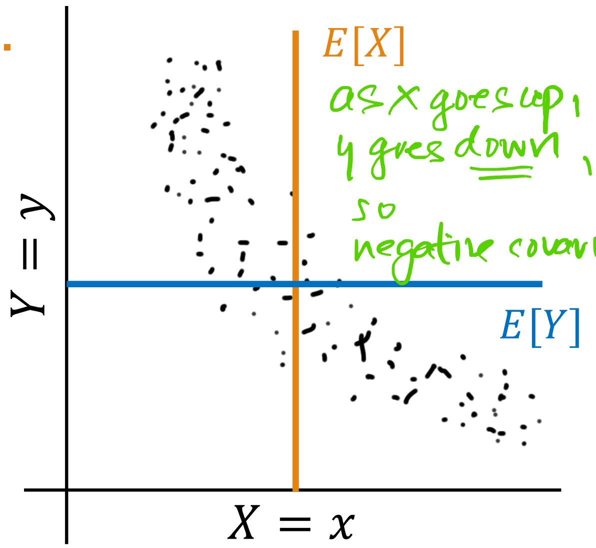
no obvious pattern in how y changes as x increases.

~0 covariance

1.

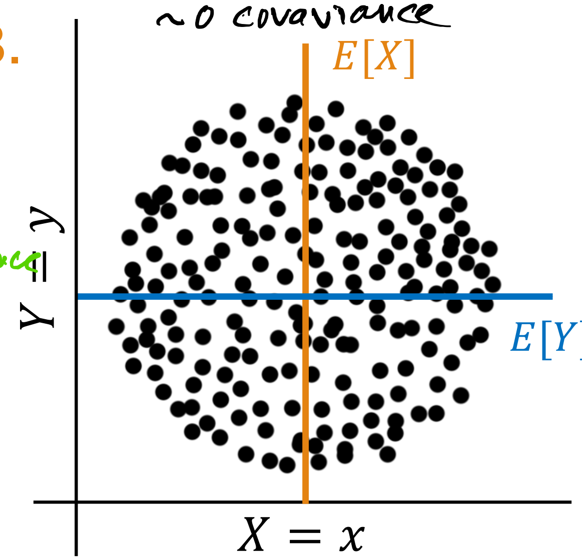


2.



as x goes up, y goes down, so negative covariance

3.



positive

negative

zero

Covarying humans

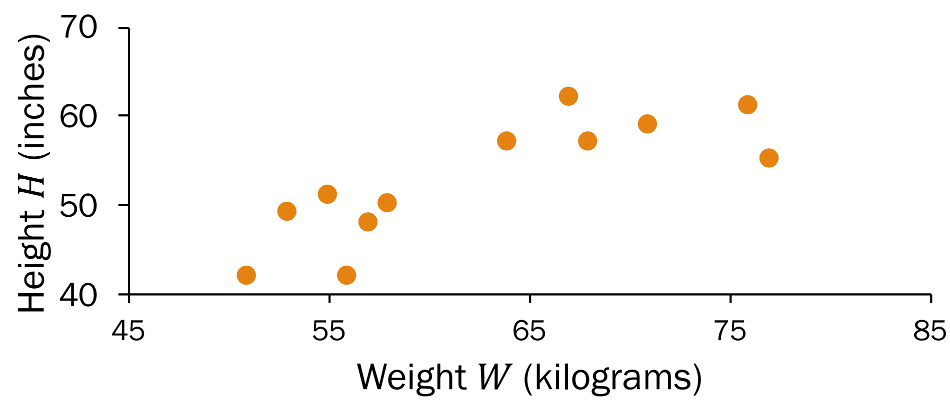
$$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Weight (kg)	Height (in)	W · H
64	57	3648
71	59	4189
53	49	2597
67	62	4154
55	51	2805
58	50	2900
77	55	4235
57	48	2736
56	42	2352
51	42	2142
76	61	4636
68	57	3876

$E[W] = 62.75$ $E[H] = 52.75$ $E[WH] = 3355.83$

What is the covariance of weight W and height H?

$$\begin{aligned} \text{Cov}(W,H) &= E[WH] - E[W]E[H] \\ &= 3355.83 - (62.75)(52.75) \\ \text{(positive)} \quad &= 45.77 \end{aligned}$$



Covariance > 0: one variable ↑, other variable ↑

Properties of Covariance

The **covariance** of two variables X and Y is:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

Properties:

1. $\text{Cov}(X, Y) = \text{Cov}(Y, X)$
2. $\text{Var}(X) = E[X^2] - (E[X])^2 = E[XX] - E[X]E[X] = \text{Cov}(X, X)$
3. Covariance of sums = sum of all pairwise covariances (proof left to you)
   $\text{Cov}(X_1 + X_2, Y_1 + Y_2) = \text{Cov}(X_1, Y_1) + \text{Cov}(X_2, Y_1) + \text{Cov}(X_1, Y_2) + \text{Cov}(X_2, Y_2)$
4. Covariance under linear transformation: $\text{Cov}(aX + b, Y) = a\text{Cov}(X, Y)$

recall that $\text{Var}(aX+b) = a^2\,\text{Var}(X)$

Zero covariance does not imply independence

Let X take on values $\{-1,0,1\}$ with equal probability $1/3$.

Define $Y = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{otherwise} \end{cases}$

What is the joint PMF of X and Y?

Zero covariance does not imply independence

Let X take on values $\{-1,0,1\}$ with equal probability $1/3$.

Define $Y = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{otherwise} \end{cases}$

Y \ X	-1	0	1	Marginal PMF of Y, $p_Y(y)$
0	1/3	0	1/3	2/3
1	0	1/3	0	1/3
Marginal PMF of X, $p_X(x)$	1/3	1/3	1/3	

1. $E[X] =$ $\qquad$ $E[Y] =$

2. $E[XY] =$

3. $\text{Cov}(X,Y) =$

4. Are X and Y independent?

Zero covariance does not imply independence

Let X take on values $\{-1,0,1\}$ with equal probability $1/3$.

Define $Y = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{otherwise} \end{cases}$

		X			
		-1	0	1	
Y	0	1/3	0	1/3	2/3
	1	0	1/3	0	1/3
		1/3	1/3	1/3	

Marginal PMF of Y, $p_Y(y)$

Marginal PMF of X, $p_X(x)$

1. $E[X] = -1\left(\frac{1}{3}\right) + 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) = 0$ $\qquad E[Y] = 0\left(\frac{2}{3}\right) + 1\left(\frac{1}{3}\right) = 1/3$

2. $E[XY] = (-1 \cdot 0)\left(\frac{1}{3}\right) + (0 \cdot 1)\left(\frac{1}{3}\right) + (1 \cdot 0)\left(\frac{1}{3}\right)$
$= 0$

3. $\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$
$= 0 - 0(1/3) = 0$ ⚠ does not imply independence!

4. Are X and Y independent? ❌
$P(Y = 0 | X = 1) = 1$
$\neq \quad P(Y = 0) = 2/3$

Variance of sums of RVs

Statistics of sums of RVs

For any random variables X and Y,

$$E[X + Y] = E[X] + E[Y]$$

$$\text{Var}(X + Y) = \text{Var}(X) + 2 \cdot \text{Cov}(X, Y) + \text{Var}(Y)$$

Variance of general sum of RVs

For any random variables X and Y,

$$\text{Var}(X+Y) = \text{Var}(X) + 2 \cdot \text{Cov}(X,Y) + \text{Var}(Y)$$

Proof:

$$\text{Var}(X+Y) = \text{Cov}(X+Y, X+Y)$$

$\text{Var}(X) = \text{Cov}(X,X)$

$$= \text{Cov}(X,X) + \text{Cov}(X,Y) + \text{Cov}(Y,X) + \text{Cov}(Y,Y)$$

covariance of all pairs

$$= \text{Var}(X) + 2 \cdot \text{Cov}(X,Y) + \text{Var}(Y)$$

Symmetry of covariance + $\text{Cov}(X,X) = \text{Var}(X)$

More generally:

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{i=1}^{n} \sum_{j=\text{i}+1}^{n} \text{Cov}\left(X_i, X_j\right)$$

(proof in extra slides)

Statistics of sums of RVs

For any random variables X and Y,

$$E[X+Y] = E[X] + E[Y]$$

$$\text{Var}(X+Y) = \text{Var}(X) + 2 \cdot \text{Cov}(X,Y) + \text{Var}(Y)$$

For independent X and Y,

$$E[XY] = E[X]E[Y]$$

(Lemma: proof in extra slides)

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Variance of sum of independent RVs

For **independent** X and Y,

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Proof:

1. $\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$ — def. of covariance

$= E[X]E[Y] - E[X]E[Y]$ — X and Y are independent

$= 0$

2. $\text{Var}(X+Y) = \text{Var}(X) + \underbrace{2 \cdot \text{Cov}(X,Y)}_{\text{this is zero when } X \text{ and } Y \text{ are independent}} + \text{Var}(Y)$

$= \text{Var}(X) + \text{Var}(Y)$

NOT bidirectional:
$\text{Cov}(X,Y) = 0$ does NOT imply independence of X and Y!

Proving Variance of the Binomial

$X \sim \text{Bin}(n, p) \quad \text{Var}(X) = np(1-p)$

To simplify the algebra a bit, let $q = 1 - p$, so $p + q = 1$.

So:

$E(X^2) = \sum_{k \geq 0}^{n} k^2 \binom{n}{k} p^k q^{n-k}$	Definition of Binomial Distribution: $p + q = 1$
$= \sum_{k=0}^{n} kn \binom{n-1}{k-1} p^k q^{n-k}$	Factors of Binomial Coefficient: $k\binom{n}{k} = n\binom{n-1}{k-1}$
$= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)}$	Change of limit: term is zero when $k - 1 = 0$
$= np \sum_{j=0}^{m} (j+1) \binom{m}{j} p^j q^{m-j}$	putting $j = k - 1, m = n - 1$
$= np \left(\sum_{j=0}^{m} j \binom{m}{j} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right)$	splitting sum up into two
$= np \left(\sum_{j=0}^{m} m \binom{m-1}{j-1} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right)$	Factors of Binomial Coefficient: $j\binom{m}{j} = m\binom{m-1}{j-1}$
$= np \left((n-1)p \sum_{j=1}^{m} \binom{m-1}{j-1} p^{j-1} q^{(m-1)-(j-1)} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right)$	Change of limit: term is zero when $j - 1 = 0$
$= np((n-1)p(p+q)^{m-1} + (p+q)^m)$	Binomial Theorem
$= np((n-1)p + 1)$	as $p + q = 1$
$= n^2p^2 + np(1-p)$	by algebra

Then:

$$\text{var}(X) = E(X^2) - (E(X))^2$$

$$= np(1-p) + n^2p^2 - (np)^2 \qquad \text{Expectation of Binomial Distribution: } E(X) = np$$

$$= np(1-p)$$

as required.

proofwiki.org

Let's instead prove this using independence and variance!

Proving Variance of the Binomial

$X \sim \text{Bin}(n, p) \quad \text{Var}(X) = np(1-p)$

Let $\quad X = \sum_{i=1}^{n} X_i$

Let X_i = ith trial is heads

$X_i \sim \text{Ber}(p)$

$\text{Var}(X_i) = p(1-p)$

X_i are independent (by definition)

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^{n} X_i\right)$$

$$= \sum_{i=1}^{n} \text{Var}(X_i)$$

yay! :)

X_i are independent, therefore variance of sum = sum of variance

$$= \sum_{i=1}^{n} p(1-p)$$

Variance of Bernoulli

$$= np(1-p)$$

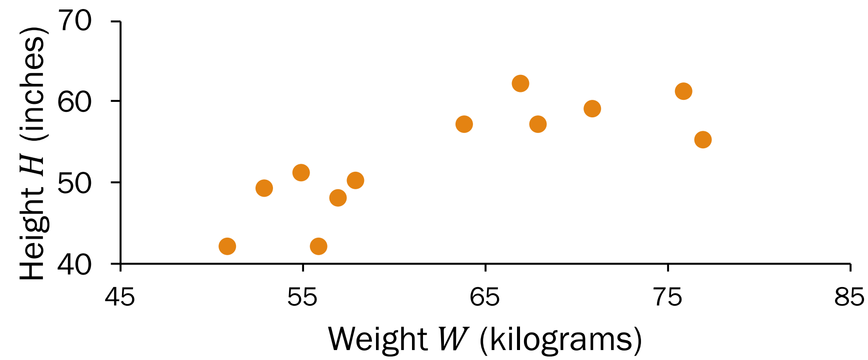
Correlation

Covarying humans

$$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

What is the covariance of weight W and height H?

$$\text{Cov}(W,H) = E[WH] - E[W]E[H]$$
$$= 3355.83 - (62.75)(52.75)$$
$$= 45.77 \quad \text{(positive)}$$

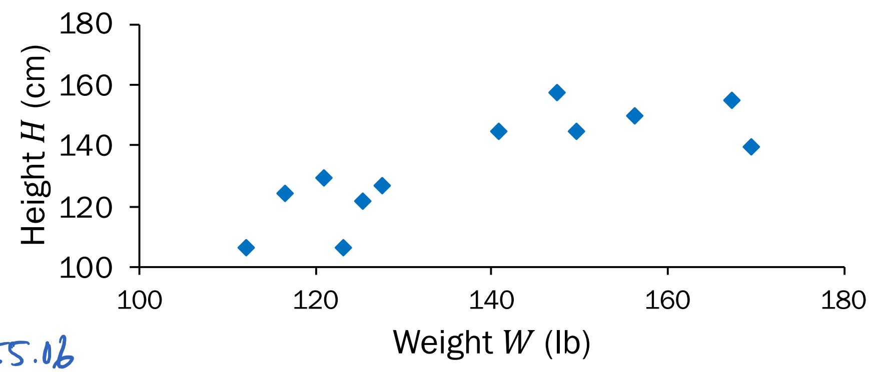


What about weight (lb) and height (cm)?

$$\text{Cov}(2.20W, 2.54H)$$
$$= E[2.20W \cdot 2.54H] - E[2.20W]E[2.54H]$$
$$= 18752.38 - (138.05)(133.99)$$
$$= 255.06 \quad \text{(positive)}$$

$2.20 \cdot 2.54 \cdot 45.77 \approx 255.06$

⚠ Covariance depends on units!



Sign of covariance (+/–) more meaningful than magnitude

Correlation

The **correlation** of two variables X and Y is:

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \, \sigma_Y}$$

$\sigma_X^2 = \text{Var}(X)$,
$\sigma_Y^2 = \text{Var}(Y)$

- Note: $-1 \leq \rho(X, Y) \leq 1$
- Correlation measures the **linear relationship** between X and Y:

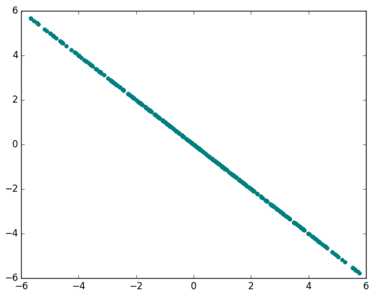
$$\begin{aligned} \rho(X, Y) &= 1 && \Rightarrow Y = aX + b, \text{where } a = \sigma_Y / \sigma_X \\ \rho(X, Y) &= -1 && \Rightarrow Y = aX + b, \text{where } a = -\sigma_Y / \sigma_X \\ \rho(X, Y) &= 0 && \Rightarrow \text{uncorrelated (absence of linear relationship)} \end{aligned}$$

Correlation reps

What is the correlation coefficient $\rho(X,Y)$?

A. $\rho(X,Y) = 1$
B. $\rho(X,Y) = -1$
C. $\rho(X,Y) = 0$
D. Other

1.



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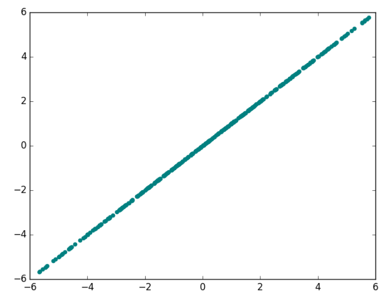
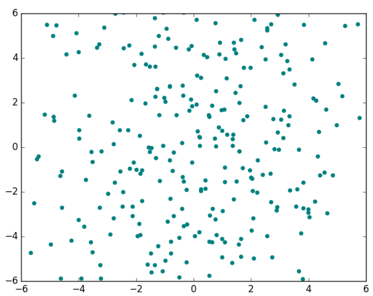
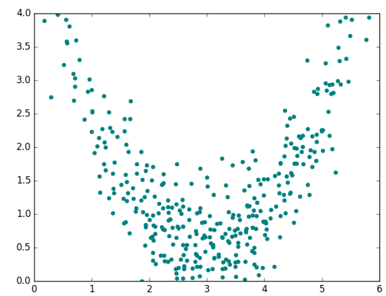


3.



4.

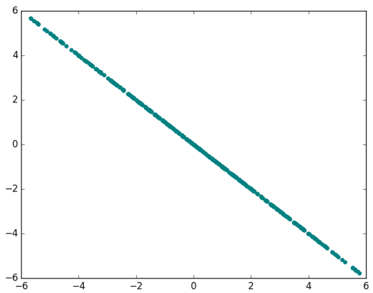


Correlation reps

A. $\rho(X,Y) = 1$
B. $\rho(X,Y) = -1$
C. $\rho(X,Y) = 0$
D. Other

What is the correlation coefficient $\rho(X,Y)$?

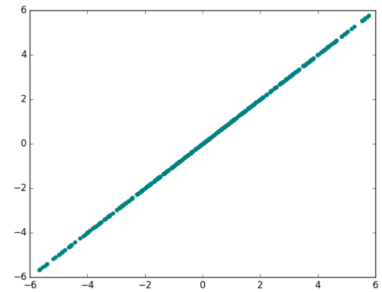
1.



B. $\rho(X,Y) = -1$

$$Y = -aX + b$$
$$a > 0$$

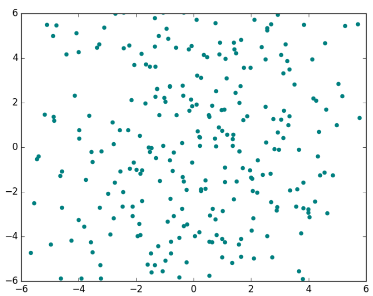
2.



A. $\rho(X,Y) = 1$

$$Y = aX + b$$
$$a > 0$$

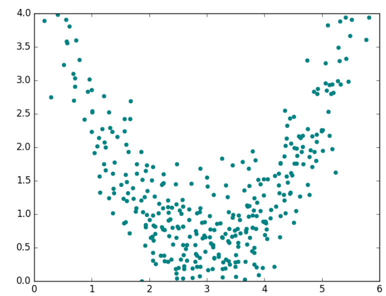
3.



C. $\rho(X,Y) = 0$

"uncorrelated"

4.

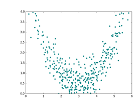


C. $\rho(X,Y) = 0$

$$Y = X^2$$

X and Y can be nonlinearly related even if $\rho(X,Y) = 0$.

Throwback to CS103: Conditional statements

Statement $P \to Q$:	Independence → No correlation	✅
Contrapositive $\neg Q \to \neg P$:	Correlation → Dependence	✅ (logically equivalent)
Inverse $\neg P \to \neg Q$:	Dependence → Correlation	❌ (not always) $Y = X^2$ $\rho(X,Y) = 0$ 
Converse $Q \to P$:	No correlation → Independence	❌ (not always)

"Correlation does not imply causation"

Extras

Expectation of product of independent RVs

If X and Y are independent, then

$$E[XY] = E[X]E[Y]$$
$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Proof: $E[g(X)h(Y)]$

$$= \sum_y \sum_x g(x)h(y)p_{X,Y}(x,y)$$

(for continuous proof, replace summations with integrals)

$$= \sum_y \sum_x g(x)h(y)p_X(x)p_Y(y)$$

X and Y are independent

$$= \sum_y \left(h(y)p_Y(y) \sum_x g(x)p_X(x) \right)$$

Terms dependent on y are constant in integral of x

$$= \left(\sum_x g(x)p_X(x) \right) \left(\sum_y h(y)p_Y(y) \right)$$

Summations separate

$$= E[g(X)]E[h(Y)]$$

Variance of Sums of Variables

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i=1}^{n}\sum_{j=\text{i}+1}^{n} \text{Cov}\left(X_i, X_j\right)$$

Proof:

Var(X) = Cov(X, X)

covariance of all pairs

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \text{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \text{Var}(X_i) + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \text{Cov}\left(X_i, X_j\right)$$

Symmetry of covariance
Cov(X, X) = Var(X)

$$= \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i=1}^{n}\sum_{j=\text{i}+1}^{n} \text{Cov}\left(X_i, X_j\right)$$

Adjust summation bounds