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11: Joint (Multivariate) Distributions

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February 2, 2024

Lecture Discussion on Ed

Normal Approximation

Normal Random Variables

$$X \sim \mathcal{N}(\underbrace{\mu}_{\text{mean}}, \underbrace{\sigma^2}_{\text{variance}})$$

- Used to model many real-life situations because it maximizes entropy (i.e., randomness) for a given mean and variance.

colloquially, this means it commits to the least amount of structure for the given mean and variance

- Also useful for approximating the Binomial random variable!

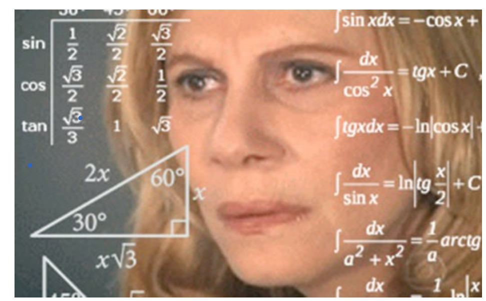
focus on this for a few slides!

Website testing

- 100 people are presented with a new website design.
- $X = \#$ people whose time on site increases
- PM assumes design has no effect, so assume P(stickier) = 0.5 independently.
- CEO will endorse the new design if $X \geq 65$.

this is technically an a prior belief about which design is better.

What is $P(\text{CEO endorses change})$? *Give a numerical approximation*.



Approach 1: Binomial

Define

$$X \sim \text{Bin}(n = 100, p = 0.5)$$

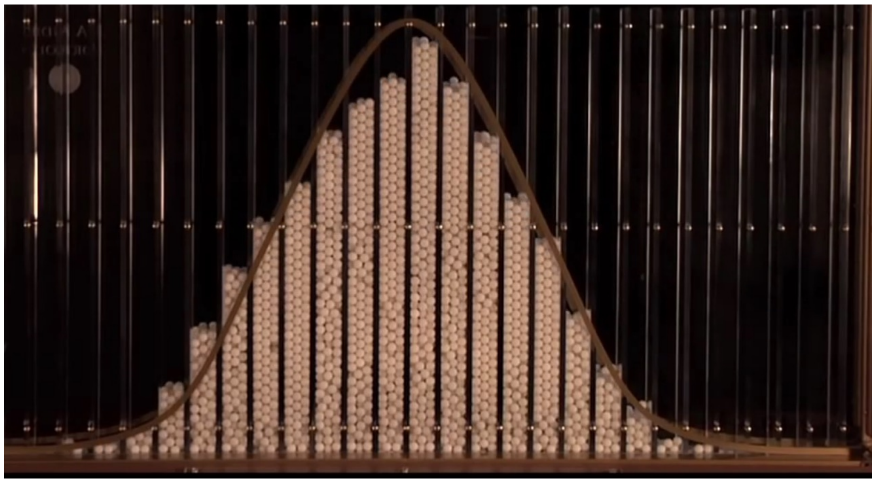
Want: $P(X \geq 65)$

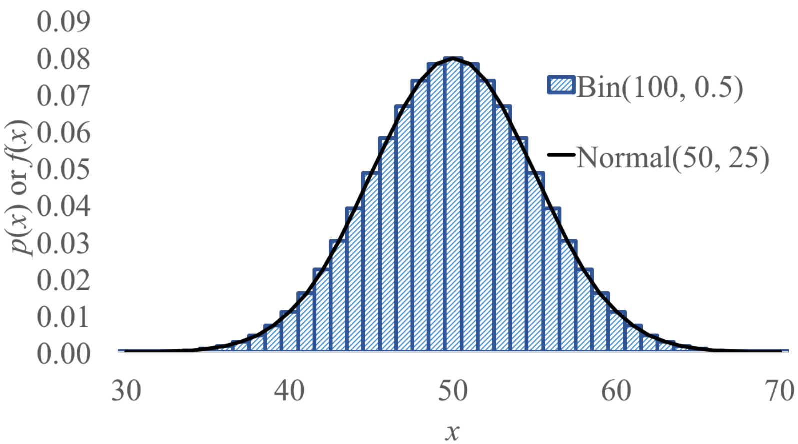
Solve

$$P(X \geq 65) = \sum_{k=65}^{100} \binom{100}{k} 0.5^k (1 - 0.5)^{100-k}$$

0.5^{100}

Don't worry, Normal approximates Binomial





Galton Board

(We'll explain **why** in 2 weeks)

Website testing

```
jerry$ python
>>> from scipy.stats import binom, norm
>>> binom.pmf(range(65, 101), n, p).sum()
0.0017588208614850058
>>> 1 - norm(50, 5).cdf(65)
0.0013498980316301035
```

- 100 people are given a new website design.
- $X = \#$ people whose time on site increases
- PM assumes design has no effect, so P(stickier) = 0.5 independently.
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What is $P(\text{CEO endorses change})$? *Give a numerical approximation*.

Approach 1: Binomial

Define

$X \sim \text{Bin}(n = 100, p = 0.5)$

Want: $P(X \geq 65)$

Solve

$P(X \geq 65) \approx 0.0018$

computed directly

⚠️⚠️ (this approach is missing something important)

Approach 2: approximate with Normal

Define

$Y \sim \mathcal{N}(\mu, \sigma^2)$

$\mu = np = 50$

$\sigma^2 = np(1-p) = 25$

$\sigma = \sqrt{25} = 5$

derived from Binomial parameters.

Solve

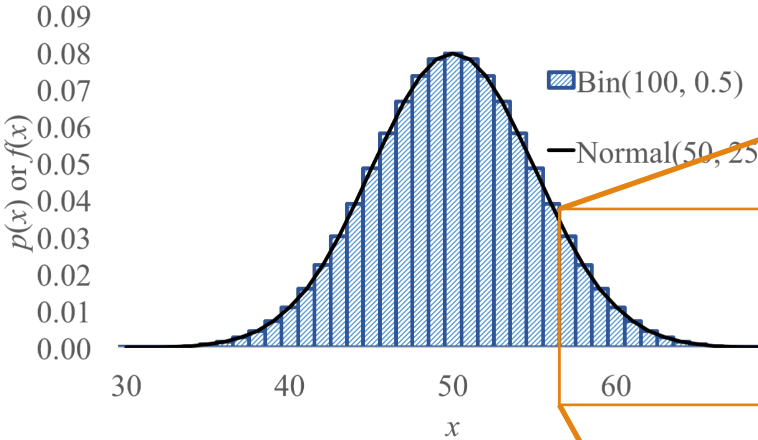
$P(X \geq 65) \approx P(Y \geq 65) = 1 - F_Y(65)$ 🤨

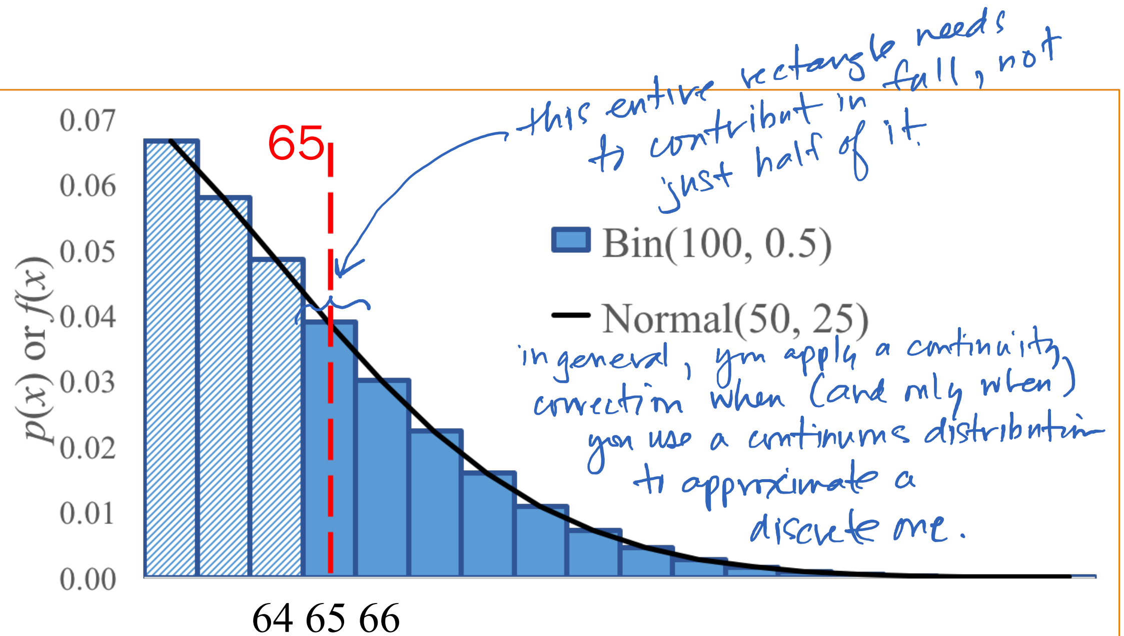
$= 1 - \Phi\left(\frac{65-50}{5}\right) = 1 - \Phi(3) \approx 0.0013$?

not terribly good! what's up?

Website testing (with continuity correction)

In our website testing, $Y \sim \mathcal{N}(50, 25)$ approximates $X \sim \text{Bin}(100,0.5)$.





$P(X \geq 65)$ Binomial

$\approx P(Y \geq 64.5)$ Normal

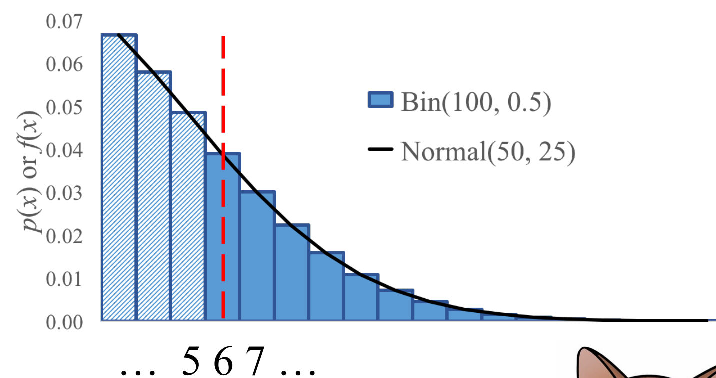
≈ 0.0018 ✅ the better Approach 2

You must perform a **continuity correction** when approximating a Binomial RV with a Normal RV.

Continuity correction

If $Y \sim \mathcal{N}(np, np(1-p))$ approximates $X \sim \text{Bin}(n, p)$, how do we approximate the following probabilities?

Discrete (e.g., Binomial) probability question	Continuous (Normal) probability question
$P(X = 6)$	
$P(X \geq 6)$	
$P(X > 6)$	
$P(X < 6)$	
$P(X \leq 6)$	



Continuity correction

If $Y \sim \mathcal{N}(np, np(1-p))$ approximates $X \sim \text{Bin}(n, p)$, how do we approximate the following probabilities?

those underlined in green should include $X=6$!

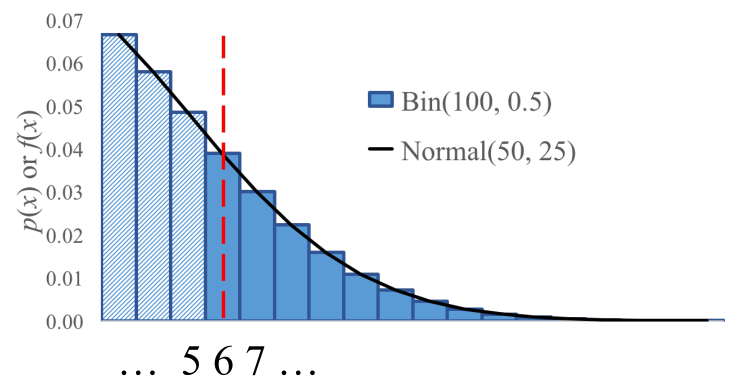
those underlined in red want nothing to do with $X=6$, so be sure to exclude its contribution when approximating.

Discrete (e.g., Binomial) probability question	Continuous (Normal) probability question
$P(X = 6)$	$P(5.5 \leq Y \leq 6.5)$
$P(X \geq 6)$	$P(Y \geq 5.5)$
$P(X > 6)$	$P(Y \geq 6.5)$
$P(X < 6)$	$P(Y \leq 5.5)$
$P(X \leq 6)$	$P(Y \leq 6.5)$

means ≥ 7 → $P(X > 6)$

≤ 5 ← $P(X < 6)$

helpful to frame boundaries in terms of $\leq$ and $\geq$, avoid $<$ and $>$ for consistency.



Who gets to approximate?

$$X \sim \text{Bin}(n, p)$$
$$E[X] = np$$
$$\text{Var}(X) = np(1-p)$$

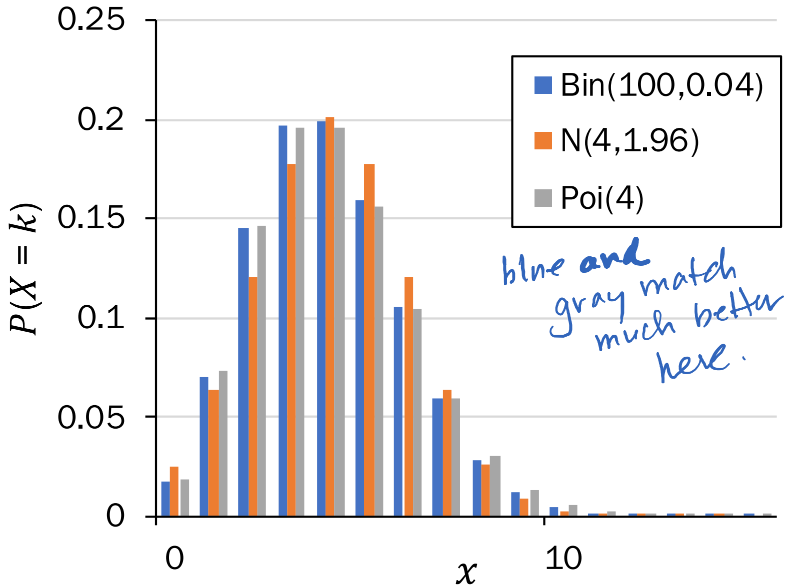
$$Y \sim \text{Poi}(\lambda)$$
$$\lambda = np$$

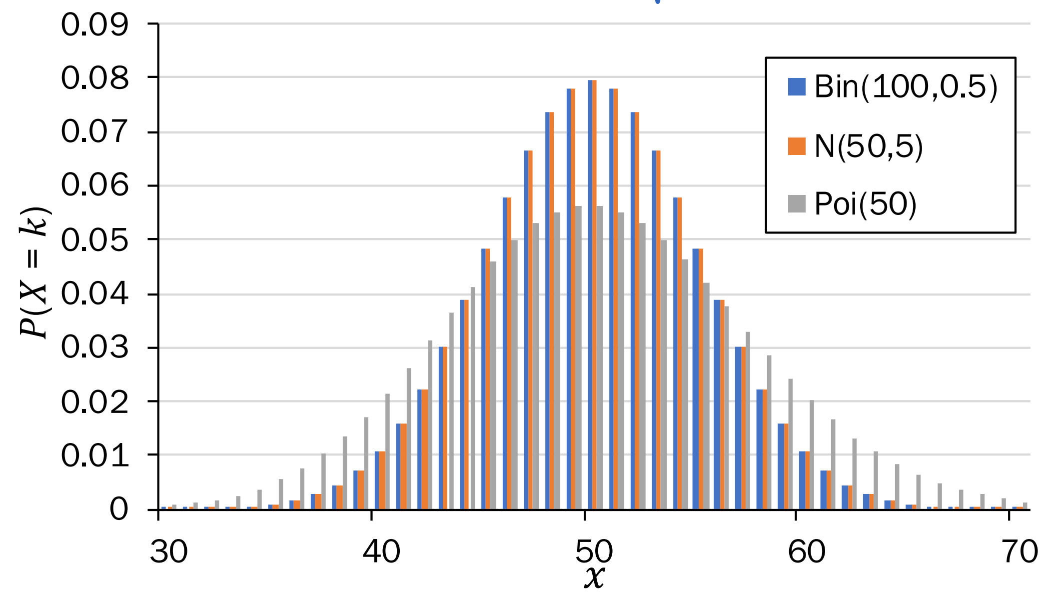
?

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$
$$\mu = np$$
$$\sigma^2 = np(1-p)$$

Who gets to approximate?

blue and orange pair up better here.





Poisson approximation
n large (> 20), p small (< 0.05)
slight dependence okay

Normal approximation
n large (> 20), p mid-ranged ($np(1-p) > 10$)
independence

1. If there is a choice, use Gaussian to approximate.
2. When using Normal to approximate a discrete RV, use a continuity correction.

Stanford Admissions (a while back)

Stanford accepts 2480 students.

- Each admitted student matriculates with p = 0.68 (independently)
- Let X = # of students who will attend

What is $P(X > 1745)$? *Give a numerical approximation*.

Strategy:

A. Just Binomial
B. Poisson
C. Normal
D. None/other

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- Each admitted student matriculates with p = 0.68 (independently)
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What is $P(X > 1745)$? *Give a numerical approximation.*

↙ still would generate the correct answer.

Strategy:

A. Just Binomial — computationally expensive (also not an approximation)
B. Poisson — $p = 0.68$, not small enough
C. Normal — ✅ Variance $np(1-p) = 540 > 10$
D. None/other

Define an approximation

Let $Y \sim \mathcal{N}\left(E[X], \text{Var}(X)\right)$

$E[X] = np = 1686$

$\text{Var}(X) = np(1-p) \approx 540 \rightarrow \sigma = 23.3$

$P(X > 1745) \approx P(Y \geq 1745.5)$ ⚠️ Continuity correction

Solve

$$P(Y \geq 1745.5) = 1 - F(1745.5)$$

$$= 1 - \Phi\left(\frac{1745.5 - 1686}{23.3}\right)$$

$$= 1 - \Phi(2.54) \approx 0.0055$$

Discrete Joint RVs

From last slide deck

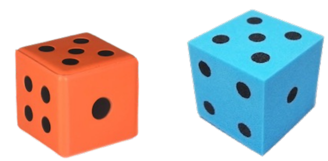
What is the probability that the Warriors win?
How do you model zero-sum games?

$$P(A_W > A_B)$$

This is a probability of an event involving **two** random variables!

Joint probability mass functions

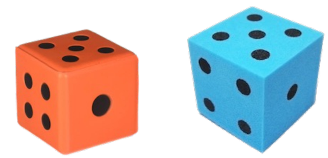
Roll two 6-sided dice, yielding values X and Y.



X	$P(X = 1)$	$P(X = k)$
random variable	probability of an event	probability mass function

Joint probability mass functions

Roll two 6-sided dice, yielding values X and Y.



X random variable	$P(X=1)$ probability of an event	$P(X=k)$ probability mass function
X, Y random variables	$P(X=1 \cap Y=6)$ $P(X=1, Y=6)$ new notation: the comma probability of the intersection of two events	$P(X=a, Y=b)$ joint probability mass function

Discrete joint distributions

For two discrete joint random variables X and Y,
the joint probability mass function is defined as:

$$p_{X,Y}(a,b) = P(X = a, Y = b)$$

The marginal distributions of the joint PMF are defined as:

$$p_X(a) = P(X = a) = \sum_y p_{X,Y}(a,y)$$

$$p_Y(b) = P(Y = b) = \sum_x p_{X,Y}(x,b)$$

Use marginal distributions to extract a 1D RV from a joint PMF.

Two dice

Roll two 6-sided dice, yielding values X and Y.

1. What is the joint PMF of X and Y?

$$p_{X,Y}(a,b) = 1/36 \qquad (a,b) \in \{(1,1), \ldots, (6,6)\}$$

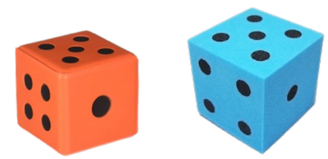
		X					
		1	2	3	4	5	6
Y	1	1/36	…	…	…	…	1/36
	2	…	…	…	…	…	…
	3	…	…	…	…	…	…
	4	…	…	…	…	…	…
	5	…	…	…	…	…	…
	6	1/36	…	…	…	…	1/36

$P(X = 4, Y = 3)$

Probability table
- All possible outcomes for several discrete RVs
- Not parametric (e.g., parameter p in Ber(p))

Two dice

Roll two 6-sided dice, yielding values X and Y.



1. What is the joint PMF of X and Y?

$$p_{X,Y}(a,b) = 1/36 \qquad (a,b) \in \{(1,1), \ldots, (6,6)\}$$

2. What is the marginal PMF of X?

X is constrained to be a everywhere, whereas Y varies over its full support.

$$p_X(a) = P(X = a) = \sum_y p_{X,Y}(a,y) = \sum_{y=1}^{6} \frac{1}{36} = \frac{1}{6} \qquad a \in \{1, \ldots, 6\}$$

A computer (or three) in every house.

Consider households in Silicon Valley.

- A household has X Macs and Y PCs.
- Each house has a maximum of 3 computers total (Macs + PCs).

1. What is $P(X = 1, Y = 0)$, the missing entry in the probability table?

Y (# PCs) \ X (# Macs)	0	1	2	3
0	.16	?	.07	.04
1	.12	.14	.12	0
2	.07	.12	0	0
3	.04	0	0	0

the symmetry above the diagonal here is a coincidence, not a requirement.

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3	.04	0	0	0

A joint PMF must sum to 1:

$$\sum_x \sum_y p_{X,Y}(x, y) = 1$$

A computer (or three) in every house.

Consider households in Silicon Valley.

- A household has X Macs and Y PCs.
- Each house has a maximum of 3 computers total (Macs + PCs).

2. How do you compute the marginal PMF of X?

Y (# PCs) \ X (# Macs)	0	1	2	3	sum cols here
0 A	.16	.12	.07	.04	.39
1	.12	.14	.12	0	.38
2	.07	.12	0	0	.19
3	.04	0	0	0	.04
B (sum rows here)	.39	.38	.19	.04	

C

A computer (or three) in every house.

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0 A	.16	.12	.07	.04	.39
1	.12	.14	.12	0	.38
2	.07	.12	0	0	.19
3	.04	0	0	0	.04
B (sum rows here)	.39	.38	.19	.04	

C

A. $p_{X,Y}(x, 0) = P(X = x, Y = 0)$

B. Marginal PMF of X $\quad p_X(x) = \sum_y p_{X,Y}(x, y)$

C. Marginal PMF of Y $\quad p_Y(y) = \sum_x p_{X,Y}(x, y)$

To find a marginal distribution over one variable, sum over all other variables in the joint PMF.

A computer (or three) in every house.

Consider households in Silicon Valley.

- A household has X Macs and Y PCs.
- Each house has a maximum of 3 computers total (Macs + PCs).

3. Let $C = X + Y$. What is $P(C = 3)$?

Y (# PCs) \ X (# Macs)	0	1	2	3
0	.16	.12	.07	.04
1	.12	.14	.12	0
2	.07	.12	0	0
3	.04	0	0	0

A computer (or three) in every house.

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- A household has X Macs and Y PCs.
- Each house has a maximum of 3 computers total (Macs + PCs).

3. Let $C = X + Y$. What is $P(C = 3)$?

Y (# PCs) \ X (# Macs)	0	1	2	3
0	.16	.12	.07	.04
1	.12	.14	.12	0
2	.07	.12	0	0
3	.04	0	0	0

$$P(C = 3) = P(X + Y = 3)$$

Law of Total Probability

$$= \sum_x \sum_y P(X + Y = 3 | X = x, Y = y) P(X = x, Y = y)$$

these probabilities are 0 when $x+y \neq 3$, 1 when $x+y=3$

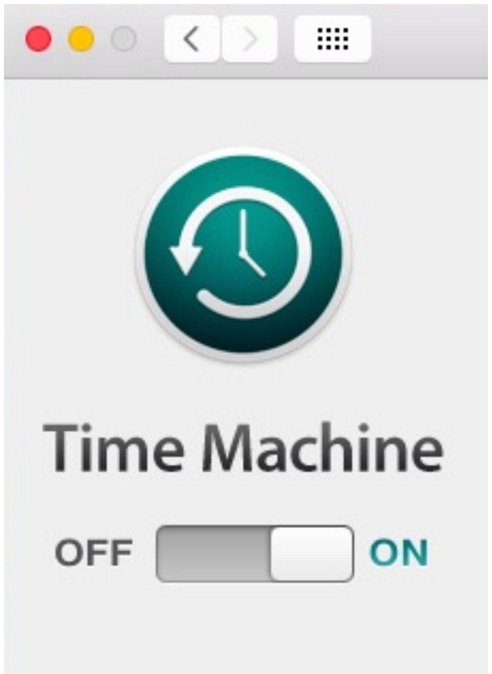
$$= P(X = 0, Y = 3) + P(X = 1, Y = 2) + P(X = 2, Y = 1) + P(X = 3, Y = 0) = 0.32$$

this is known as a convolution, and we'll go in depth next lecture.

We'll come back to sums of RVs next lecture!

Multinomial RV

Recall the good times



Permutations

$n!$

How many ways are there to order n objects?

Counting unordered objects

Binomial coefficient

How many ways are there to group n objects into two groups of size k and $n-k$, respectively?

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Called the binomial coefficient because of something from aLgEbRa

Multinomial coefficient

How many ways are there to group n objects into r groups of sizes n_1, n_2, …, n_r respectively?

assume, $n_1 + n_2 + \cdots + n_r = n$

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Multinomials generalize Binomials for counting.

Probability

Binomial RV

What is the probability of getting k successes and $n-k$ failures in n trials?

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial # of ways of ordering the successes

Probability of each ordering of k successes is equal + mutually exclusive

Multinomial RV

What is the probability of getting c_1 of outcome 1, c_2 of outcome 2, ..., and c_m of outcome m in n trials?

Multinomial RVs also generalize Binomial RVs for probability!

Multinomial Random Variable

Consider an experiment of n independent trials:

- Each trial results in one of m outcomes. $P(\text{outcome } i) = p_i$, $\sum_{i=1}^{m} p_i = 1$
- Let X_i= # trials with outcome i

Joint PMF

$$P(X_1 = c_1, X_2 = c_2, \dots, X_m = c_m) = \binom{n}{c_1, c_2, \dots, c_m} p_1^{c_1} p_2^{c_2} \cdots p_m^{c_m}$$

where $\sum_{i=1}^{m} c_i = n$ and $\sum_{i=1}^{m} p_i = 1$

generalization of Binomial.

Multinomial # of ways of ordering the outcomes

Probability of each ordering is equal + mutually exclusive

Hello dice rolls, my old friends

A fair, six-sided die is rolled 7 times.

What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

Hello dice rolls, my old friends

A fair, six-sided die is rolled 7 times.

What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3)$$

$$= \binom{7}{1,1,0,2,0,3} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^3 = 420 \left(\frac{1}{6}\right)^7$$

Hello dice rolls, my old friends

A fair, six-sided die is rolled 7 times.

What is the probability of getting:

- 1 one
- 1 two
- 0 threes
- 2 fours
- 0 fives
- 3 sixes

# of times a six appears

$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3)$$

$$= \binom{7}{1,1,0,2,0,3} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^3 = 420 \left(\frac{1}{6}\right)^7$$

choose where the sixes appear

probability of rolling a six

this many times

Probabilistic text analysis

Ignoring the order of words...

What is the probability of any given word that you write in English?

- $P(\text{word} = \text{"the"}) > P(\text{word} = \text{"susurration"})$
- $P(\text{word} = \text{"Stanford"}) > P(\text{word} = \text{"Cal"})$

Probabilities of **counts** of words = Multinomial distribution 👉

A document is a large multinomial.

(according to the Global Language Monitor, there are 988,968 words in the English language used on the internet.)

Probabilistic text analysis

Probabilities of **counts** of words = multinomial distribution

Example document: #words: $n = 48$

"When my late husband was alive he deposited some amount of Money with overseas Bank in which the amount will be declared to you once you respond to this message indicating your interest in helping to receive the fund and use it for Heavens work as my wish."

$$P\left(\begin{array}{l}\text{bank} = 1\\ \text{fund} = 1\\ \text{money} = 1\\ \text{wish} = 1\\ \dots\\ \text{to} = 3\end{array}\middle|\,\text{spam}\right) = \frac{48!}{1!\,1!\,1!\,1!\cdots 3!}\,p_{\text{bank}}^{1}\,p_{\text{fund}}^{1}\cdots p_{\text{to}}^{3}$$

sum of exponents is also 48

these sum to 48

not concerned with order, just frequency

Note: $P\left(\text{bank}\,\middle|\,\begin{array}{c}\text{spam}\\ \text{writer}\end{array}\right) \gg P\left(\text{bank}\,\middle|\,\begin{array}{c}\text{writer=}\\ \text{you}\end{array}\right)$

Old and New Analysis

Authorship of the Federalist Papers

- 85 essays advocating ratification of the US constitution
- Written under the pseudonym "Publius" (really, Alexander Hamilton, James Madison, John Jay)

Who wrote which essays?

- Analyze probability of words in each essay and compare against word distributions from known writings of three authors
- Curious what the analysis is? Read [this](#)!

Statistics of Two RVs

Expectation and Covariance

In real life, we often have many RVs interacting at once.

- We've seen some simpler cases (e.g., sum of independent Bernoullis).
- Come Monday, we'll discuss sums of Binomials, Poissons, etc.
- In general, manipulating joint PMFs is difficult.
- Fortunately, you don't need to model joint RVs completely all the time.

Instead, we'll focus next on reporting **statistics** of multiple RVs:

- Expectation of sums (you've seen some of this, more on Monday)
- Covariance: measure of how two random variable vary with each other (more next Wednesday and Friday)

Properties of Expectation, extended to two RVs

1. Linearity:

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

2. Expectation of a sum = sum of expectation:

$$E[X + Y] = E[X] + E[Y]$$

(we've seen this; we'll prove today!)

3. Unconscious statistician:

$$E[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$$

True for both independent and dependent random variables!

Proof of expectation of a sum of RVs

$E[X+Y] = E[X] + E[Y]$

$$E[X+Y] = \sum_x \sum_y (x+y) p_{X,Y}(x,y)$$

LOTUS,
$g(X,Y) = X + Y$

$$= \sum_x \sum_y x p_{X,Y}(x,y) + \sum_x \sum_y y p_{X,Y}(x,y)$$

switch :)

$$= \sum_x x \sum_y p_{X,Y}(x,y) + \sum_y y \sum_x p_{X,Y}(x,y)$$

Linearity of summations
(cont. case: linearity of integrals)

$$= \sum_x x p_X(x) + \sum_y y p_Y(y)$$

Marginal PMFs for X and Y

$$= E[X] + E[Y]$$