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05: Independence

Jerry Cain

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Lecture Discussion on Ed

Independence I

Independence

Two events E and F are defined as independent if:

$$P(EF) = P(E)P(F)$$

Otherwise E and F are called dependent events.

If E and F are independent, then:

$$P(E|F) = P(E)$$

Intuition through proof

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$

Statement:

If E and F are independent, then $P(E|F) = P(E)$.

Proof:

$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of conditional probability

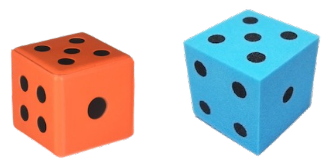
$$= \frac{P(E)\cancel{P(F)}}{\cancel{P(F)}}$$ Independence of E and F

$$= P(E)$$ Taking the bus to cancellation city

Knowing that F happened does not change our belief that E happened.

Dice, our misunderstood friends

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$, $P(E|F) = P(E)$



- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 5$

$G = \{(1,4), (2,3), (3,2), (4,1)\}$

$|G| = 4$

1. Are E and F independent?

$EF = E \cap F = \{(1,6)\}$

$P(E) = 1/6$

$P(F) = 1/6$ $\quad P(E)P(F) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

$P(EF) = 1/36$ $\quad P(EF) = \frac{1}{36}$

✅ <u>independent</u>

2. Are E and G independent?

$EG = E \cap G = \{(1,4)\}$

$P(E) = 1/6$

$P(G) = 4/36 = 1/9$

$P(EG) = 1/36 \neq P(E)P(G)$

$\frac{1}{6} \cdot \frac{1}{9} = \frac{1}{54}$

❌ <u>dependent</u>

Generalizing independence

Three events E, F, and G are independent if:

$P(EFG) = P(E)P(F)P(G)$, and
$P(EF) = P(E)P(F)$, and
$P(EG) = P(E)P(G)$, and
$P(FG) = P(F)P(G)$

we need pairwise independence

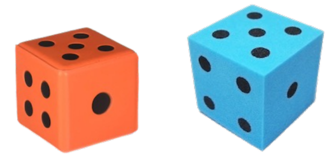
n events $E_1, E_2, \dots, E_n$ are independent if:

for $r = 1, \dots, n$:
for every subset $E_1, E_2, \dots, E_r$:
$P(E_1 E_2 \dots E_r) = P(E_1)P(E_2) \cdots P(E_r)$

- we need pairwise independence
- we need trio-wise independence
- we need quartet-wise independence

etc.

Dice, increasingly misunderstood (still our friends)



- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 7$

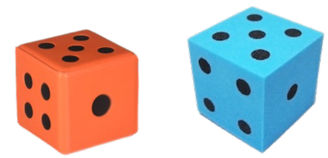
$G = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

1. Are E and F independent? ✅
2. Are E and G independent?
3. Are F and G independent?
4. Are E, F, G independent?

EF is still $\{(1,6)\}$

$P(EF) = 1/36$

Dice, increasingly misunderstood (still our friends)



- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 7$

$G = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

1. Are E and F ✅ independent?

$P(EF) = 1/36$

2. Are E and G ✅ independent?

$P(EG) = \frac{1}{36}$

$P(E)P(G) = \frac{1}{6} \cdot \frac{1}{6}$

3. Are F and G ✅ independent?

$P(FG) = \frac{1}{36}$

$P(F)F(G) = \frac{1}{6} \cdot \frac{1}{6}$

4. Are E, F, G ❌ independent?

$P(EFG) = \frac{1}{36}$

$P(E)P(F)P(G) = (\frac{1}{6})^3 \neq \frac{1}{36}$

note that $EG = \{(1,6)\}$, $FG = \{(1,6)\}$, $EFG = \{(1,6)\}$

Pairwise independence is not sufficient to prove independence of >2 events!

Independence II

Independent trials

We often are interested in experiments consisting of n **independent trials**.

- n trials, each with the same set of possible outcomes
- n-way independence: an event in one subset of trials is independent of events in other subsets of trials

Examples:

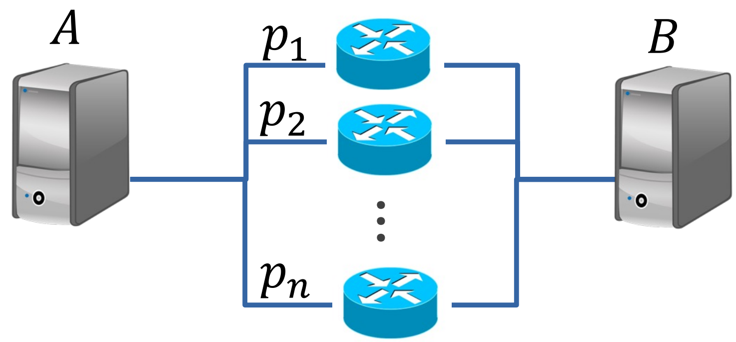
- Flip a coin n times
- Roll a die n times
- Send a multiple-choice survey to n people
- Send n web requests to k different servers

Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- E = functional path from A to B exists.

What is $P(E)$?



Network reliability

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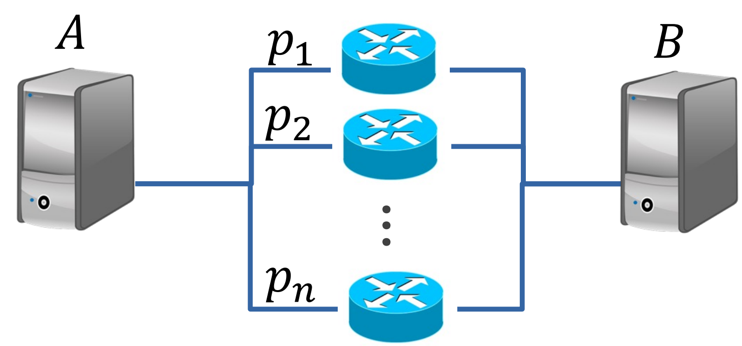


router i functions with probability p_i

router i fails us with probability $(1-p_i)$

$$
\begin{aligned}
P(E) &= P(\geq 1 \text{ one router works}) \\
&= 1 - P(\text{all routers fail}) = 1 - P(\text{router 1 fails } \underline{\text{AND}} \text{ router 2 fails } \underline{\text{AND}} \ldots) \\
&= 1 - (1-p_1)(1-p_2)\cdots(1-p_n) \\
&= 1 - \prod_{i=1}^{n}(1-p_i)
\end{aligned}
$$

≥ 1 with independent trials: take complement

Exercises

Independence?

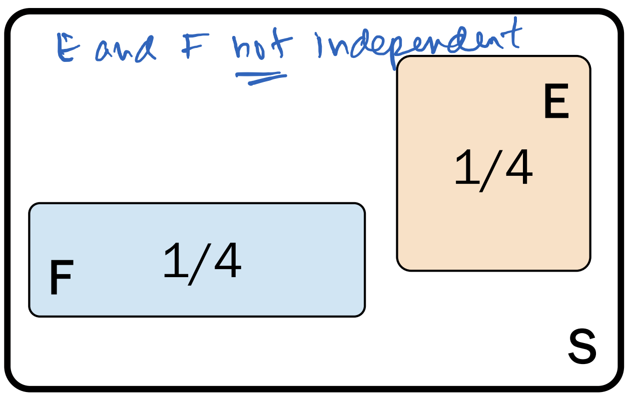
Independent events E and F ⟺ $P(EF) = P(E)P(F)$, $P(E|F) = P(E)$

assume $P(E), P(F) > 0$

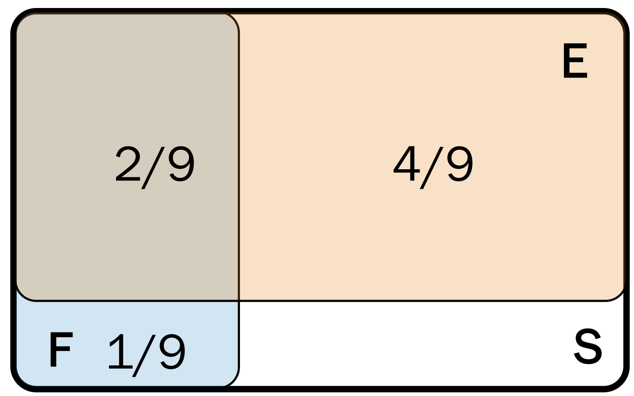
1. True or False? Two events E and F are independent if:
   A. no Knowing that F happens means that E can't happen. $P(E|F) = 0 \neq P(E)$
   B. yes Knowing that F happens doesn't change probability that E happened. $P(E|F) = P(E)$ definition of independence

2. Are E and F independent in the following pictures?

A.



B.



E and F are independent!

$EF = \emptyset$

$P(EF) = 0$

$P(E) = 1/4$

$P(F) = 1/4$

product of the two $= 1/16 \neq 0$

$P(E) = \frac{2}{9} + \frac{4}{9} = \frac{6}{9} = \frac{2}{3}$

$P(F) = \frac{2}{9} + \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$

$P(EF) = 2/9$

$P(E) \cdot P(F) = \frac{2}{3} \cdot \frac{1}{3} = 2/9$

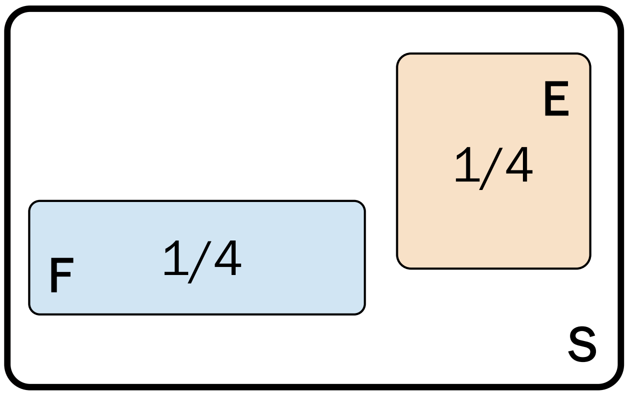
Independence?

Independent events E and F ⟺ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

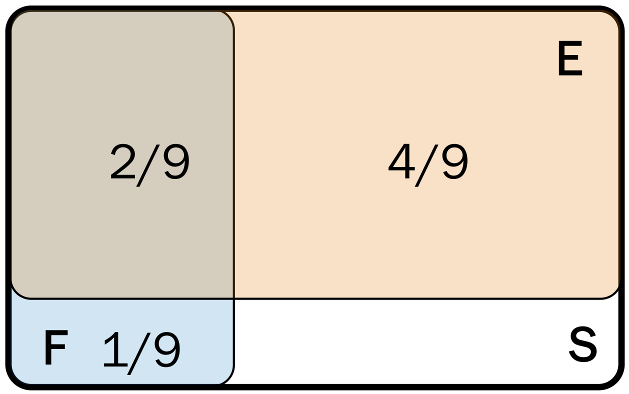
1. True or False? Two events E and F are independent if:
   A. Knowing that F happens means that E can't happen.
   B. Knowing that F happens doesn't change probability that E happened.

2. Are E and F independent in the following pictures?

A.



B.



Be careful:
- Independence is NOT mutual exclusion.
- Independence is difficult to visualize graphically.

Independence

Two events E and F are defined as independent if:

$$P(EF) = P(E)P(F)$$

For independent events E and F,

- $P(E|F) = P(E)$
- E and F^C are independent.

new

Independence of complements

Statement:

If E and F are independent, then E and F^C are independent.

Proof:

$$\begin{aligned} P(EF^C) &= P(E) - P(EF) && \text{Intersection} \\ &= P(E) - P(E)P(F) && \text{Independence of } E \text{ and } F \\ &= P(E)[1 - P(F)] && \text{Factoring} \\ &= P(E)P(F^C) && \text{Complement} \end{aligned}$$

E and F^C are independent — Definition of independence

mathematically: $P(E|F^C) = P(E)$

Knowing that F **did or didn't happen** does not change our belief that E happened.

(biased) Coin Flips

Suppose we flip a coin n times. Each coin flip is an **independent trial** with probability p of coming up heads. Write an expression for the following:

1. $P(n$ heads on n coin flips)
2. $P(n$ tails on n coin flips)
3. $P($first k heads, then $n - k$ tails)
4. $P($exactly k heads on n coin flips)

(biased) Coin Flips

Suppose we flip a coin n times. Each coin flip is an **independent trial** with probability p of coming up heads. Write an expression for the following:

1. $P(n$ heads on n coin flips$)$
2. $P(n$ tails on n coin flips$)$
3. $P($first k heads, then $n-k$ tails$)$
4. $P($exactly k heads on n coin flips$)$

n consecutive heads $\underset{p}{H}\underset{p}{H}\underset{p}{H}\underset{p}{H}\cdots\underset{p}{H} \Rightarrow \boxed{p^n}$

n consecutive tails $\underset{1-p}{T}\underset{1-p}{T}\underset{1-p}{T}\cdots\underset{1-p}{T} \Rightarrow (1-p)^n = \boxed{q^n}$ where $q = 1-p$

$\underbrace{HH\cdots H}_{k}\ \underbrace{TTT\cdots T}_{n-k} \Rightarrow p^k(1-p)^{n-k}$ $\boxed{= p^k q^{n-k}}$

any single sequence of n flips with k heads somewhere is $p^k(1-p)^{n-k}$

there are $\binom{n}{k}$ such sequences

total probability is $\boxed{\binom{n}{k} p^k (1-p)^{n-k}}$

$$\binom{n}{k} \quad p^k(1-p)^{n-k}$$

\# of mutually exclusive outcomes

$P($a particular outcome's k heads on n coin flips$)$

Make sure you understand #4! It will come up again.

Probability of events

E or F
$P(E \cup F)$

Mutually exclusive?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

E and F
$P(EF)$

Independent?

Just multiply!

Chain Rule

Probability of events

E or F

$P(E \cup F)$

Mutually exclusive?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

E and F

$P(EF)$

Independent?

Just multiply!

$P(E)P(F)$

Chain Rule

$P(E)P(F|E)$

or

$P(F)P(E|F)$

Probability of events

E or F
$P(E \cup F)$

De Morgan's

E and F
$P(EF)$

Mutually exclusive?

Just add!

Inclusion-Exclusion Principle

Independent?

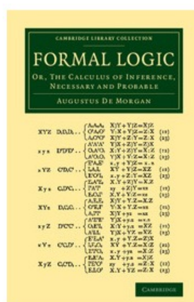
Just multiply!

Chain Rule

Augustus De Morgan

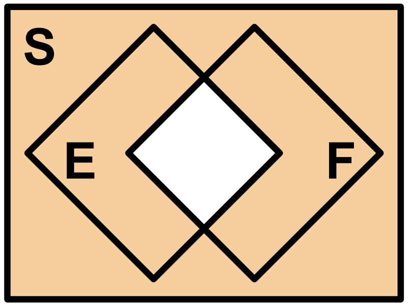
Augustus De Morgan (1806–1871):

British mathematician who wrote the book *Formal Logic* (1847).



De Morgan's Laws

De Morgan's lets you switch between AND and OR.



$$(E \cap F)^C = E^C \cup F^C$$

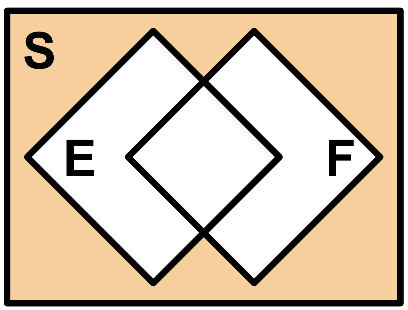
$$\left(\bigcap_{i=1}^{n} E_i\right)^C = \bigcup_{i=1}^{n} E_i^C$$

when $n=4$, $(E_1 \cap E_2 \cap E_3 \cap E_4)^c = E_1^c \cup E_2^c \cup E_3^c \cup E_4^c$

In probability:

$$\begin{aligned} P(E_1 E_2 \cdots E_n) &= 1 - P\big((E_1 E_2 \cdots E_n)^C\big) \\ &= 1 - P(E_1^C \cup E_2^c \cup \cdots \cup E_n^c) \end{aligned}$$

Great if E_i^C mutually exclusive!



$$(E \cup F)^C = E^C \cap F^C$$

$$\left(\bigcup_{i=1}^{n} E_i\right)^C = \bigcap_{i=1}^{n} E_i^C$$

$n=4$?

$(E_1 \cup E_2 \cup E_3 \cup E_4)^c = E_1^c \cap E_2^c \cap E_3^c \cap E_4^C$

In probability:

$$\begin{aligned} P(E_1 \cup E_2 \cup \cdots \cup E_n) &= 1 - P\big((E_1 \cup E_2 \cup \cdots \cup E_n)^C\big) \\ &= 1 - P(E_1^c E_2^c \cdots E_n^c) \end{aligned}$$

Great if E_i independent!

Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets. $\sum_{i=1}^{n} p_i = 1$
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

for each string s

$p_1 \quad p_2 \quad p_3 \quad p_4 \quad \ldots \quad p_n$

$1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad n$

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

Define S_i = string i is hashed into bucket 1
S_i^C = string i is not hashed into bucket 1

$$P(S_i) = p_1$$
$$P(S_i^C) = 1 - p_1$$

Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

Define S_i = string i is hashed into bucket 1
S_i^C = string i is not hashed into bucket 1

$P(S_i) = p_1$
$P(S_i^C) = 1 - p_1$

WTF (not-real acronym for Want To Find):

$$P(E) = P(S_1 \cup S_2 \cup \cdots \cup S_m)$$

$$= 1 - P\left((S_1 \cup S_2 \cup \cdots \cup S_m)^C\right) \quad \text{Complement}$$

$$= 1 - P\left(S_1^C S_2^C \cdots S_m^C\right) \quad \text{De Morgan's Law}$$

$$= 1 - P\left(S_1^C\right)P\left(S_2^C\right) \cdots P\left(S_m^C\right) = 1 - \left(P\left(S_1^C\right)\right)^m \quad S_i \text{ independent trials}$$

$$= 1 - (1 - p_1)^m$$

More hash table fun: Possible approach?

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

Define F_i = bucket i has at least one string in it

$$
\begin{aligned}
P(E) &= P(F_1 \cup F_2 \cup \cdots \cup F_k) \\
&= 1 - P\left((F_1 \cup F_2 \cup \cdots \cup F_k)^C\right) \\
&= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right) \\
? &= 1 - P\left(F_1^C\right) P\left(F_2^C\right) \cdots P\left(F_k^C\right)
\end{aligned}
$$

⚠ F_i bucket events are *dependent*!

So we cannot approach with complement.

More hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

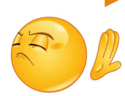
Define F_i = bucket i has at least one string in it

$$P(E) = P(F_1 \cup F_2 \cup \cdots \cup F_k)$$

$$= 1 - P\left((F_1 \cup F_2 \cup \cdots \cup F_k)^C\right)$$

$$= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right)$$

$= P(\text{buckets 1 to } k \textbf{ all denied strings})$
$= \left(P(\text{each string hashes to } k+1 \text{ or higher})\right)^m$
$= (1 - p_1 - p_2 \ldots - p_k)^m$



$$= 1 - (1 - p_1 - p_2 \ldots - p_k)^m$$

The fun never stops with hash tables

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it? ✅
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it? ✅

Looking for a challenge? ☺

The fun never stops with hash tables

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?
3. E = each of buckets 1 to k has $\geq$ 1 string hashed into it?

Hint: Use Part 2's event definition:

Define F_i = bucket i has at least one string in it