CS103X: Discrete Structures
Homework Assignment 6

Due March 7, 2008

Exercise 1 (10 points)**.** How many simple directed (unweighted) graphs on the set of vertices $\{v_1, v_2, \ldots, v_n\}$ are there that have at most one edge between any pair of vertices? (That is, for two vertices a, b, only at most one of the edges (a, b) and (b, a) is in the graph.) For this question vertices are distinct and isomorphic graphs are not the same. Substantiate your answer.

Exercise 2 (20 points)**.** Given a connected graph $G = (V, E)$, the *distance* $d_G(u, v)$ of two vertices u, v in G is defined as the length of a shortest path between u and v. The *diameter* $\mathrm{diam}(G)$ of G is defined as the greatest distance among all pairs of vertices in G. (That is, $\max_{u,v \in V} d_G(u, v)$.) The *eccentricity* $\mathrm{ecc}(v)$ of a vertex v of G is defined as $\max_{u \in V} d_G(u, v)$. Finally, the *radius* $\mathrm{rad}(G)$ of G is defined as the minimal eccentricity of a vertex in G, namely $\min_{v \in V} \mathrm{ecc}(v)$. Prove:

(a) $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G)$.

(b) For every $n \in \mathbb{N}^+$, there are connected graphs G_1 and G_2 with $\mathrm{diam}(G_1) = \mathrm{rad}(G_1) = n$ and $\mathrm{diam}(G_2) = 2\mathrm{rad}(G_2) = 2n$.

Exercise 3 (10 points)**.** Let G be a graph in which all vertices have degree at least d. Prove that G contains a path of length d.

Exercise 4 (15 points)**.** Given a graph $G = (V, E)$, an edge $e \in E$ is said to be a *bridge* if the graph $G' = (V, E \setminus \{e\})$ has more connected components than G. Prove that if all vertex degrees in a graph G are even then G has no bridge.

Exercise 5 (10 points)**.** Prove that given a connected graph $G = (V, E)$, the degrees of all vertices of G are even if and only if there is a set of edge-disjoint cycles in G that cover the edges of G. (That is, the edge set of G is the disjoint union of the edge sets of these cycles.)

Exercise 6 (10 points)**.** Given a graph G, its *line graph* $L(G)$ is defined as follows:

- Every edge of G corresponds to a unique vertex of $L(G)$.
- Any two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G share a common endpoint.

Prove that if G is regular and connected then $L(G)$ is Eulerian.

Exercise 7 (10 points)**.** Prove that if a graph has at most m vertices of degree at most n and all other vertices have degree at most k, with $k < n$ and $m < n$, then the graph is colorable with $m + k + 1$ colors.

Exercise 8 (15 points)**.** Let G be a simple graph with n vertices. Prove that if G does not have K_3 as an induced subgraph, G has at most $\lfloor \frac{n^2}{4} \rfloor$ edges.