CS103X: Discrete Structures
Homework 4 Solutions

Due February 22, 2008

Exercise 1 (10 points)**.** Silicon Valley questions:

(a) How many possible six-figure salaries (in whole dollar amounts) are there that contain at least three distinct digits?

(b) Second Silicon Valley question: What is the number of six-figure salaries that are not multiples of either 3, 5, or 7.

Solution

(a) First note that there are $999,999 - 100,000 + 1 = 900,000$ total possible salaries. Let us now count the complement case - number of six-figure salaries that contain at most two distinct digits. There are 9 salaries with 1 distinct digit (111111, 222222,..., 999,999). Now let us count the salaries that have exactly two distinct digits. Since the first digit cannot be 0, we need to consider the case when one of the two distinct digits is 0 differently. So suppose one of the two distinct digits is 0, then we have 9 choices for the other digit. The first position needs to contain this non-zero digit and we get to place either of the two in the last 5 positions however we want. There are $2^5 - 1$ different ways to do this, since we can consider a bijection between 5 digit binary numbers where 0 means that 0 is in the position, 1 means that the other digit is in the position. So there are in total $9 \times (2^5 - 1)$ different salaries with two distinct digits, one of which is 0 (since a binary value of 11111 gives all 1 digit, we take it out). For the case of two distinct digits when neither digit is 0, we get $\binom{9}{2} \times (2^6 - 2)$. First we choose two digits, and then similar to the 0 case, we have $2^6 - 2$ different ways of arranging the two numbers for two distinct digits. Notice that we have have to disclude the values 000000 and 111111, which would only include 1 distinct digit. Therefore, in total, we have $900,000 - 9 - 9 \times (2^5 - 1) - \binom{9}{2} \times (2^6 - 2) = 897,480$

(b) From (a) there are 900,000 possible six-figure salaries. Now let us count the number of possible six-figure salaries that are multiples of either 3, 5, or 7. We can subtract this number from the total to obtain the number of six-figure salaries that are not multiples of either 3, 5, or 7.
Let D_3, D_5, D_7 be the set of six-figure salaries that are multiples of 3,5,7 respectively. We therefore need to find $|D_3 \cup D_5 \cup D_7|$. We will use the inclusion-exclusion principle to compute this.

$|D_3| = \frac{(999{,}999-100{,}002)}{3} + 1 = 300,000$

$|D_5| = \frac{(999{,}995-100{,}000)}{5} + 1 = 180,000$

$|D_7| = \frac{(999{,}999-100{,}002)}{7} + 1 = 128,572$

$|D_3 \cap D_5| = \frac{(999{,}990-100{,}005)}{3\times5} + 1 = 60,000$

$|D_3 \cap D_7| = \frac{(999{,}999-100{,}002)}{3\times7} + 1 = 42,858$

$|D_5 \cap D_7| = \frac{(999{,}990-100{,}030)}{5\times7} + 1 = 25,714$

$|D_3 \cap D_5 \cap D_7| = \frac{(999{,}915-100{,}065)}{3\times5\times7} + 1 = 8,571$

From the inclusion-exclusion principle:

$|D_3 \cup D_5 \cup D_7| = |D_3| + |D_5| + |D_7| - |D_3 \cap D_5| - |D_3 \cap D_7| - |D_5 \cap D_7| + |D_3 \cap D_5 \cap D_7| = 488,571$

Therefore, the number of six-figure salaries that are not multiples of either 3, 5, or 7 are given by:
$900,000 - 488,571 = 411,429$

Exercise 2 (15 points)**.** A rook on a chessboard is said to put another chess piece under attack if they are in the same row or column.

(a) How many ways are there to arrange 8 rooks on a chessboard (the usual 8×8 one) so that none are under attack?

(b) How many ways are there to arrange k rooks on an $n \times n$ chessboard so that none are under attack?

(c) Imagine a three-dimensional chess variant played on a $8 \times 8 \times 8$ board. (512 cells overall.) Call it Weir-D Chess. A *battleship* is a Weir-D Chess piece that can attack any piece that is in the same two-dimensional layer, along some coordinate. (For example, a battleship in position $(5, 2, 6)$ puts cell $(8, 2, 1)$ under attack, but not cell $(8, 3, 1)$.) How many ways are there to arrange 8 battleships on a Weir-D Chess board so that none are under attack?

Give solutions with no summation.

Solution

(a) 8!. Consider 8×8 board as 8 columns. In the first column, you can choose 8 different positions for a rook. For the second column, there are only 7 remaining positions available, etc. The last column will force the last rook into position.

(b) $(n)_k \binom{n}{k}$. Consider $n \times n$ board as n columns. In the first column, you can choose n different positions for a rook. For the second column, there are $n-1$ remaining positions available, etc down to $n-k+1$ remaining positions for the last rook. We are placing k rooks, so we need to choose which columns they are in. Thus, it is $(n)_k \binom{n}{k}$

(c) $(8!)^2$. Consider an $8 \times 8 \times 8$ cube and consider placing a rook on each of the 8 planes horizontally (or vertically). The first rook placed has 8×8 positions in its plane. This rook eliminates a cross section of all the others such that there are effectively 7×7 valid positions in each remaining plane. Similarly, the next rook has 7×7 positions and leaves 6×6 valid positions in each remaining plane. If we continue in this fashion, we will have $8^2 \times 7^2 \times \ldots \times 2^2 \times 1^2 = (8!)^2$ ways of placing the rooks.

Exercise 3 (15 points)**.** A function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ is called monotone nondecreasing if $1 \leq i < j \leq n \;\Rightarrow\; f(i) \leq f(j)$.

(a) How many such functions are there?

(b) How many such functions are there that are surjective?

(c) How many such functions are there that are injective?

Solution

(a) There are $\binom{n+m-1}{n}$ such functions. Consider the codomain $\{1, 2, ..., m\}$ as bins and the domain $\{1, 2, ..., n\}$ as balls. If a bin (codomain element) contains a ball, it means that one of the elements in the domain maps to it. Thus, if we repesent this as bins and balls, any ordering of the bins and balls will give us a unique mapping from domain to codomain since it has to be monotone nondecreasing (The bijection as follows: the smallest valued bin that has a ball must must map to the minimal domain element. Remove that ball and the new smallest valued bin that could be the same value as the bin in the previous step has a ball must map to the second minimal domain element, and so on). Now we can apply the canonical "unordered with repetition" formula.

(b) There are $\binom{(n-m)+m-1}{n-m} = \binom{n-1}{n-m}$ such functions. Again, if we use the balls and bins analogy, we have to first allocate 1 ball for each bin, and then choose positions for the rest of the balls. Thus, $n - m$ balls are left for us to put into bins, as in the canonical unordered with repetition problem.

(c) There are $\binom{m}{n}$. Once we choose the a set of n elements from m, we will know the exact mapping because the function must be monotone nondecreasing. Thus, we need to determine in how many ways can we choose n elements from m.

Exercise 4 (10 points)**.** How many ways are there to express a positive integer n as:

(a) A sum of k natural numbers? (For example, if $n = 2$ and $k = 3$ the answer is 6, since $2 = 2+0+0 = 0+2+0 = 0+0+2 = 1+1+0 = 1+0+1 = 0+1+1$.)

(b) A sum of positive integers?

The order of the summands is important. (Imagine the summation written down.)

Solution

(a) $\binom{n+k-1}{n}$. Consider each x_i as a box and there are n balls. Then we have the canonical "unordered with repetition".

(b) Consider the formula in part (a). We now have k ranging from 1 to n, with atleast 1 ball in each box. We allocate 1 ball in each box first, so we have $n - k$ balls left to place. Thus, we have

$$\sum_{k=1}^{n} \binom{n-k+k-1}{n-k} = \sum_{k=1}^{n} \binom{n-1}{n-k} = 2^{n-1}$$

Exercise 5 (10 points)**.** Prove either algebraically or combinatorically:

(a) For $p, n \geq 0$, $\displaystyle\sum_{k=p}^{n} \binom{k}{p} = \binom{n+1}{p+1}$

(b) $\displaystyle\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n}$

Solutions

(a) Consider what the RHS is counting. We are choosing $p+1$ elements from $n+1$ possible elements. Let us number these elements $1, 2, 3, \ldots, n, n+1$. Let us count the number of ways to select $p+1$ elements in a particular way. Suppose the maximal element picked is going to be $p+1$. Then we have $\binom{p}{p}$ ways of choosing the last p elements. Now suppose the maximal element picked is going to be $p+2$. Then we have $\binom{p+1}{p}$ ways of choosing the last p elements. In general, if the maximal element picked is $k+1$, then we have $\binom{k}{p}$ ways of choosing the last p elements. So we have

$$\sum_{k=p}^{n} \binom{k}{p}$$

as desired.

(b) Consider the RHS. Let's apply Pascal's rule repeatedly:

$$\begin{aligned}
\binom{m+n+1}{n} &= \binom{m+n}{n} + \binom{m+n}{n-1} \\
&= \binom{m+n}{n} + \binom{m+(n-1)}{n-1} + \binom{m+n-1}{n-2} \\
&= \binom{m+n}{n} + \binom{m+(n-1)}{n-1} + \binom{m+(n-2)}{n-2} + \binom{m+n-2}{n-3} \\
&\vdots \\
&= \binom{m+n}{n} + \binom{m+(n-1)}{n-1} + \binom{m+(n-2)}{n-2} + \ldots + \binom{m-1}{1} + \binom{m}{0} \\
&= \sum_{k=0}^{n} \binom{m+k}{k}
\end{aligned}$$

as desired.

Exercise 6 (10 points)**.** Give a closed-form expression (without summation) for the following:

$$\sum_{k=0}^{n} 2^k \binom{n}{k}.$$

Solution $\sum_{k=0}^{n} 2^k \binom{n}{k} = (1+2)^n = 3^n$

Exercise 7 (10 points)**.** In a mathematics contest with three problems, 80% of the participants solved the first problem, 75% solved the second and 70% solved the third. Prove that at least 25% of the participants solved all three problems. (The claim might seem obvious — find a *proof*.)

Solution Let the total number of participants be $n > 0$ (if $n = 0$, the proof is trivial). Denote the set of people who missed the first problem by A, the set of people who missed the second by B, and the set who missed the third by C. We know that $|A| = n - 0.8n = 0.2n$, $|B| = n - 0.75n = 0.25n$ and $|C| = n - 0.7n = 0.3n$. We also know, from the lecture notes, that

$$|A \cup B \cup C| \leq |A| + |B| + |C| = 0.2n + 0.25n + 0.3n = 0.75n$$

The set of people who solved all three problems is the complement of $A \cup B \cup C$ (the set who missed at least one problem), so it has size

$$n - |A \cup B \cup C| \geq n - 0.75n = 0.25n$$

Therefore at least 25% of the participants solved all three problems.

Exercise 8 (10 points)**.** What is the number of integer solutions of the equation

$$x_1 + x_2 + x_3 = 50,$$

such that $0 \leq x_i \leq 20$ for each $1 \leq i \leq 3$?

Solution Let us count the complement. At least 1 bin has more than 20 balls. This problem may be thought of as the "unordered with repetition" problem. Consider 3 bins and 50 balls. Without loss of generality, let the first bin contain at least 21 balls. Then there are 29 balls remaining. There are effectively 3 bins and 29 balls, which makes for $\binom{29+3-1}{29}$ ways of having the first bin contain at least 21 balls. Without loss of generality, let the first two bins contain at least 21 balls. Then there are 8 balls remaining. Still, there are effectively 3 bins and 8 ball,s which makes for $\binom{8+3-1}{8}$. There cannot be at any time all three bins having at least 21 balls, as $21 \times 3 = 63 > 50$. Now we can apply the inclusion-exclusion principle. Let $P(\{1\})$ be the number ways that we can have bin 1 have more than 20 balls, $P(\{1,2\})$ be the number of ways that we can have bins 1 and 2 have more than 20 balls, etc.

$$P(\{1\}) + P(\{2\}) + P(\{3\}) - P(\{1,2\}) - P(\{1,3\}) - P(\{2,3\}) + P(\{1,2,3\})$$

$$= 3 \times \binom{29+3-1}{29} - 3 \times \binom{8+3-1}{8} + 0$$

This is the complement. In total, the number of integer solutions to the equation is $\binom{50+3-1}{50}$ with no restrictions. Therefore, the number of ways to make $0 \le x_i \le 20$ for each $1 \le i \le 3$ is:

$$\binom{50+3-1}{50} - (3 \times \binom{29+3-1}{29} - 3 \times \binom{8+3-1}{8}) = 66$$

Exercise 9 (10 points)**.** There are n people at a party, and each person has arrived in a different hat. The revelry leaves them slightly tipsy, so each of them goes home wearing someone else's hat. Find the number of ways of putting n hats on n people so that no person is wearing his/her own hat. *Give the full proof.*

Solution Refer section 11.2 (Derangements) in lecture notes.