CS103X: Discrete Structures
Homework Assignment 3

Due February 15, 2008

Exercise 1 (15 points)**.** For each of the following relations, state whether they fulfill each of the 4 main properties - reflexive, symmetric, antisymmetric, transitive. Briefly substantiate each of your answers.

(a) The coprime relation on $\mathbb{Z}$. (Recall that a, $b \in \mathbb{Z}$ are coprime if and only if $\gcd(a, b) = 1$.)

(b) Divisibility on $\mathbb{Z}$.

(c) The relation T on $\mathbb{R}$ such that aTb if and only if $ab \in Q$.

Exercise 2 (10 points)**.** In a partially ordered set, a chain is a totally ordered subset. For example, in the set 1, 2, 3, 4, 5, 6, the divisibility relation is a partial order and 1, 2, 4 and 1, 3, 6 are chains.

(a) What is the longest chain on the set $\{1, 2, \ . \ . \ . \ , n\}$ using the divisibility relation? How many distinct chains have this length? For the second part, make sure to consider all positive values of n.

(b) What is the longest chain on the powerset of a set A with $|A| = n$ with the $\subseteq$ relation? How many distinct chains have this length?

Exercise 3 (25 points)**.** Formulate each of the below as a *single* statement (proposition or predicate), using only mathematical and logical notation that has been defined in class. For example, the use of logical quantifiers and connectives, and arithmetic, number-theoretic, and set-theoretic operations is allowed, as is the use of operators like gcd or sets like $\mathbb{Q}$, $\mathbb{R}$, etc., but *not* the use of English-language words or informal shorthand like $\{1, 2, \ldots, n\}$.

(a) There are infinitely many primes.

(b) If a and b are integers and $b \neq 0$, then there is a unique pair of integers q and r, such that $a = qb + r$ and $0 \leq r < |b|$.

(c) If a and n are coprime then there exists exactly one $x \in \mathbb{Z}_n$ for which $ax \equiv_n b$, for any $b \in \mathbb{Z}$.

(d) Two integers are coprime if and only if every integer can be expressed as their linear combination.

(e) The principle of strong induction

Exercise 4 (25 points)**.** After completing the previous exercise, write the negation of each of your logical statements, such that the symbol $\neg$ does not appear in your statements. (That is, eliminate negated quantifiers and negated compounds as you have learned in class, and then replace statements such as $\neg(a|b)$ by statements like $a \nmid b$.) Read the negations out in natural language and check for yourself that you understand why these are the right negations for the statements in the previous exercise.

Exercise 5 (15 points)**.** (a) Prove that the logical connectives $\{\neg, \vee\}$ are a universal set of connectives. That is, show that propositions like $P \rightarrow Q$, $P \leftrightarrow Q$, and $P \wedge Q$ can be expressed in terms of $\neg$ and $\vee$ alone.

(b) Prove that $\neg, \oplus$ are not a universal set.

Exercise 6 (10 points)**.** You are given the following predicate on the set P of all people who ever lived:

Parent(x, y): true if and only if x is the parent of y.

(a) Rewrite in the language of mathematical logic (you may assume the equality/inequality operators):

All people have two parents.

(b) We will recursively define the concept of *ancestor*:

An ancestor of a person is one of the person's parents or the ancestor of (at least) one of the person's parents.

Rewrite this definition using the language of mathematical logic. Specifically, you need to provide a necessary and sufficient condition for the predicate Ancestor(x, y) to be true. (Note that you can inductively use the Ancestor$(\cdot, \cdot)$ predicate in the condition itself.)