CS 103X: Discrete Structures
Midterm Solutions

February 8, 2007

Exercise 1 (20 points)**.** Prove or disprove: For any three sets A, B, C,

(a) $C \setminus (A \cap B) = (C \setminus A) \cap (C \setminus B)$.

(b) $C \cup (A \cap B) = (C \cup A) \cap (C \cup B)$.

Solution

(a) The statement is false. A counterexample is: $A = \{1\}$, $B = \{2\}$, $C = \{1, 2\}$. Then $C \setminus (A \cap B) = \{1, 2\}$ but $(C \setminus A) \cap (C \setminus B) = \emptyset$.

(b) The statement is true. We will first show that $C \cup (A \cap B) \subseteq (C \cup A) \cap (C \cup B)$. Consider any e which is an element of the $C \setminus (A \cap B)$. Then $e \in C$ or $e \in A \cap B$. If $e \in C$, then $e \in C \cup A$ and $e \in C \cup B$, so e is an element of the $(C \cup A) \cap (C \cup B)$. If $e \notin C$, then e must be in $A \cap B$ to still be in the $(C \cup A) \cap (C \cup B)$. But then $e \in A$, which implies $e \in C \cup A$, and similarly $e \in C \cup B$, so e is still an element of the $(C \cup A) \cap (C \cup B)$. This proves $C \setminus (A \cap B) \subseteq (C \cup A) \cap (C \cup B)$.

Now we show that $C \cup (A \cap B) \supseteq (C \cup A) \cap (C \cup B)$. Consider any element e in the $(C \cup A) \cap (C \cup B)$. Since e must be an element of $C \cup A$, it must be in C or in A. If $e \in C$, then e must be in the $C \setminus (A \cap B)$ too. If $e \notin C$, then we necessarily require $e \in A$ for it to still be in $C \cup A$. Also, e must be in $C \cap B$, so if $e \notin C$ then e must be in B. So $e \in A \cap B$, and hence e is in the $C \setminus (A \cap B)$. So $C \setminus (A \cap B) \supseteq (C \cup A) \cap (C \cup B)$.

Putting the two results together, we have proved that $C \cup (A \cap B) = (C \cup A) \cap (C \cup B)$.

Exercise 2 (20 points)**.** You have been appointed Postmaster General for a new nation. Your job is to determine what denominations to issue stamps in. Unfortunately, the government wants to be able to charge an amount for each letter or package that corresponds exactly to its weight, so the postage fee could be any integral number of cents. The government is also cheap and wants to reduce costs by only printing a minimal number of types of stamps, and stamps of value less than 5 cents will not be allowed.

(a) What is the minimum number n of integer values $v_1, v_2, \ldots, v_n$ (where $v_i \geq 5$ for all i) such that all positive integers can be expressed as linear combinations of them? What is the precise condition that such a minimal set $v_1, v_2, \ldots, v_n$ has to satisfy? Prove. Does your answer apply to real life?

(b) Suppose the government caps the lowest possible package charge at c cents, for some $c > 50$. In a "real life" scenario, what is now the minimum number of stamp values $v_1, v_2, \ldots$ (where $v_i \geq 5$ for all i) that you can get away with so as to cover all package weights greater or equal to c? Prove.

Solution

(a) The minimum number of values is $n = 2$. The precise condition is that v_1 and v_2 are coprime, namely $\gcd(v_1, v_2) = 1$. To prove, invoke Theorem 4.4.2, that an integer is a linear combination of v_1 and v_2 iff it is a multiple of $\gcd(v_1, v_2)$. Thus if $\gcd(v_1, v_2) = 1$, all positive integers can be a linear combination of v_1 and v_2 since all positive integers are a multiple of 1. And if $\gcd(v_1, v_2) > 1$, then 1 is not a multiple of the GCD and cannot be represented. (Note that almost everyone forgot to show this: this is what the comments "prove other direction" or "prove necessity" referred to.) Finally, it won't work in "real life" because to represent values such as 1 requires negative coefficients, and in real life you can't to put negative amounts of stamps on letters.

(b) The minimum number is still 2. The simplest way is to use 5- and 6-cent stamps. To prove that this works, note that all integers $c > 50$ can be written as $5n+1, 5n+2, 5n+3, 5n+4, 5n+5$ for some integer $n \geq 10$. We can prove by induction on these five types that they can all be a linear combination of 5 and 6 with nonnegative coefficients. For base cases ($n = 10$):

$$\begin{aligned} 51 &= 9 \times 5 + 1 \times 6 \\ 52 &= 8 \times 5 + 2 \times 6 \\ 53 &= 7 \times 5 + 3 \times 6 \\ 54 &= 6 \times 5 + 4 \times 6 \\ 55 &= 5 \times 5 + 5 \times 6 \end{aligned}$$

Now assume that for $n = k$, that $5k+1$ is a linear combination of 5 and 6 with nonnegative coefficients, so $5k + 1 = a \times 5 + b \times 6$ with $a, b \in \mathbb{N}$. If we consider $n = k + 1$, then $5(k+1)+1 = 5k+6$ can be represented as $(a+1) \times 5 + b \times 6$, so it is also a linear combination with nonnegative coefficients. The inductive step is exactly the same for the other four cases, so if $5k+1, 5k+2, 5k+3, 5k+4, 5k+5$ can be represented than so can $5(k+1)+1, 5(k+1)+2, 5(k+1)+3, 5(k+1)+4, 5(k+1)+5$. This concludes the proof by induction.

Exercise 3 (20 points)**.** Given a nonzero rational number r and two irrational numbers a and b, prove or disprove:

(a) ab is irrational.

(b) ar is irrational.

(c) a^b is irrational.

Solution

(a) The statement is false. An immediate counterexample is $a = b = \sqrt{2}$. We know $\sqrt{2}$ is irrational, but the product $ab = 2$ is rational.

(b) The statement is true. Proof by contradiction: Assume ar is a rational number $\frac{p}{q}$, where $p, q \in \mathbb{Z}, q \neq 0$. Since r is rational, it can be written as $\frac{s}{t}$, $s, t \in \mathbb{Z}, t \neq 0$. Then $a = \frac{pt}{qs}$. Now $r \neq 0$, so $s \neq 0$, and hence $qs \neq 0$. pt and qs are both integers, and the latter is non-zero, so $\frac{pt}{qs}$ is rational. This implies a is rational, which contradicts our assumption. Hence the product is always irrational.

(c) The statement is false. Here are two ways to disprove it:

i. Consider $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$. Then $a^b = 2$, which is rational. We know b to be irrational, but a's status is unclear. If a is irrational, then we have an immediate counterexample. If a is rational, then we have a counterexample c^d, where $c = d = \sqrt{2}$ are irrational but the product (a) is rational. Either way, we have shown that an irrational power of an irrational may be rational.

ii. Consider $a = \sqrt{2}$, $b = 2\log_2 3$. Then $a^b = 3$, which is rational. We know a is irrational, and so is $\log_2 3$ from Theorem 3.1.2 in the lecture notes. By part (b), b is irrational and we have a counterexample. (A very similar method can be used to show that $a = \sqrt{3}$, $b = \log_{\sqrt{3}} 2$, which was one of the solutions you came up with during the midterm, works.)

Note: Euler's identity, $e^{i\pi} = -1$, is *not* a counterexample: irrational numbers are a subset of the real numbers, so an imaginary number like $i\pi$ does not count as irrational.

Exercise 4 (20 points)**.** A perfect number is a natural number $n \geq 2$ with the property that the sum of all of n's divisors (including 1, but not n itself) is n. 6 and 28 are the first two examples. Prove that if $(2^p - 1)$ is prime, then $2^{p-1}(2^p - 1)$ is a perfect number. Using this property, find another perfect number besides 6 and 28.

Solution The only prime factors are 2 and $(2^p - 1)$. The powers of 2 range from 0 to $p - 1$. Thus the only factors are $1, 2, \ldots, 2^{p-1}$ and $(2^p - 1), (2^p - 1) \times 2, \ldots, (2^p - 1) \times 2^{p-2}$ (The second sequence stops at 2^{p-2} since we are omitting the number itself.) We can then write the sum of factors as

$$\sum_{i=0}^{p-1} 2^i + \sum_{i=0}^{p-2} (2^p - 1)2^i$$

The right sum can be factored out giving

$$\sum_{i=0}^{p-1} 2^i + (2^p - 1)\sum_{i=0}^{p-2} 2^i$$

Now we can use a fact about sums of powers of 2 (this is a specific example of the sum of a *geometric series*):

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

(In your solutions, you were not required to prove this. You can prove it fairly simply by induction). If we apply this identity twice, we have

$$(2^p - 1) + (2^p - 1) \times (2^{p-1} - 1)$$

This simplifies to

$$2^{p-1}(2^p - 1)$$

so the sum of the number's factors is the number itself and thus $2^{p-1}(2^p - 1)$ is a perfect number.

To find another perfect number, note that the next prime $2^p - 1$ is 31 when $p = 5$. Plugging this into the formula gives $16 \times 31 = 496$.

Exercise 5 (20 points)**.** Consider n planes in 3-dimensional space so that no two are parallel, any three have exactly one point in common, and no four have a common point. What is the number of 3-dimensional parts into which these planes partition the space? Prove. (You can use the 2-dimensional result without proof.)

Solution The problem is solved much like the 2-dimensional version. Consider adding the i-th plane (call it plane P) and counting how many new regions it creates. Two nonparallel planes intersect along a line, so the other $i-1$ planes all produce intersection lines on P. The conditions that no two planes are parallel, any three have one point in common, and no four have a common point imply that on P, no intersection lines are parallel and no three lines have a common point. Thus we can apply the two-dimensional result and say that plane P is divided into $\frac{(i-1)i}{2}+1$ regions by the $i-1$ lines produced from the intersections of the other planes. Each region of P divides one of the existing regions of 3-space created by the first $i-1$ planes in two, so adding the ith plane adds $\frac{(i-1)i}{2}+1$ regions. Recalling that there is one region before any planes are added, we can then write the number of regions as

$$1+\sum_{i=1}^{n}\left(1+\frac{i^2-i}{2}\right).$$

We can break up this sum into a sum over 1, i and i^2. Obviously $\sum_{i=1}^{n} 1 = n$. We know from the lecture notes that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$

and from midterm practice that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

(As in problem 4, you were allowed to use these without proof.) Substituting these in, the number of regions for n planes is

$$1+n+\frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right),$$

which simplifies to

$$\frac{n^3+5n+6}{6}.$$

Much like the two-dimensional version, this can be formalized as a proof by induction.