CS 103X: Discrete Structures
Homework Assignment 2 — Solutions

Exercise 1 (10 points)**.** Show that the sum of the first n odd natural numbers is n^2.

Solution The proof is by induction. The induction basis is trivial, the first odd natural number is 1 so the sum is 1, and $1^2 = 1$ of course. Now assume that the sum of the first k odd numbers is k^2. The $(k+1)$-st odd number is $2k+1$, and by the induction hypothesis the sum of the first $k+1$ odd numbers is k^2+2k+1. This is equal to $(k+1)^2$ and the claim thus holds for $k+1$. This completes the proof.

Exercise 2 (10 points)**.** What's wrong with the following induction proof?

> We prove that for any $n \in \mathbb{N}$ and any $a \in \mathbb{R}$, $a^n = 1$. The proof proceeds by strong induction. For the induction basis, $a^0 = 1$ and the claim holds. Assume that the claim holds for all k up to n. Then
>
> $$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1.$$
>
> This proves the claim.

Solution In the very first induction step we are assuming that the claim holds for $n = 0$ and need to prove correctness for $n = 1$. At that point the proof assumes that the claim also holds for $n = -1$, which was never proved and is not correct in general.

Exercise 3 (10 points)**.** Prove by induction that, for any set A, $|2^A| = 2^{|A|}$.

Solution The induction is on the cardinality. For the induction basis, any set of cardinality 0 is the empty set; its power set is $\{\emptyset\}$, so $|2^{\emptyset}| = 1 = 2^{|\emptyset|}$. Assume the claim holds for $n = k$, so for any set A such that $|A| = k$, $|2^A| = 2^{|A|}$. Any set A' of $k+1$ elements can be seen as a set A of k elements plus one new element e. A set $S \in 2^{A'}$ either contains e or not. Sets that do not contain e are subsets of A and their number is $2^{|A|}$ by the induction hypothesis. For a set $S \in 2^{A'}$ that contains e, consider $S_0 = S \setminus \{e\}$. $S_0 \in 2^A$ and every element of 2^A can be uniquely obtained in this way. Thus the number of sets $S \in 2^{A'}$ that contain e is also $2^{|A|}$. Together this implies that $|2^{A'}| = 2 \times 2^{|A|} = 2^{|A|+1} = 2^{|A'|}$. This concludes the induction step and proves the claim.

Exercise 4 (10 points)**.** Prove Bernoulli's inequality: For any $n \in \mathbb{N}$ and $r \in \mathbb{R}$, such that $r > -1$,

$$(1+r)^n \geq 1 + rn.$$

Solution By induction on n. For the induction basis, $n = 0$ and $(1+r)^0 \geq 1 + r0$ since $1 \geq 1$. Assume that the claim holds for $n = k$, so $(1+r)^k \geq 1 + rk$. We need to prove the inequality to $k+1$. The induction hypothesis implies

$$(1+r)^{k+1} \geq (1+rk)(1+r) = 1 + rk + r + r^2k \geq 1 + rk + r = 1 + r(k+1)$$

The first inequality follows from the assumption $1 + r \geq 0$, which allows us to multiply the inequality by $1 + r$ without changing the sign. The second inequality follows since $r^2k \geq 0$. This proves the induction step and concludes the proof.

Exercise 5 (10 points)**.** Prove the strong induction principle from the principle of induction. Conclude that the two principles are equivalent. (That is, anything that can be derived from one, can also be derived from the other.)

Solution Given a set A of positive integers, assume the conditions for the strong induction principle:

- $1 \in A$. (1)
- If $\{1, 2, \ldots, k\} \subseteq A$ then $k + 1 \in A$. (2)

hold for the set A. We must show that $A = \mathbb{N}^+$, and we will do this by ordinary induction. Let $P(n)$ be the following proposition:

"$\{1, 2, \ldots, n\} \subseteq A$."

For the base case, $P(1)$ follows trivially from (1). Now assume $P(n)$ holds, and consider $P(n+1)$. From our induction hypothesis and (2), $n+1 \in A$, or in other words $P(n)$ implies that $\{1, 2, \ldots, n+1\} \subseteq A$, which means $P(n+1)$ is true. By the induction principle, $P(n)$ holds for all positive integers n, so $\mathbb{N}^+ \subseteq A$. But A is a set of positive integers, so $A = \mathbb{N}^+$. This proves the result.

The other direction (proving the induction principle from the principle of strong induction) is similar but easier.

Exercise 6 (10 points)**.** Suppose $f(i, j)$ is a function of i and j, and $n \in \mathbb{N}^+$. Prove or give a counterexample:

$$\sum_{i=1}^{n} \sum_{j=1}^{i} f(i,j) \;=\; \sum_{j=1}^{n} \sum_{i=j}^{n} f(i,j)$$

If the sums are replaced with products, does your conclusion change?

Solution They are indeed equal, as we'll show by induction on n. For the base case, $n = 1$ and both sides of the equation are $f(1,1)$, so they're equal. Assume the claim holds for $n = k$. For $n = k+1$, the left hand side is

$$\sum_{i=1}^{k+1} \sum_{j=1}^{i} f(i,j) \;=\; \sum_{i=1}^{k} \sum_{j=1}^{i} f(i,j) \;+\; \sum_{j=1}^{k+1} f(k+1,j)$$

The last (single) sum is obtained by separating the case $i = k+1$ from the double sum. Also, the right hand side is

$$\begin{aligned}
\sum_{j=1}^{k+1}\sum_{i=j}^{k+1} f(i,j) &= \sum_{j=1}^{k+1}\sum_{i=j}^{k} f(i,j) \;+\; \sum_{j=1}^{k+1} f(k+1,j) \\
&= \sum_{j=1}^{k}\sum_{i=j}^{k} f(i,j) \;+\; \sum_{i=k+1}^{k} f(i,k+1) \;+\; \sum_{j=1}^{k+1} f(k+1,j) \\
&= \sum_{j=1}^{k}\sum_{i=j}^{k} f(i,j) \;+\; \sum_{j=1}^{k+1} f(k+1,j)
\end{aligned}$$

Note that the second sum in the second line evaluates to 0 because it has no terms: the starting value $i = k+1$ is greater than the ending value $i = k$.

By our induction hypothesis, $\sum_{i=1}^{k}\sum_{j=1}^{i} f(i,j) = \sum_{j=1}^{k}\sum_{i=j}^{k} f(i,j)$, so the left and right hand sides are equal in the case $n = k+1$ and the claim holds. This proves the result by induction.

If the sums are replaced with products, a completely analogous proof is possible (we just replace $\sum$ with $\prod$ and $+$ with $\times$). So the result holds in this case too.

Exercise 7 (20 points)**.** Many roots are irrational:

(a) Prove that $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{6}$ are irrational. (Hint: For $\sqrt{3}$, use the fact that every integer is of the form $3n$, $3n+1$, or $3n+2$.) Why doesn't the same proof technique imply that $\sqrt{4}$ is irrational?

(b) Prove that $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}+\sqrt{6}$ are irrational.

Solution

(a) Let's start with the proof that $\sqrt{3}$ is irrational. Following the proof that $\sqrt{2}$ is irrational in the notes, we start by assuming $\sqrt{3}$ is actually a rational number p/q, where p and q are integers with no common divisor and $q \neq 0$, and show this leads to a contradiction. Squaring,

$$\begin{aligned}
3 &= \frac{p^2}{q^2} \\
\text{or } p^2 &= 3q^2
\end{aligned}$$

Now (from the hint) every integer is of the form $3n$, $3n+1$ or $3n+2$, and the squares in the three cases are $9n^2$, $9n^2+6n+1$ and $9n^2+12n+4$. The first is divisible by 3 and the second and third leave a remainder of 1 when divided by 3. So a perfect square is a multiple of 3 if and only if its square root is divisible by 3. In other words, p must be divisible by 3, say $p = 3k$ for some integer k. But then

$$\begin{aligned}
9k^2 &= 3q^2 \\
\text{or } q^2 = 3k^2
\end{aligned}$$

and by the same argument, q must also be divisible by 3. This contradicts our assumption that p and q have no common divisor, and so $\sqrt{3}$ must, in fact, be irrational.

Identical proofs work for $\sqrt{5}$ and $\sqrt{6}$ once we have proved that a number is divisible by 5 if and only if its square is divisible by 5, and likewise for 6.

This technique does *not* work for $\sqrt{4}$ because an integer of the form $4n+2$ (which is not divisible by 4) has the square $16n^2+16n+4$ (which *is* divisible by 4). So our "if and only if" condition cannot be proved. Of course, we know that $\sqrt{4}=2$ is very rational indeed.

(b) For $\sqrt{2}+\sqrt{3}$, we square to obtain $\sqrt{2}^2+\sqrt{3}^2+2\sqrt{2}\sqrt{3}=5+2\sqrt{6}$. If this was a rational number p/q, we would have $\sqrt{6}=(p-5q)/2q$, which is a rational number. But we've proved in the first part that $\sqrt{6}$ is irrational, which proves the result by contradiction. For $\sqrt{2}+\sqrt{6}$, the same method works, only we use the fact that $\sqrt{3}$ is irrational.

Exercise 8 (20 points)**.** Consider n lines in the plane so that no two are parallel and no three intersect in a common point. What is the number of regions into which these lines partition the plane? Prove.

Solution The answer must be some function of n, which we'll denote by $F(n)$. We can consider putting down the lines one after the other on the plane (in any arbitrary order), with the total number of regions increasing at each step. After putting the first $n-1$ lines we have $F(n-1)$ regions. When we draw the nth line, we know it must intersect all of the other $n-1$ lines since no two lines are parallel. Since no three share an intersection, this new line has exactly $n-1$ intersection points with the other lines. These intersections mean that the new line goes through n of the existing regions of the plane (at every intersection point, the line leaves one existing region and enters another, and doesn't change regions before the first intersection, after the last, or between any two intersections). The new line divides each region through which it passes into two, so n new regions are created. So

$$F(n)=F(n-1)+n$$

It's easy enough to expand this sum as:

$$\begin{aligned}
F(n) &= F(n-1)+n \\
&= (F(n-2)+(n-1))+n \\
&= (F(n-3)+(n-2))+(n-1)+n \\
&\;\;\vdots \\
&= F(0)+1+2+\cdots+(n-1)+n \\
&= F(0)+\frac{n(n+1)}{2} \qquad \text{(recall the sum of the first } n \text{ natural numbers)}
\end{aligned}$$

Now when there are no lines, there is only one region (the whole plane), so $F(0)=1$. So

$$F(n)=\frac{n(n+1)}{2}+1.$$

This, however, is not a formal proof! Rather, that was an outline of the kind of reasoning you could use to realize that the right answer is $n(n+1)/2+1$. The actual proof proceeds by induction.

For the basis, with 0 lines there is 1 region so the claim holds. Now assume it holds for $n = k$. When we draw the $(k+1)$-st line, we know it must intersect all of the other k lines since no lines are parallel. Since no three share an intersection, this new line has exactly k intersection points with the other lines. These intersections mean that the new line goes through $k+1$ of the existing regions of the plane and divides each of those in two, such that $k+1$ new regions are created. Adding this to the existing tally gives a total of $k+1+k(k+1)/2+1 = (k^2+3k+2)/2+1 = (k+1)(k+2)/2+1 = (k+1)((k+1)+1)/2+1$ lines, so the claim holds for $k+1$. This completes the proof by induction.