CS 103X: Discrete Structures
Homework Assignment 5

Due February 22, 2007

Exercise 1 (25 points). Formulate each of the below as a *single* statement (proposition or predicate), using only mathematical and logical notation that has been defined in class. For example, the use of logical quantifiers and connectives, and arithmetic, number-theoretic, and set-theoretic operations is allowed, as is the use of operators like gcd or sets like $\mathbb{Q}$, $\mathbb{R}$, etc., but *not* the use of English-language words or informal shorthand like $\{1, 2, \ldots, n\}$.

(a) Every integer multiple of 4 can be expressed as the difference of two perfect squares.

(b) Bernoulli's inequality holds.

(c) Two integers are coprime if and only if every integer can be expressed as their linear combination.

(d) The principle of induction.

(e) Goldbach's conjecture. ("Every even integer greater than 2 can be written as the sum of two primes." You should *not* rely on an externally defined set $\mathbb{P}$ of primes.)

Exercise 2 (25 points). After completing the previous exercise, write the negation of each of your logical statements, such that the symbol $\neg$ does not appear in your statements. (That is, eliminate negated quantifiers and negated compounds as you have learned in class, and then replace statements such as $\neg(a|b)$ by statements like $a \nmid b$.) Read the negations out in natural language and check for yourself that you understand why these are the right negations for the statements in the previous exercise.

Exercise 3 (10 points). Let's formulate the famous "Barber of Seville" paradox in the notation of first-order logic (i.e. the sort of logic described in the lecture notes). In English, the paradox may be stated as:

"The barber of Seville shaves precisely those residents of Seville who do not shave themselves."

(Convince yourself that this is indeed a paradox.) Assume S is the set of all residents of Seville, including the barber. We have the following predicates over elements of the set S:

- *Shaves*(x, y): true if x shaves y, false otherwise.

- *Barber*(x): true if x is the barber of Seville (you may assume that Seville has just one barber), false otherwise.

Rewrite the statement of the paradox using only these predicates, along with the notation of mathematical logic.

Exercise 4 (20 points). Assuming P, Q and R are logical propositions, which of the following statements are tautologies, which are contradictions, and which are neither? No proof is necessary, but for every statement that you think is neither a tautology nor a contradiction you should provide one set of truth assignments to its variables that makes it true, and another set that makes it false.

(a) $(P \lor Q) \rightarrow (P \land Q)$

(b) $(P \lor \neg Q) \land (\neg P \land Q)$

(c) $((P \lor Q) \land R) \leftrightarrow ((P \land R) \lor (Q \land R))$

(d) $(P \leftrightarrow Q) \land (Q \leftrightarrow R) \land \neg(P \leftrightarrow R)$

Exercise 5 (20 points). You are given the following (all letters are logical propositions):

$$(t \rightarrow (r \lor p)) \rightarrow ((\neg r \lor k) \land \neg k).$$

Prove that this implies $\neg r$. Write down a proof using inference rules like we did in class. Manufacture inference rules from tautologies as needed.