CS 103X: Discrete Structures
Homework Assignment 3

Due February 1, 2007

Exercise 1 (20 points)**.** On well-ordering and induction:

(a) Prove the induction principle from the well-ordering principle.

(b) Prove the well-ordering principle from the induction principle.

Conclude that the principles of induction, strong induction, and well-ordering are equally powerful.

Exercise 2 (20 points)**.** (a) Let's develop another proof that $\sqrt{2}$ is irrational. Assume as we did in class that there exist two numbers $p, q \in \mathbb{Z}$, with $q \neq 0$, such that

$$\frac{p}{q} = \sqrt{2}.$$

Show that

$$\frac{2q - p}{p - q} = \sqrt{2}.$$

Use the well-ordering principle to complete the argument, and write the whole proof formally.

(b) Use the Fundamental Theorem of Arithmetic to prove that for $n \in \mathbb{N}$, $\sqrt{n}$ is irrational unless n is a perfect square, that is, unless there exists $a \in \mathbb{N}$ for which $n = a^2$.

Exercise 3 (20 points)**.** Prove or disprove, for integers a, b, c and d:

(a) If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

(b) If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

(c) If a and b are perfect squares and $a \mid b$, then $\sqrt{a} \mid \sqrt{b}$.

(d) If $ab \mid cd$, then $a \mid c$ or $a \mid d$.

Exercise 4 (25 points)**.** On Euclid's algorithm:

(a) Write the algorithm in pseudo-code. (10 points)

(b) State a theorem that asserts the correctness of the algorithm and prove the theorem. (10 points)

(c) Use the algorithm to calculate $\gcd(5924, 6892)$. Write out the complete sequence of derivations. (5 points)

Exercise 5 (15 points)**.** Some prime facts:

(a) Prove that for every positive integer n, there exist at least n consecutive composite numbers. (10 points)

(b) Prove that if an integer $n \geq 2$ is such that there is no prime $p \leq \sqrt{n}$ that divides n, then n is a prime. (5 points)