CS 103X: Discrete Structures
Final Exam – Solutions

March 21, 2007

Exercise 1 (10 points)**.** Prove that all odd perfect squares are congruent to 1 modulo 4.

Solution The square roots of odd perfect squares are of course odd, and all odd numbers are congruent to either 1 or 3 modulo 4. The square of a number that is 1 modulo 4 is also 1 modulo 4, while the square of a number that is 3 modulo 4 is congruent to (3×3) modulo 4, or 1 modulo 4.

(**Alternative solution:** The square of an even integer is even, so an odd perfect square is the square of an odd integer — let this integer be $n = 2k + 1$. Then $n^2 = 4n^2 + 4n + 1 = 4(n^2 + n) + 1$, which obviously leaves a remainder of 1 when divided by 4.)

Exercise 2 (10 points)**.** Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$. Is $\propto$ an equivalence relation? A partial order? A total order? Prove.

Solution It is none of the above. Recall that an equivalence relation is reflexive, symmetric, and transitive. $\propto$ is reflexive and transitive but not symmetric — let $f(n) = n$, $g(n) = n^2$. Here $f = O(g)$ but $g \neq O(f)$. It is also clearly not antisymmetric; if $f(n) = n$ and $g(n) = 2n$, $f = O(g)$ and $g = O(f)$ but $f \neq g$. This prevents $\propto$ from being a partial order, and thus it is not a total order also.

Exercise 3 (10 points)**.** You are given the following predicate on the set P of all people who ever lived:

Parent(x, y): true if and only if x is the parent of y.

(a) Rewrite in the language of mathematical logic (you may assume the equality/inequality operators):

All people have two parents.

(b) We will recursively define the concept of *ancestor*:

An ancestor of a person is one of the person's parents or the ancestor of (at least) one of the person's parents.

Rewrite this definition using the language of mathematical logic. Specifically, you need to provide a necessary and sufficient condition for the predicate Ancestor(x, y) to be true. (Note that you can inductively use the Ancestor$(\cdot, \cdot)$ predicate in the condition itself.)

Solution

(a)

$$\forall x \in P \;\; \exists y, z \in P : \big(\text{Parent}(y, x) \wedge \text{Parent}(z, x) \wedge y \neq z\big)$$

(b)

$$\forall x, y \in P : \Big(\text{Ancestor}(x, y) \leftrightarrow \big(\text{Parent}(x, y) \vee (\exists z \in P : (\text{Parent}(z, y) \wedge \text{Ancestor}(x, z)))\big)\Big)$$

Exercise 4 (10 points)**.** The drama club has m members and the dance club has n members. For an upcoming musical, a committee of k people needs to be formed with at least one member from each club. If the clubs have exactly r members in common, what is the number of ways the committee may be chosen? Substantiate.

Solution There are $m+n-r$ total people to choose from, so without the restriction the number of ways is $\binom{m+n-r}{k}$. Then we subtract the ways that won't work, which is when no people from one club are chosen. There are $m-r$ only in the dance club and $n-r$ only in the drama club. Thus there are $\binom{m-r}{k}$ ways to choose while having no one from the drama club chosen, and $\binom{n-r}{k}$ ways to pick no one from the dance club. Subtracting these gives a final answer of $\binom{m+n-r}{k} - \binom{m-r}{k} - \binom{n-r}{k}$.

Exercise 5 (10 points)**.** How many nonnegative integers less than or equal to 300 are coprime with 144? Substantiate.

Solution 144 has a prime factorization of all 2's and 3's. So, by inclusion-exclusion, the answer is 300 − (number divisible by 2) − (number divisible by 3) + (number divisible by 2 and 3). Of course, the last is the same as the number divisible by 6. Since 300 is divisible by 2,3, and 6, the formula is $300 - \frac{300}{2} - \frac{300}{3} + \frac{300}{6} = 300 - 150 - 100 + 50 = 100$.

Exercise 6 (10 points)**.** How many simple directed (unweighted) graphs on the set of vertices $\{v_1, v_2, \ldots, v_n\}$ are there that have at most one edge between any pair of vertices? (That is, for two vertices a, b, only at most one of the edges (a, b) and (b, a) is in the graph.) For this question vertices are distinct and isomorphic graphs are not the same. Substantiate your answer.

Solution There are $\binom{n}{2}$ possible unordered pairs of vertices. For each pair $\{a, b\}$, we may have only the edge (a, b), or only the edge (b, a), or no edge at all between vertices a and b, giving a total of 3 mutually exclusive possibilities. So the required number of graphs is $3^{\binom{n}{2}} = 3^{n(n-1)/2}$.

(**Note:** A number of students proposed the following solution — if we remove the directionality of the edges, the resulting graph is a simple undirected graph by the given conditions. Let it have k edges, out of a possible maximum of $\binom{n}{2}$. There are $\binom{\binom{n}{2}}{k}$ ways to pick these edges, and 2^k ways to assign directions to them, so the total number of possible graphs is

$$\sum_{k=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{k} 2^k$$

This is a correct solution and recieved full credit. However, the answer can be drastically simplified by observing that the above sum is nothing but the expansion, by the Binomial Theorem, of $(2+1)^{\binom{n}{2}} = 3^{\binom{n}{2}}$.)

Exercise 7 (10 points)**.** You already know from Bezout's Identity that if a and b are coprime integers, then there are integers x and y such that $ax + by = 1$. Now prove the same result using the Pigeonhole Principle. (You may assume that a and b are positive.)
Hint: Take the remainders, modulo b, of the first $b-1$ positive multiples of a, and consider what happens if 1 is not in this set.

Solution a and b are coprime, so at most one of a and b can be 1. Without loss of generality assume $b \neq 1$ — this ensures $b \nmid a$. We can rewrite $ax + by = 1$ as $ax = (-y)b + 1$. This suggests that we consider the remainders of multiples of a modulo b, i.e. the integers $a \text{ rem } b, 2a \text{ rem } b, 3a \text{ rem } b, \ldots, (b-1)a \text{ rem } b$. Assume, for the sake of contradiction, that none of them is 1. Then, since there are $b-1$ of them and they all lie in the set $\{2, 3, \ldots, b-1\}$ (0 is absent since $b \nmid a$), which has $b-2$ elements, the Pigeonhole Principle tells us that two of them must be equal. Say $pa \text{ rem } b = qa \text{ rem } b$, for $1 \leq p, q < b$. This implies $pa \equiv_b qa$, or $(p-q)a \equiv_b 0$. But since a and b are coprime, this means $p-q$ is a multiple of b, which is impossible since p and q are unequal positive integers less than b (so $0 < |p-q| < b$). Hence we have reached a contradiction and 1 must be one of the remainders, say $pa \text{ rem } b = 1$. Then $pa = qb + 1$ for some q, and choosing $x = p$, $y = -q$ we get the required result.

Exercise 8 (10 points)**.** Prove that at a cocktail party with ten or more people, there are either three mutual acquaintances or four mutual strangers.

Solution The proof is analogous to the cocktail party example in the Pigeonhole Principle chapter in the lecture notes. Pick an arbitrary person a — if a knows at least 4 people then of these 4 either all are mutual strangers or at least two of them know each other (giving, with a, a set of 3 mutual acquaintances) and we are done. If a knows at most 3 people, then there are 6 people a does not know. By the result in the lecture notes, this set of six people contains either 3 mutual acquaintances or 3 mutual strangers. If the former, we are done. If the latter, then along with a, we have a set of 4 mutual strangers.

Exercise 9 (10 points)**.** Given a (simple, undirected) graph G, its *line graph* $L(G)$ is defined as follows:

- Every edge of G corresponds to a unique vertex of $L(G)$.
- Any two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G share a common endpoint.

Prove that if G is regular and connected then $L(G)$ is Eulerian.

Solution If v is a vertex and e an edge of G, let $L(e)$ and $L(v)$ represent their corresponding edge and vertex, respectively, in $L(G)$. We will first show that the line graph is connected. Take any pair of edges e and e' and let v be an endpoint of e and v' be an endpoint of e'. Then by connectivity of G, there is a path $v, e_1, v_2, e_2, \ldots, v_m, e_m, v'$ in G. In $L(G)$, by definition, $L(e)$ and $L(e_1)$ are adjacent (linked by the edge $L(v)$), e_1 and e_2 are adjacent, and so on. So there is a path $L(e), L(v), L(e_1), L(v_2), \ldots, L(e_m), L(v'), L(e')$, and $L(e)$ and $L(e')$ are connected. Since the two edges were arbitrarily chosen, $L(G)$ is connected.

Now observe that if all vertices in G have the same degree k in G, then every edge shares a vertex with $k-1$ other edges at *each* of its two endpoints. Since the graph is simple, these two sets of $k-1$ edges are mutually exclusive. So each edge shares an endpoint with *exactly* $2(k-1)$ other edges in G, i.e. each vertex of $L(G)$ has exactly $2(k-1)$ neighbours, or in other words has even degree. Now, applying Theorem 15.1.1, we see that $L(G)$ must have an Eulerian tour.

Exercise 10 (10 points)**.** Let G be a (simple, undirected) graph that has no induced subgraphs that are P_4 or C_3. Prove that G is bipartite.

Solution Since we know a graph is bipartite if and only if it has no odd cycles, we can equivalently prove that G has no odd cycles. Then by taking the contrapositive, it is equivalent to prove that any graph with an odd cycle has either P_4 or C_3 as an induced subgraph. From here we proceed by contradiction, assume there exists some graph G with an odd cycle and no induced subgraphs that are P_4 or C_3. Obviously, if this cycle is length 3, then C_3 is an induced subgraph. If the length is 5 or greater, select any four adjacent points in the cycle (i.e. points A, B, C, D such that edges AB, BC, CD are part of the cycle) and consider the induced subgraph on those four. If the cycle edges are the only ones present, P_4 is an induced subgraph. If not, then one of the edges AC, BD, AD must be in the original graph. If AC is, then we have an induced C_3 subgraph on A, B, C, and if BD is then there is an induced C_3 on B, C, D. The only remaining possibility is that AD is present; then we can make a new cycle by removing AB, BC, CD from the original cycle and adding AD. This new cycle has length 2 less than the original, so it is still odd. Since this is the only possibility that does not immediately produce a C_3 or P_4, we can repeat this process to examine progressively smaller cycles. But then eventually we will create a cycle of length 3, the shortest possible odd-cycle length, which produces a C_3 induced subgraph. This is a contradiction, and thus any graph with an odd cycle must have either P_4 or C_3 as an induced subgraph. This completes the proof.

(**Note:** The word "induced" is very important here! See the definition of "induced subgraph" in the lecture notes — it is *not* the same as a "subgraph".)