CS 103X: Discrete Structures
Final Exam

March 21, 2007

THE STANFORD UNIVERSITY HONOR CODE

- The Honor Code is an undertaking of the students, individually and collectively:

  - that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

  - that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

- The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid as far as practicable, academic procedures that create temptations to violate the Honor Code.

- While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to create optimal conditions for honorable academic work.

Exams are to be done individually and must represent original work—it is a violation of the honor code to copy or derive exam question solutions from other students or anyone at all, textbooks, or previous instances of this course.

I acknowledge and accept the honor code:

Signature: _____

Name (print): _____

EXAM RULES

- **Do not write "shot in the dark" solutions. If you don't know the solution, don't submit guesswork hoping to get partial credit. Empty solutions are guaranteed 30% credit by default. Completely off-the-mark or "let's write everything I know about this area hoping something will sound relevant" solutions will get none. If you can't find the answer, you are better off with 30% credit.**
- You have 3 hours to complete this exam.
- Do not include your scratch work with your exam. Please work the solutions out on another sheet of paper and then write your solutions neatly on the exam.
- You may use Prof. Koltun's lecture notes and two double-sided cheat-sheets. You may *not* use any other material, such as your own course notes, homeworks, distributed homework solutions, other sets of lecture notes, books, computers, cell phones, crystal balls, Tarot cards, etc.
- Write clearly and neatly.
- Stagger your seats.

GOOD LUCK!

Exercise 1 (10 points)**.** Prove that all odd perfect squares are congruent to 1 modulo 4.

Exercise 2 (10 points). Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$. Is $\propto$ an equivalence relation? A partial order? A total order? Prove.

Exercise 3 (10 points)**.** You are given the following predicate on the set P of all people who ever lived:

Parent(x, y): true if and only if x is the parent of y.

(a) Rewrite in the language of mathematical logic (you may assume the equality/inequality operators):

All people have two parents.

(b) We will recursively define the concept of *ancestor*:

An ancestor of a person is one of the person's parents or the ancestor of (at least) one of the person's parents.

Rewrite this definition using the language of mathematical logic. Specifically, you need to provide a necessary and sufficient condition for the predicate Ancestor(x, y) to be true. (Note that you can inductively use the Ancestor$(\cdot, \cdot)$ predicate in the condition itself.)

Exercise 4 (10 points). The drama club has m members and the dance club has n members. For an upcoming musical, a committee of k people needs to be formed with at least one member from each club. If the clubs have exactly r members in common, what is the number of ways the committee may be chosen? Substantiate.

Exercise 5 (10 points). How many nonnegative integers less than or equal to 300 are coprime with 144? Substantiate.

Exercise 6 (10 points). How many simple directed (unweighted) graphs on the set of vertices $\{v_1, v_2, \ldots, v_n\}$ are there that have at most one edge between any pair of vertices? (That is, for two vertices a, b, only at most one of the edges (a, b) and (b, a) is in the graph.) For this question vertices are distinct and isomorphic graphs are not the same. Substantiate your answer.

Exercise 7 (10 points)**.** You already know from Bezout's Identity that if a and b are coprime integers, then there are integers x and y such that $ax + by = 1$. Now prove the same result using the Pigeonhole Principle. (You may assume that a and b are positive.)

Hint: Take the remainders, modulo b, of the first $b - 1$ positive multiples of a, and consider what happens if 1 is not in this set.

Exercise 8 (10 points)**.** Prove that at a cocktail party with ten or more people, there are either three mutual acquaintances or four mutual strangers.

Exercise 9 (10 points). Given a graph G, its *line graph* $L(G)$ is defined as follows:

- Every edge of G corresponds to a unique vertex of $L(G)$.

- Any two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G share a common endpoint.

Prove that if G is regular and connected then $L(G)$ is Eulerian.

Exercise 10 (10 points). Let G be a graph that has no induced subgraphs that are P_4 or C_3. Prove that G is bipartite.