CS 103X: Discrete Structures
Homework Assignment 8

Due March 10, 2006

Exercise 1 (50 points)**.** Give a concise proof:

(a) In a collection of $n+1$ integers, at least two are congruent modulo n.

(b) Among any five points with integer coordinates in the plane, there are two, such that the center of the line segment that connects them also has integer coordinates.

(c) Among five points chosen from within a square with side length 1, at least two lie within distance $\sqrt{2}/2$.

(d) In a graph with at least 2 vertices, there are two that have the same degree.

(e) In a clique on 10 vertices in which every edge is colored red or blue, there is either a red clique on 3 vertices or a blue clique on 4 vertices.

Exercise 2 (20 points)**.** Order the following functions by asymptotic growth rates. That is, list them as $f_1(n), f_2(n), \ldots, f_{15}(n)$, such that $f_i(n) = O(f_{i+1}(n))$ for all i. Give a brief argument to support the inequality $f_i(n) = O(f_{i+1}(n))$ for every i. (You don't have to give all the details.)

$$n, n^2, \log_{10} n, n \log_2 n, n^{\ln n}, (\ln n)^n, \ln(n^n), (\ln n)^{\ln n}, (\ln n)^{\ln \ln n}, (\ln \ln n)^{\ln n}, 2^{\ln n}, 2^{\sqrt{\ln n}}, 2^n, 3^n, 3^{n/2}.$$

Hint: Take logarithms to bring the exponents down. Use basic properties of logarithms.

Exercise 3 (10 points)**.** What's wrong with the following induction proof?

> We prove by induction that $n^2 = O(n)$. When $n = 1$, $1^2 = 1$ and the claim holds. Assume that $k^2 = O(k)$ for some $k \geq 1$. Then $(k+1)^2 = k^2 + 2k + 1$. Each of the summands on the right side is $O(k)$, and thus $(k+1)^2 = O(k)$. This completes the proof by induction.

Exercise 4 (20 points)**.** How many unweighted graphs with n vertices and m edges are there that are

(a) Undirected and simple.

(b) Directed and simple.

(c) Undirected and not necessarily simple.

(d) Directed and not necessarily simple.

Exercise 5 (Extra credit)**.** Consider the numbers $1, 2, \ldots, 2n$, and take any $n+1$ of them. Prove that there are two numbers i, j in this sample such that $i | j$.