P(n): Every planar graph is 5-colorable.

Base case…

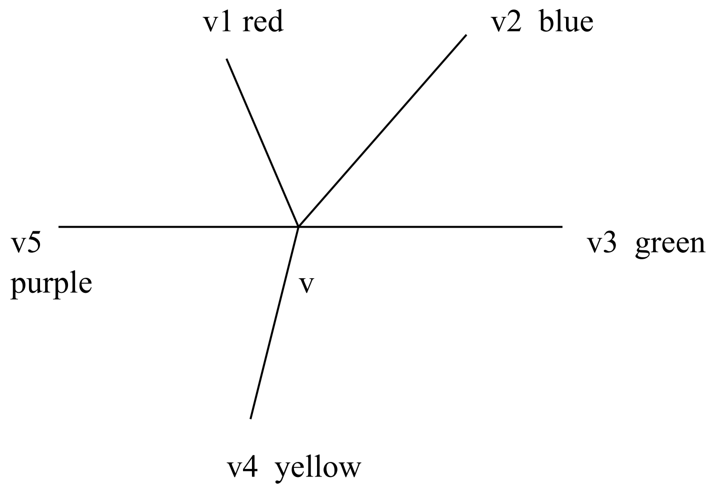
Inductive hypothesis…

Proof:

Let G be a planar graph with n+1 vertices. We know that G contains a vertex v whose degree is at most 5. The graph G – v is a planar graph with n vertices and is covered by the inductive hypothesis, so can be colored with 5 colors.

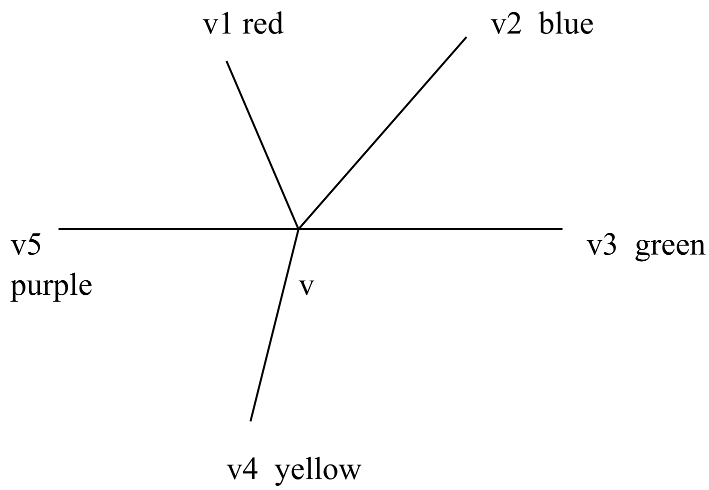
We may assume that v has exactly 5 neighbors and that they are differently colored, since otherwise there would be at most 4 vertices adjacent to v. This leaves a spare color for coloring v.

Here is the situation:




Define H(x,y) to be the two-colored subgraph of G induced by all vertices colored x or y. Consider H(red, green), the colors of v1 and v3:

1) If v1 and v3 lie in different components of H, then we can interchange the colors red and green of all the vertices in H containing v1. The result of this recoloring is v1 and v3 will both be green so v can be colored red.




2) If v1 and v3 lie in the same component of H then there is a circuit of the form v -> v1 -> … -> v3 -> v with alternating colors red and green. Notice that v2 lies entirely in the circuit and v4 lies outside, so there cannot be any two-color path between v2 and v4 lying entirely in H(blue, yellow) – the graph is planar and such a path would cross an edge of the circuit. We can therefore interchange the colors of the vertices in the component of H(blue, yellow) containing v2. Then, vertex v2 and v4 would both be colored yellow enabling v to be colored blue.

This completes the proof.