Handout #69
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CS103A
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Final Exam Review Session Solutions

Number Theory

1. (20 points) Prove that for all positive integers n: $6 \mid (n(n+1)(2n+1))$. Induction is not required for this proof (but is allowed).

Non-Induction Proof: To show this, we need to show divisibility both by 2 and by 3.

If n is odd, then $n+1$ is even and divisible by 2. If n is even, then it is divisible by 2. In either case $n(n+1)$ is divisible by 2 and so then is $n(n+1)(2n+1)$. We're halfway done.

Now we also know that $n \bmod 3$ is 0, 1, or 2, so we will show that in each of the three cases, one of the factors is divisible by 3. If $n \bmod 3 = 0$, then n is divisible by 3. If $n \bmod 3 = 1$, then for some m, $n = 3m + 1$. In this case, $2n + 1 = 6m + 3$, which is divisible by 3, and so $2n + 1$ is divisible by 3. If $n \bmod 3 = 2$, then for some m, $n = 3m + 2$. In this case, $n + 1 = 3m + 3$, which is divisible by 3, and so $n + 1$ is divisible by 3. So whatever the case is, one of the three factors is divisible by 3 and so $n(n+1)(n+2)$ must be divisible by 3.

Thus, we find that $n(n+1)(n+2)$ is divisible both by 2 and by 3 no matter what, and so it must be divisible by 6. Therefore, $6 \mid n(n+1)(n+2)$.

Induction Proof: Because we want to prove this for all positive integers n, we can prove this via induction on the integer k, beginning with the base case $k = 1$.

Let $P(n)$: $6 \mid n(n+1)(n+2)$ or, equivalently, $n(n+1)(n+2) = 6c$ for some constant c

BASE CASE: $k = 1$

$$\begin{aligned} 1(1+1)(2(1)+1) &= 1(2)(3) \\ &= 6 \\ &= 6c \qquad \text{for } c = 1 \end{aligned}$$

INDUCTIVE CASE:

Assume as our inductive hypothesis that $P(k)$ holds, namely that $6 \mid k(k+1)(2k+1)$, or equivalently, that $k(k+1)(2k+1) = 6c$ for some constant c. Now show that this implies that $P(k+1)$ holds, namely that $6 \mid (k+1)(k+2)(2k+3)$, or equivalently, that $(k+1)(k+2)(2k+3) = 6d$ for some constant d.

We shall begin with our formula and then expose our inductive hypothesis in order to show this.

$$\begin{aligned} (k+1)(k+2)(2k+3) &= (k^2 + 3k + 2)(2k + 3) \\ &= 2k^3 + 6k^2 + 4k + 3k^2 + 9k + 6 \\ &= 2k^3 + 9k^2 + 13k + 6 \\ &= (2k^3 + 3k^2 + k) + (6k^2 + 12k + 6) \\ &= k(k+1)(2k+1) + 6(k+1)^2 \\ &= 6c + 6(k+1)^2 \end{aligned}$$

$$= 6(c + (k + 1)^2)$$
$$= 6d \qquad \text{for } d = c + (k + 1)^2)i$$

We have established that *P(k)* implies *P(k+1)*. Together with the base case *P(1)*, by the principle of mathematical induction this means that *P(n)* holds for all positive integers *n*.

Induction

2. (20 points) Let S be the set of strings defined recursively as follows:

- a, b is in S
- if $x \in S$ and $y \in S$, then $(x + y)$ is in S

Prove that every element of S has twice as many parentheses as '+' signs. When counting parentheses, each '(' or ')' counts once, so for example, (a + b) has two parentheses and one '+' sign.

Note that in the definition of S, a and b are symbols that appear in strings of S, and x and y are variables used to explain how to construct a new member of S from other members. Examples of strings in S: a, b, (a + b), (b + (a + b)), ((a + b) + (a + b)).

Proof by Structural Induction

We want to show that every member of S has the property
P(s): s has twice as many parentheses as '+' signs.

BASE CASE: The strings a and b are in S, and both have 0 parentheses and 0 plus signs. Since $2 \cdot 0 = 0$, both P(a) and P(b) are true.

INDUCTIVE CASE: We need to show that the recursive rule preserves the property. So, suppose x and y have the property P. Then

$$\text{NPAREN}(x) = 2 \cdot \text{NPLUS}(x) \qquad \text{(i)}$$
$$\text{NPAREN}(y) = 2 \cdot \text{NPLUS}(y) \qquad \text{(ii)}$$

But

$$\text{NPAREN}(\text{"}(x+y)\text{"}) = \text{NPAREN}(x) + \text{NPAREN}(y) + 2 \qquad \text{(iii)}$$

and

$$\text{NPLUS}(\text{"}(x+y)\text{"}) = \text{NPLUS}(x) + \text{NPLUS}(y) + 1 \qquad \text{(iv)}$$

Substituting (i) and (ii) into (iii) and factoring gives

$$\text{NPAREN}(\text{"}(x+y)\text{"}) = 2 \cdot (\text{NPLUS}(x) + \text{NPLUS}(y) + 1) \qquad \text{(v)}$$

and substituting (iv) into (v) gives

$$\text{NPAREN}(\text{"}(x+y)\text{"}) = 2 \cdot \text{NPLUS}(\text{"}(x+y)\text{"})$$

Thus the recursive rule preserves the property, and by the principle of structural induction all members of S have the property.

NOTE: some students mistakenly attempted to prove the wrong thing for this problem. You are asked to prove that every member of S has a certain property, NOT that every string with the property is in S. The moral: read the question carefully!

Sequences

3. (20 points) In this problem we consider the Fibonacci sequence, which is defined as follows: $F_1 = 1$, $F_2 = 1$, and for $n > 2$, $F_n = F_{n-1} + F_{n-2}$.

Suppose that m is a positive integer. Prove that for all positive integers n, $F_m \mid F_{nm}$.

You are allowed to use the following Lemma (which you do not have to prove):

> If x and y are positive integers, $F_{x+y} = F_{x-1}F_y + F_xF_{y+1}$.

Because we need to prove this for all positive integers, we will prove this via induction on the integer *k*, beginning with base case *n = 1*.

Let *P(n)*: $F_m \mid F_{km}$, or equivalently, $F_{km} = F_k * c$ for some constant *c*.

BASE CASE: *k = 1*

$$\begin{aligned} F_{km} &= F_{(1)m} \\ &= F_m \\ &= F_m * c \qquad \text{for } c = 1 \end{aligned}$$

Thus, *P(1)* holds.

INDUCTIVE CASE:

Assume as our inductive hypothesis that *P(k)* holds, namely that $F_m \mid F_{km}$, or equivalently, that $F_{km} = F_m * c$ for some constant *c*. Now show that *P(k+1)* holds, namely that $F_m \mid F_{(k+1)m}$, or equivalently, that $F_{(k+1)m} = F_m * d$ for some constant *d*. We shall begin with $F_{(k+1)m}$ and expose the inductive hypothesis in order to show the desired conclusion.

$$\begin{aligned} F_{(k+1)m} &= F_{km + m} \\ &= F_{m-1} * F_{nm} + F_m * F_{nm+1} \\ &= F_{m-1} * F_m{*}c + F_m * F_{nm+1} \\ &= F_m * (F_{m-1} * c + F_{nm+1}) \\ &= F_m * d \qquad \text{for } d = F_{m-1} * c + F_{nm+1} \end{aligned}$$

We have established that *P(k)* implies *P(k+1)*. Together with the base case *P(k)*, by the principle of mathematical induction this means that *P(n)* holds for all positive integers *n*.

Recursion

4. (20 points) Your answers to parts (a) and (b) should consist of one or more base cases and one or more recursive rules.

(a) (6 points) Give a recursive definition for the set S of all strings that contain only the letters a and b, with all the a's appearing before all the b's. Here are some examples of strings in the set S: a, aabbbbb, aaaaaaabbb, bbbbbb. The empty string is not in S.

Base Cases: $a \in S$, $b \in S$

Recursive Rules: If $x \in S$, then $ax \in S$ and $xb \in S$.

(b) (8 points) Give a recursive definition for the set S of ordered pairs of positive integers such that $(a, b) \in S$ if and only if $a + b$ is odd. Here are some examples of members of S: (2, 5), (7, 32), (16, 15).

Base Cases: $(1, 2) \in S$, $(2, 1) \in S$

Recursive Rules: if $(a, b) \in S$, then $(a + 2, b) \in S$, and $(a, b + 2) \in S$

(c) (6 points) Integer multiplication can be accomplished by repeated addition. For example, 4*5 = 4 + 4 + 4 + 4 + 4. In general, m*n can be calculated by adding n values of m. Define a recursive function Multiply(m, n) that uses this approach to accomplish multiplication. You should not use the multiplication operator anywhere in your definition. You may assume that both arguments to your function are non-negative integers.

The form of your answer should be similar to the following definition of factorial:

$$\text{Factorial}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot \text{Factorial}(n-1) & \text{if } n > 1 \end{cases}$$

$$\text{Multiply}(m,n) = \begin{cases} 0 & \text{if } n = 0 \\ m + \text{Multiply}(m,n-1) & \text{if } n > 0 \end{cases}$$

Combinatorics

5. (20 points) Five couples go to the movies together and sit in a row of ten seats. In how many ways can the 10 people be arranged if:

(a) (2 points) They may sit in any order.

This is a basic permutation of all the seats:

$$P(10, 10) \quad = \quad 10!$$

(b) (6 points) All the men sit together and all the women sit together.

In this case, we break the problem up into two parts. First, we consider the fact that we can place the gender blocks in one of two ways: men left, women right, or vice versa. Second, we consider the arrangements within those gender blocks, each of which is a smaller version of the same problem as part (a), a simple permutation. Thus, we get:

$$2 \cdot P(5, 5) \cdot P(5, 5) \quad = \quad 2 \cdot 5!^2$$

(c) (6 points) Each couple sits together (i.e., for each couple, the two people are in adjacent seats).

This problem is similar to part (b) in that we are seating in blocks, except the blocks here are couples. First, we consider the different ways of arranging the couples themselves, which is a simple permutation of five blocks. Second, within each couple, there are two ways to seat individuals. Since there are five couples, we multiply times two five times:

$$P(5, 5) * 2 * 2 * 2 * 2 * 2 \quad = \quad 5! \cdot 2^5$$

(d) (6 points) One couple is arguing and they refuse to sit together. The other couples can sit in any way—together or not.

Here we want to find the number of seating arrangements with the fighting couple sitting together and then subtract that from the total number of seating arrangements. We know the total number from part (a). The number of seating arrangements with the fighting couple sitting together is a three part problem. First, consider the number of places that couple can be seated (there are nine). Next, consider the number of ways to seat the remaining eight individuals (a simple permutation of eight). Finally, multiply by two for seating within the couple itself:

$$P(10, 10) - 9 \cdot P(8, 8) \cdot 2 \quad = \quad 10! - 9 \cdot 8! \cdot 2$$