Handout #68 CS103A
December 7, 2008 Robert Plummer

Final Exam Review Session

Number Theory

1. (20 points) Prove that for all positive integers n: $6 \mid (n (n + 1) (2n +1))$. Induction is not required for this proof (but is allowed).

Induction

2. (20 points) Let S be the set of strings defined recursively as follows:

- a, b is in S
- if $x \in S$ and $y \in S$, then (x + y) is in S

Prove that every member of S has twice as many parentheses as '+' signs. When counting parentheses, each '(' or ')' counts once, so for example, (a + b) has two parentheses and one '+' sign.

Note that in the definition of S, a and b are symbols that appear in strings of S, and x and y are variables used to explain how to construct a new member of S from other members. Examples of strings in S: a, b, (a + b), (b + (a + b)), ((a + b) + (a + b)).

Sequences

3. (20 points) In this problem we consider the Fibonacci sequence, which is defined as follows: $F_1 = 1$, $F_2 = 1$, and for $n > 2$, $F_n = F_{n-1} + F_{n-2}$.

Suppose that m is a positive integer. Prove that for all positive integers n, $F_m \mid F_{nm}$.

You are allowed to use the following Lemma (which you do not have to prove):

If x and y are positive integers, $F_{x+y} = F_{x-1}F_y + F_xF_{y+1}$.

Recursion

4. (20 points) Your answers to parts (a) and (b) should consist of one or more base cases and one or more recursive rules.

(a) (6 points) Give a recursive definition for the set S of all strings that contain only the letters a and b, with all the a's appearing before all the b's. Here are some examples of strings in the set S: a, b, aabbbbb, aaaaaaabbb, bbbbbb. The empty string is not in S.

(b) (8 points) Give a recursive definition for the set S of ordered pairs of positive integers such that (a, b) $\in$ S if and only if a + b is odd. Here are some examples of members of S: (2, 5), (7, 32), (16, 15).

(c) (6 points) Integer multiplication can be accomplished by repeated addition. For example, 4*5 = 4 + 4 + 4 + 4 + 4. In general, m*n can be calculated by adding n values of m. Define a recursive function Multiply(m, n) that uses this approach to accomplish multiplication. You should not use the multiplication operator anywhere in your definition. You may assume that both arguments to your function are non-negative integers.

The form of your answer should be similar to the following definition of factorial:

$$\text{Factorial}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot \text{Factorial}(n-1) & \text{if } n > 1 \end{cases}$$

Combinatorics

5. (20 points) Five couples go to the movies together and sit in a row of ten seats. In how many ways can the 10 people be arranged if:

(a) (2 points) They may sit in any order.

(b) (6 points) All the men sit together and all the women sit together.

(c) (6 points) Each couple sits together (i.e., for each couple, the two people are in adjacent seats).

(d) (6 points) One couple is arguing and they refuse to sit together. The other couples can sit in any way—together or not.