Problem Set #10 Solutions

1) Product rule: 30 * 12 * 4 *6

2)

a) The two kinds of numbers don't overlap (e.g., if there are exactly four 0's then there are four 1's, not three), so it's the number of ways to place four 0's plus the number of ways to place three 1's, i.e., C(8,4) + C(8,3).

b) Let's first determine how many numbers have five consecutive 0's (the number will be the same for give consecutive 1's). If the 0's start in the first bit, then the following five bits can vary, giving us a total of 2^5. If the 0's start in the second bit, then the first bit must be a 1 (or else the 0's start in the first bit), but the following four bits can vary, giving a total of 2^4. If the 0's start in the third bit, then the first bit and trailing three bits can vary, giving us 2^4 possible numbers. Similarly, starting in the fourth, fifth, and sixth bits gives 2^4 possibilities each time. This gives a grand total of $2^5 + 5*2^4 = 112$ distinct numbers with five consecutive 0's.

Similarly, there will be 112 distinct numbers with 5 consecutive 1's. However, when we add these two numbers together, we double count those strings with both, namely 0000011111 and 1111100000. Thus, we need to subtract two, to get

$$112 + 112 - 2 = 222$$

3)

a) This is a combination with repetition, so we do

$$C(6 + 36 - 1, 36) = C(41, 36)$$

b) Of our two dozen scones, 12 are being set for us, while the second dozen we can choose freely. Those we are choosing 12 scones from among 6 different kinds with repetition, so we do

$$C(6 + 12 - 1, 12) = C(17, 12)$$

c) This is a complicated number to compute directly so perhaps the complementation method (subtracting from a larger set) will work better. In fact, it does. The strategy here is to first calculate the number of combinations with the first five constraints only (i.e., minus the cherry constraint) and then subtract the number of combinations possible with the first five constraints plus having four or more cherry scones. This will leave us with the desired number of combinations.

Therefore, forget the cherry constraint for the time being. We are constrained to have at least one plain, at least two raisin, at least three blueberry, at least one raspberry, and at least two apple. In our two dozen total, we can choose freely only 15 scones, from among our six kinds, with repetition. Thus, we do C(6 + 15 – 1, 15) = C(20, 15); alternatively, we can do C(6 + 15 – 1, 6 – 1) = C(20, 5). Next we know that if we have at least four cherry scones, then we can select freely only 11 scones, from among our six kinds, with repetition. Thus, we do C(6 + 11 – 1, 11) = C(16, 11); again, we can alternatively do C(6 + 11 – 1, 6 – 1) = C(16, 5).

Therefore, we can get the total number of scones meeting all six original constraints:

C(20, 15) – C(16, 11)

4)

a) P(10, 5), which is 10*9*8*7*6. Another way to look at it is that there are C(10,5) ways to select the 5 digits for the number, then 5! ways to permute each selection.

b) Going right-to-left, there are 5 ways to choose the 1's digit, then for each of those there are 9 ways to choose the 10's digit, etc., so instead of the product in part (a) it's 5*9*8*7*6. Another way to look at it is 5*P(9,4). Alternately, since the number must be odd there can only be half as many numbers as part (a).

c) Now there are only 4 choices for the 1's digit, then 8 for the 1's digit, etc., so it's 4*8*7*6*5, or 4*P(8,4).

5) Since we use each digit twice, we are constructing 20-digit numbers. When choosing places for any pair (e.g. the two 1's), we have to be sure that we don't let the order matter. Without the "properly written" constraint, we would say that there are C(20,2) places to put the 0's, and for each such selection, there are C(18,2) places to put the 1's, etc. Adding the constraint limits the first choice to C(19,2), but the others stay the same, so it's C(19,2)*C(18,2)*...*C(4,2)*C(2,2). Or, just leave off the last term, which is 1, since making choices for the first 9 pairs determines where the last pair will go.

6) NOTE: recognize that C(x, y) = C(x, x – y) (e.g., see part **(a)**).

a) The way to think about this is that we are selecting four tosses out of 10 in which to toss coins heads up. The rest will be tails. Thus, our final answer is

C(10, 4)

Doing this from the point of view of the tails would've given the same answer because C(10, 6) = C(10, 4).

b) We can compute this directly for 3 heads, 4 heads, …, 10 heads. Or we can start from the total number of possible flip combinations and then subtract 0 heads, 1 head, and 2 heads from it. Let's do the latter. There are 2^{10} possible flip orders. For 0 heads there is one possible way (C(10, 0) = 1), for one there is C(10, 1) = 10 ways, for two there are C(10, 2) = 45 possible ways. Thus, our answer is

$$2^{10} – C(10, 2) – C(10, 1) – C(10, 0)$$

7) First, calculate the number of possible seatings of the five good kids, and then figure out how to seat the brats with them without causing a problem. We can order the five good kids in 5! ways. For each of these arrangements, the brats now have to be seated. There are six possible positions (the two ends and the four spaces between two good boys) for the brats, and only one brat can go in any one of these positions. There are P(6, 3) ways of ordering the 3 bad boys in the 6 positions, so the total number of arrangements is 5! * P(6, 3) = 5! * 6 * 5 * 4. Another way to look at this is that there are C(6, 3) ways to choose the 3 spots the bad boys will go in, and there are 3! orderings of the 3 boys for each, for a total of 5! * 3! * C(6,3). Since P(6, 3) = 3! * C(6, 3), we get the same answer as the first method:

$$5! * 6 * 5 * 4$$

In case you're curious, this works out to be 14,400.

8) The problem is that we are double-counting every hand once. The correct number is actually 13^4 * 24. Let us show where the problem arises.

Suppose we begin with the suits separated into distinct piles and select one card from each suit pile before reshuffling the suits into one stack and selecting a single card from there. This is the selection method suggested by the answer 13^4 * 48. We have 13 possibilities for each suit pile and then once those are selected and the cards reshuffled, we have 48 possibilities. However, using this method, we can get any hand one of two different ways. Take for example the following hand: A of clubs, K of diamonds, 2 of hearts, 3 of hearts, and 6 of spades. We could get this in one of two ways:

CLUBS:	A	CLUBS:	A
DIAMONDS:	K	DIAMONDS:	K
HEARTS:	2	HEARTS:	3
SPADES:	6	SPADES:	6
RANDOM:	3 of hearts	RANDOM:	2 of hearts

Thus, every hand can be obtained in one of two ways and is being double-counted by this method.

The proper way to understand this question is to understand it as never reshuffling the cards but instead choosing one suit from which to select two cards, so that their order of selection is never important. Thus the above hand is selected as follows

CLUBS:	A
DIAMONDS:	K
HEARTS:	2, 3
SPADES:	6

In the case where we select two cards from hearts, there are $C(13, 2) * 13^3$ possible distinct hands. For the total number of distinct hands, we multiply that times four (once per suit), giving us $4*C(13, 2)*13^3$. If we work out this math, we get

$$\begin{aligned} 4*C(13, 2)*13^3 &= 4* 13!/(11!2!) * 13^3 \\ &= 4* 6 * 13^4 \\ &= 13^4 * 24 \end{aligned}$$

Given that we said that 13^4*48 double counts, this answer makes complete sense. Furthermore, we observe that the answer 13^4*48 can be obtained from $4*P(13, 2)*13^3$

$$\begin{aligned} 4*P(13, 2)*13^3 &= 4* 13!/(11!) * 13^3 \\ &= 4* 12 * 13^4 \\ &= 13^4 * 48 \end{aligned}$$

So the reason cards were double counted was that order was taken to be important when it should not have been. This suggests an interesting relationship between permutations and combinations: a combination is equal to its corresponding permutation divided by the number of variations in the order of a particular choice of elements, which is itself equal to the number of elements chosen factorial.

9)

a) P(118, 16) = 118! / 102!

b) C(118, 10) = 118! / (108! · 10!)

c) C(10, 2) · C(8, 2) · C(6, 2) · C(4, 2) / 5!

d) C(10, 2) · C(8, 2) · C(6, 2) · C(4, 2) / 4!

10)

a) C(13,1) · C(4,3) · C(12,1) · C(4,2)
b) (52·48·44·40·36) / 5! Another way to get the result is $C(13, 5) * 4^5$

11)

a) C(7, 4), or equivalently, C(7, 3)
b) Every shortest route consists of 7 segments between junctions, with 4 horizontal segments and 3 vertical segments. If we use '0' to denote a horizontal segment and '1' to denote a vertical segment, then each route corresponds to a binary sequence of length 7 that contains four 0's and three 1's and vice versa. Thus there is a bijection from the set of such sequences to the set of shortest routes, and the number of routes is the same as the answer for part (a).