Handout #65 CS103A
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Problem Set #9 Solutions

1) $a_1 = 6$, $a_{n+1} = a_n + 6$ for $n \geq 1$

2) P(n): $$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{\ }$$

Base Case: P(0) asserts that $x^0 = \frac{1 - x^1}{1 - x}$ Since both sides are 1, this is true.

Inductive step: Assume P(k): $$\sum_{i=0}^{k} x^i = \frac{1 - x^{k+1}}{1 - x}$$

Show P(k+1): $$\sum_{i=0}^{k+1} x^i = \frac{1 - x^{k+2}}{1 - x}$$

By the definition of summation

$$\sum_{i=0}^{k+1} x^i = \sum_{i=0}^{k} x^i + x^{k+1}$$

Applying the Inductive Hypothesis

$$\sum_{i=0}^{k+1} x^i = \frac{1 - x^{k+1}}{1 - x} + x^{k+1}$$

$$= \frac{1 - x^{k+1} + x^{k+1}(1-x)}{1 - x}$$

$$= \frac{1 - x^{k+1} + x^{k+1} - x^{k+2}}{1 - x} = \frac{1 - x^{k+2}}{1 - x}$$

QED

3) We know the recursive formula $H_1 = 1$, $H_n = 2 \cdot H_{n-1} + 1$. By writing out the first few values $H_1 = 1$, $H_2 = 3$, $H_3 = 7$, $H_4 = 15$ … we can guess that closed-form formula $H_n = 2^n - 1$ will hold for all n. We now prove this is correct inductively.

Base Case: For n = 1, we get $H_1 = 2^1 – 1 = 2 – 1 = 1$, which is correct by the definition of the recursive formula.

Inductive case: Assume P(k), that is, $H_k = 2^k – 1$ for some $k \geq 1$. We must prove P(k + 1), that is, $H_{k+1} = 2^{k+1} – 1$.

$H_{k+1} = 2 \cdot H_k + 1$	def. of H_{k+1}, since $k + 1 > 1$
$H_{k+1} = 2 \cdot (2^k – 1) + 1$	inductive hypothesis
$H_{k+1} = 2^{k+1} – 2 + 1$	algebra
$H_{k+1} = 2^{k+1} – 1$	algebra

Therefore, $P(k) \rightarrow P(k + 1)$ for all $k \geq 1$. Since we have proved P(1) holds, P(n) therefore holds for all $n \geq 1$, and therefore $H_n = 2^n – 1$ for all $n \geq 1$.

4)

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Add(a, b):
    if (a = 0)
        return b
    else
        return Add(a -1, b+ 1)
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There are other equally good solutions.

5)

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Largest(S):
    if (Length(S) = 1)
        return First(S)
    else if (First(S) ≥ Largest(Rest(S))
        return First(S)
    else
        return Largest(Rest(S))
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6)

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Concat(S1, S2):
    if (S1 = ∅)
        return S2
    else
        return Cons(First(S1), Concat(Rest(S1), S2))
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7) PROOF BY NORMAL INDUCTION:

$P(N)$: $5 \mid (a+b)$ when $(a,b) \in S$

Induction will be on the number of applications of the recursive step.

Base case: $P(0)$ is true since $5 \mid 0$ (after 0 applications we have the only element in the set we start with $(0,0)$).

Inductive Hypothesis: Assume $P(i)$ where $0 \le i \le k$ and prove $P(k+1)$: on the $k+1$ application of the recursive step $5 \mid (a+b)$ when $(a,b) \in S$

Proof:

Consider an element obtained with $k+1$ applications of the recursive step. Since the final application to an element (a,b) must be applied to an element created with fewer applications of the recursive step, we know that $5 \mid a+b$ by the inductive hypothesis. We just need to check that this holds when the final recursive step is applied: $5 \mid a + 2 + b + 3$ or $5 \mid a + 3 + b + 2$. Both are true since $5 \mid a+b$ and $5 \mid 5$. QED

PROOF BY STRUCTURAL INDUCTION:

We want to prove $P(t)$: $5 \mid (a+b)$ whenever $t = (a, b) \in S$

Base case: The property $P((0, 0))$ is true since $5 \mid 0$, therefore P is true for the only primitive element of S, $(0, 0)$.

Inductive Step: Assume $P((a, b))$ holds for some (a, b). We must show that application of either recursive rule to (a, b) preserves the property P.

Rule 1: $(a + 2, b + 3) \in S$, so we must prove $P(a + 2, b + 3)$.

$(a + 2) + (b + 3) = a + b + 5$	algebra
$a + b = 5d$ for some d	(by inductive hypothesis that $P((a, b))$ holds)
$(a + 2) + (b + 3) = 5d + 5$	combine previous two lines
$5 \mid (a + 2) + (b + 3)$	property of divides

Rule 2: $(a + 3, b + 2) \in S$, so we must prove $P(a + 3, b + 2)$.

$(a + 3) + (b + 2) = a + b + 5$	algebra
$a + b = 5d$ for some d	(by inductive hypothesis that $P((a, b))$ holds)
$(a + 3) + (b + 2) = 5d + 5$	combine previous two lines
$5 \mid (a + 3) + (b + 2)$	property of divides

Therefore, since P holds for the primitive element and applying the recursive rules preserves P, P holds for all elements in S by the principle of structural induction.

8) $P_m(n)$ is the property that $F_{m+n} = F_{m-1} \cdot F_n + F_m \cdot F_{n+1}$, given any positive integer m. We want to show that for any positive integer m, $P_m(n)$ holds for all $n \geq 1$.

Base case #1-$P_m(1)$:	$F_{m+1} = F_{m-1} + F_m$	definition of Fibonacci sequence
	$F_{m+1} = F_{m-1} \cdot 1 + F_m \cdot 1$	algebra
	$F_{m+1} = F_{m-1} \cdot F_1 + F_m \cdot F_2$	replace with $F_1 = 1$ with $F_2 = 1$

Base case #2-$P_m(2)$:	$F_{m+2} = F_{m+1} + F_m$	definition of Fibonacci sequence
	$F_{m+2} = (F_{m-1} + F_m) + F_m$	definition of Fibonacci sequence
	$F_{m+2} = F_{m-1} \cdot 1 + F_m \cdot 2$	algebra
	$F_{m+2} = F_{m-1} \cdot F_2 + F_m \cdot F_3$	replace with $F_2 = 1$ with $F_3 = 2$

Inductive step: Assume $P_m(i)$ holds for $1 \leq i \leq k$ for some $k \geq 2$. We will prove the property $P_m(k + 1)$, which is that $F_{m+k+1} = F_{m-1} \cdot F_{k+1} + F_m \cdot F_{k+2}$

$F_{m+k+1} = F_{m+k-1} + F_{m+k}$	definition of Fib. sequence
$F_{m+k+1} = (F_{m-1} \cdot F_{k-1} + F_m \cdot F_k) + (F_{m-1} \cdot F_k + F_m \cdot F_{k+1})$	inductive hypothesis
$F_{m+k+1} = (F_{m-1} \cdot F_{k-1} + F_{m-1} \cdot F_k) + (F_m \cdot F_k + F_m \cdot F_{k+1})$	algebra
$F_{m+k+1} = F_{m-1} \cdot (F_{k-1} + F_k) + F_m(F_k + F_{k+1})$	factoring
$F_{m+k+1} = F_{m-1} \cdot F_{k+1} + F_m \cdot F_{k+2}$	factoring

Therefore, since we've proved $P_m(k + 1)$ for $k \geq 2$ if $P_m(i)$ holds for $1 \leq i \leq k$, and P(1) and P(2) hold, then by the principle of strong mathematical induction $P_m(n)$ holds for all $n \geq 1$.

NOTE: A number of students did not include P(2) as a base case. This is needed, however, as the inductive proof relies on the inductive hypothesis applying to the previous two numbers in sequence. This is similar to the flaw in the inductive proof in Problem Set 8, #8.

9) Base case $() \in S$.

Recursive rule #1: If $r \in S$, then $(r) \in S$
Recursive rule #2: If $q, r \in S$, then $qr \in S$

10) Base cases: $\varnothing \in S$
$0 \in S$
$1 \in S$

Recursive rule #1: If $r \in S$, then $0r0 \in S$
Recursive rule #2: If $r \in S$, then $1r1 \in S$

11) We are given that $a_1 = 3$ and $a_n = a_{n-1} + 2n$ for $n > 1$. We want to prove that $a_n = n^2 + n + 1$ is a correct closed-form formula, we will do so inductively.

Base case: $a_1 = 1^2 + 1 + 1 = 3$ (true by definition of a_1)

Inductive case: Assume P(k) is true for some $k \geq 1$, that is $a_k = k^2 + k + 1$. We will prove P(k + 1), that is $a_{k+1} = (k + 1)^2 + (k + 1) + 1$.

$a_{k+1} = a_k + 2(k+1)$	recursive definition of sequence
$a_{k+1} = k^2 + k + 1 + 2(k+1)$	application of inductive hypothesis
$a_{k+1} = (k^2 + 2k + 1) + (k+1) + 1$	algebra
$a_{k+1} = (k + 1)^2 + (k+1) + 1$	algebra

Therefore, since $P(k) \rightarrow P(k + 1)$ and P(1) holds, P(n) is true for all $n \geq 1$ by the principle of mathematical induction, and therefore the closed-form formula is correct.