Gödel's Incompleteness Theorem

Kurt Gödel (1906 – 1978)

Gödel Numbering

$(\; \exists \; x \;) \; (\; x \; = \; s \; y \;)$

$2^{8} \cdot 3^{4} \cdot 5^{11} \cdot 7^{9} \cdot 11^{8} \cdot 13^{11} \cdot 17^{5} \cdot 19^{7} \cdot 23^{13} \cdot 29^{9}$

This scheme allows us to represent every formula with a unique number. Given a number, we can determine whether it is a Gödel number, and if so, we can recover the formula, since every number has a unique prime factorization.

Correspondence between syntactic properties of sentences and arithmetic properties of Gödel numbers

S: $(\; \exists \; x \;) \; (\; x \; = \; s \; y \;)$

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Syntactic remark:	S begins with '('
Arithmetic statement:	The Gödel number is S is divisible by 2^8 but not by 2^9

Correspondence between syntactic properties of sentences and arithmetic properties of Gödel numbers

Gödel showed that all of the important syntactic notions of first-order logic can be represented in the language of Peano Arithmetic, such as:

n is the Gödel number of a wff

n is the Gödel number of a sentence

n is the Gödel number of an axiom of Peano Arithmetic

n is the Gödel number of a proof

Now consider the statement

'The sequence of formulas with the Gödel number x is a proof of the formula with Gödel number z.'

This statement is represented by a definite formula in the arithmetic calculus that expresses a purely arithmetic relation between x and z.

So the truth or falsity of the statement hangs on whether x and z bear the proper arithmetic relationship.

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So the truth or falsity of the statement hangs on whether x and z bear the proper arithmetic relationship.

We will denote this relationship with the name 'Dem', so we will write the sentence above as

Dem(x, z)

Note that even though the statement at the top is a meta-mathematical statement, there is a (complicated) formula for Dem in the arithmetic calculus.

We need one more notation, concerning substitution for a variable:

Suppose m is the Gödel number of the formula (∃x)(x = sy)

Subsitute in this formula for the variable with Gödel number 13 (i.e., y) the numeral for m.

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We could calculate the Gödel number of that formula, but we can also identify it by the unambiguous meta-mathematical characterization:

> The Gödel number of the formula that is obtained from the formula with Gödel number m by substituting the numeral m for the variable with Gödel number 13.

This determines a definite number that is a certain arithmetic function of the numbers m and 13. We will denote it by

sub(m, 13, m)

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More generally, we will write

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Example:

x = s y has Gödel number $2^{11} \cdot 3^{5} \cdot 5^{7} \cdot 7^{13}$

Unlike the pattern above, suppose we just substitute the numeral for 1 for the variable with Gödel number 13

sub($2^{11} \cdot 3^{5} \cdot 5^{7} \cdot 7^{13}$, 13, 1) is $2^{11} \cdot 3^{5} \cdot 5^{7} \cdot 7^{7} \cdot 11^{6}$ since the numeral for 1 is s0

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Example:

x = s y

If we did sub($2^{11} \cdot 3^{5} \cdot 5^{7} \cdot 7^{13}$, 13, numeral for $2^{11} \cdot 3^{5} \cdot 5^{7} \cdot 7^{13}$), we would get the Gödel number of a formula with a lot of s's at the end followed by one 0.

Now consider the formula

(1) (x)~Dem(x, sub(y, 13, y)) where (x) means "for all x"

(1) says that the formula with Gödel number sub(y, 13, y) is not demonstrable, i.e., that it cannot be proved

(1) has a Gödel number, which we will call n.

We now construct the sentence G, named for Gödel, by substituting n for the variable y in (1):

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13 is the numeral for y

n is the Gödel number for (1)

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What is sub(n, 13, n)?

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So G says:

The formula with Gödel number the Gödel number of G is not provable.

I.e., G is true iff G is not provable.

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Suppose Peano Arithmetic is a sound theory, i.e., that it proves no falsehoods (because its axioms are true and its logic is truth-preserving).

If G (which is true if it is not provable) could be proved in PA, then the theory would prove a false theorem, contradicting our supposition of soundness.

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Thus G is true.

Thus ~G is false, and ~G cannot be proved in PA either.

In Gödel's words, G is a "formally undecidable" sentence in PA, i.e,

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What if we drop the assumption of soundness?

A theory is inconsistent if for some sentence φ, we can prove both φ and $\sim\varphi$.

In the First Incompleteness Theorem, Gödel goes on to show that the consistency of a formal theory T sufficient to encompass arithmetic guarantees that there is a sentence G_T , couched in the language of basic arithmetic, such that

(i) Neither G_T nor $\sim G_T$ can be derived in T, and yet
(ii) from the way G was constructed, we can recognize that G_T is true.

Furthermore, if we try to patch up T by adding G_T as an axiom, we could just repeat Gödel's procedure to come up with another such sentence. So T is not just incomplete, but incompletable.

Gödel's Second Incompleteness Theorem

Gödel was able to show that the claim that "if T is consistent, then G_T is unprovable" can itself be encoded as a numerical proposition,

$$\mathrm{Con}_T \to G_T$$

and that we can, if T is consistent, derive that sentence inside T.

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and that we can, if T is consistent, derive that sentence inside T.

But the First Theorem showed that if T is consistent, we can't derive G_T in T. So it follows that we can't derive Con_T in T, i.e., that

Theories that include basic arithmetic can't prove their own consistency.

A little history of the announcement

Gödel announced his incompleteness result on Sept. 7, 1930, at a meeting on the Epistemology of the Exact Sciences at Königsberg, Germany. Only John von Neumann, who had worked with Hilbert, immediately grasped the significance.

Ironically, Hilbert was at the meeting, and delivered a famous lecture, "Logic and the undertanding of nature" the next day. It was his last major public appearance, and he concluded with *"Wir müssen wissen! Wir werden wissen!"*

Apparently, the two men never spoke to each other.

The German quotation alludes to the saying "ignoramus et ignorabimus", "we do not know and we shall never know", which was used to express a pessimistic view of the limits of scientific knowledge in the 19th century.

Minds and Computers

J.R. Lucas (1961):

> However complicated a machine we construct, it will, if it is a machine, correspond to a formal system, which in turn will be liable to the Gödel procedure for finding a formula unprovable in that system. This formula the machine will unable to produce as true, although the mind can see it is true.

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This rests on the mistaken idea that we can see the Gödel formula to be true. But the theorem says no such thing. It says that the formula is true if and only if the system is consistent. But in general, we don't know if a formal system is consistent, and so we are uncertain as to the truth of the Gödel formula.

Lucas' remark is intended to place some sort of restriction on what is possible in the field of Artificial Intelligence. But humans were behaving intelligently for thousands of years before they invented mathematics, so it is unlikely that mathematical reasoning plays more that a peripheral role in intelligence. (Russell and Norvig)

Incompleteness Outside Mathematics (from Franzén)

Many references to the incompleteness theorem outside the field of formal logic are rather obviously nonsensical and appear to be based on gross misunderstandings or some process of free association.

Examples:

" Gödel's incompleteness theorem shows that it is not possible to prove that an objective reality exists."

"By Gödel's incompleteness theorem, all information is innately incomplete and self-referential."

"By equating existence and consciousness, we can apply Gödel's incompleteness theorem to evolution."

Incompleteness Outside Mathematics (from Franzén)

"Consistent", "inconsistent, "complete", "incomplete", and "system" are words used not only in a technical sense and logic, but in various senses in ordinary language, and so it is not surprising that the incompleteness theorem has been thought to have a great many applications outside mathematics.

Example:

"As Gödel demonstrated, all consistent formal systems are incomplete, and all complete formal systems are inconsistent. The U.S. Constitution is a formal system, after a fashion. The founders made the choice of incompleteness over inconsistency, and the Judicial Branch exists to close that gap of incompleteness."

What are the implications?

Gödel's work did send shockwaves through Mathematics. After reigning for centuries as the embodiment of certainty, Mathematics had lost that role.

Gödel's work is certainly among the most important in Mathematics, and perhaps in the history of thought, but as we have seen, it does not support many of the conclusions that people attempt to draw from it.

In a way, the most interesting outcome is how little Mathematics has changed. Most mathematicians find that in their everyday work, they don't encounter sentences that state that they themselves are unprovable.

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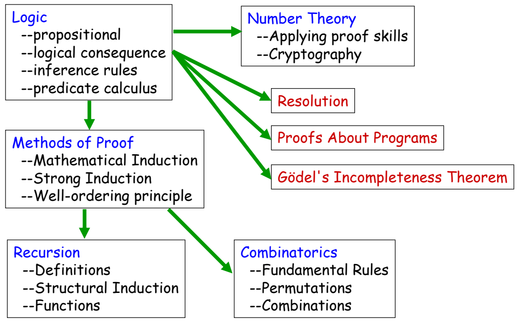
Mathematics is still useful, beautiful, and interesting. It compels us to think about great ideas and difficult problems.

Summary of CS103A



Logic
--propositional
--logical consequence
--inference rules
--predicate calculus

Number Theory
--Applying proof skills
--Cryptography

Resolution

Proofs About Programs

Gödel's Incompleteness Theorem

Methods of Proof
--Mathematical Induction
--Strong Induction
--Well-ordering principle

Recursion
--Definitions
--Structural Induction
--Functions

Combinatorics
--Fundamental Rules
--Permutations
--Combinations

Summary of CS103A

What you should come away with:

- An understanding of logic, consequence, and proof
- The ability to do real mathematical proofs
- A set of powerful proof techniques:
  - -- The rules of logic, and how to structure proofs
  - -- Proof by contradiction, and proving the contrapositive
  - -- Proof by induction
  - -- The well-ordering principle
  - -- Rules for solving combinatoric problems
  - -- An understanding of recursive definitions and functions
- Knowledge of some great ideas: cryptography, the incompleteness theorems
- A realization that much of the progress in Computer Science comes from theory and proofs, not programming.

Final Exam

Number theory through and including Gödel

Wednesday, December 10, 8:30 AM

Gates B3

Review session on Friday, 12:45 – 2:05, Gates B3
(available later online)