


CS103A


12/1/08

Gödel's Incompleteness Theorem



Kurt Gödel (1906 – 1978)

Gödel's Incompleteness Theorem



Kurt Gödel (1906 – 1978)

👉

Gödel, Kurt (1931). Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik*, 38, 173-198.

Gödel, Kurt (1931). Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik*, 38, 173-198.

On formally undecidable propositions of Principia Mathematica and related systems I

References:

Gödel's Proof
by Ernest Nagel, James R. Newman, Douglas R. Hofstadter (Editor)
NYU Press; Revised edition, 2001

Wikipedia
Gödel
Gödel's incompleteness theorems

An Introduction to Gödel's Theorems
by Peter Smith
Cambridge University Press, 2007

Gödel's Theorem: An Incomplete Guide to Its Use and Abuse
by Torkel Franzén
A K Peters, 2005

God Created the Integers
by Stephen Hawking (Editor) (contains Gödel's paper)
Running Press, 2007

Mathematics is deductive, not experimental

The axiomatic method and the notion of logical proof goes back to the Greeks

--A small number of axioms and some rules of inference are used to produce all other theorems in the system

It was hoped that adequate systems of axioms could be found for each area of mathematics. For example, Euclid's axioms of geometry, or Peano's axioms of arithmetic.

Mathematics is deductive, not experimental

The axiomatic method and the notion of logical proof goes back to the Greeks

--A small number of axioms and some rules of inference are used to produce all other theorems in the system

It was hoped that adequate systems of axioms could be found for each area of mathematics. For example, Euclid's axioms of geometry, or Peano's axioms of arithmetic.

Gödel showed that this could not be done. There are inherent limitations in such systems.

Mathematical Progress in the 19th Century

The Greeks had proposed three problems in geometry that had not been solved for 2000 years:

- to trisect any angle with a compass and straight edge
- to construct a cube with a volume twice the volume of a given cube
- to construct a square equal in area to that of a given circle.

Mathematical Progress in the 19th Century

The Greeks had proposed three problems in geometry that had not been solved for 2000 years:

- to trisect any angle with a compass and straight edge
- to construct a cube with a volume twice the volume of a given cube
- to construct a square equal in area to that of a given circle.

In the 19th century it was proved that these constructions are logically impossible.

Mathematical Progress in the 19th Century

Even more important was another problem:

Euclid's fifth axiom is equivalent to the assumption that through a point outside a given line only one parallel line can be drawn.

Unlike the other axioms, this did not seem self-evident to the Greeks.

They tried, unsuccessfully, to deduce it from the other for axioms.

Mathematical Progress in the 19th Century

Even more important was another problem:

Euclid's fifth axiom is equivalent to the assumption that through a point outside a given line only one parallel line can be drawn.

Unlike the other axioms, this did not seem self-evident to the Greeks.

They tried, unsuccessfully, to deduce it from the other for axioms.

In the 19th century, the impossibility of deducing the parallel axiom from the others was demonstrated.

This was important because

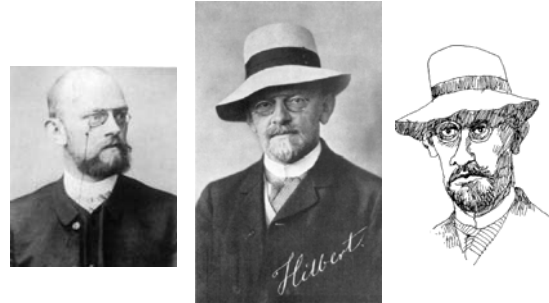
- it showed that it was possible to **prove the impossibility of proving certain propositions within** a given system
- it showed that there was more to geometry than Euclid

Mathematical Progress in the 19th Century

The emergence of non-Euclidean geometries, where Euclid's axioms were not true, raised interesting questions:

- Were the axioms for the non-Euclidean geometries consistent? (How could you prove that they didn't lead to contradictory theorems?)
- Were the Euclidean axioms consistent?

David Hilbert (1862 – 1943)



Hilbert tried to show the consistency of Euclid's axioms by transforming them into algebraic truths.

- point \Leftrightarrow pair of coordinates
- straight line \Leftrightarrow first-degree equations with two unknowns
- circle \Leftrightarrow quadratic equation of a certain form

The geometric statement that two distinct points uniquely determine a straight-line has been transformed in the algebraic truth that two distinct pairs of numbers uniquely determine a linear relation.

This just changed the problem: geometry is consistent if algebra is consistent. It was a proof of relative consistency, not absolute consistency.

Hilbert moved away from the belief that axioms can be chosen because they are self-evident .

He realized that the role of the mathematician is to draw conclusions that logically follow from a set of axioms.

It is not the concern of the mathematician to determine whether the axioms are actually true.

Hilbert moved away from the belief that axioms can be chosen because they are self-evident .

He realized that the role of the mathematician is to draw conclusions that logically follow from a set of axioms.

It is not the concern of the mathematician to determine whether the axioms are actually true.

The validity of a mathematical inference does not depend on any special meaning that may be associated with the terms or expressions contained in the axioms.

Mathematics was seen as abstract and formal rather than the "science of quantity".

Hilbert sought to prove absolute consistency by formalizing the deductive system.

--Remove the meaning from expressions occurring in the system; regard them simply as empty signs.

--Precisely state the rules by which expressions can be combined and manipulated.

Hilbert sought to prove absolute consistency by formalizing the deductive system.

- Remove the meaning from expressions occurring in the system; regard them simply as empty signs.
- Precisely state the rules by which expressions can be combined and manipulated.

Once this is done, proving a theorem from axioms is just a transformation, according to the rules, of one set of strings into another.

Hilbert hoped to show that contradictory formulas could not be obtained within a formal system by examining the structural properties of the system.

In 1920, Hilbert proposed a research project that came to be known as **Hilbert's Program**. He wanted to achieve:

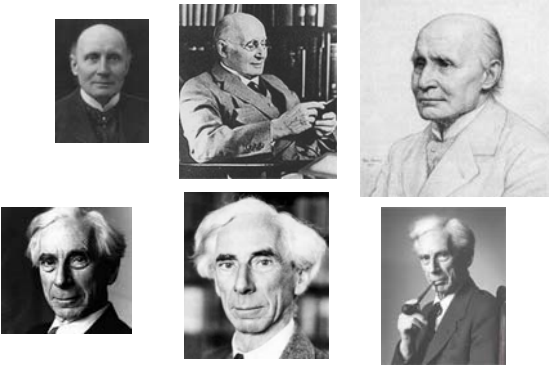

- A formalization of all of mathematics
- A proof that mathematics is consistent (no contradictions can be obtained).
- A proof that mathematics is complete (all true statements can be proved in the formalism).
- An algorithm for deciding the truth or falsity of any statement and thus the correctness of any proof.

In 1920, Hilbert proposed a research project that came to be known as **Hilbert's Program**. He wanted to achieve:

- A formalization of all of mathematics
- A proof that mathematics is consistent (no contradictions can be obtained).
- A proof that mathematics is complete (all true statements can be proved in the formalism).
- An algorithm for deciding the truth or falsity of any statement and thus the correctness of any proof.

Gödel showed that this was not possible.


Alfred North Whitehead (1861-1947) and Bertrand Russell (1872-1970)

Written as a defense of logicism (i.e., the view that mathematics is in some significant sense reducible to logic).

Instrumental in developing and popularizing modern mathematical logic.

Served as a major impetus for research in the foundations of mathematics throughout the twentieth century.



Written as a defense of logicism (i.e., the view that mathematics is in some significant sense reducible to logic).


Instrumental in developing and popularizing modern mathematical logic.

Served as a major impetus for research in the foundations of mathematics throughout the twentieth century.


Logicism

- all mathematical truths can be translated into logical truths
- all mathematical proofs can be recast as logical proofs

The logicist's goal is "to show that all pure mathematics follows from purely logical premises and uses only concepts definable in logical terms."



Whitehead and Russell thought the project might take a year to complete.



Whitehead and Russell thought the project might take a year to complete.
They were done in a decade.

Completeness

Logical consequence: a sentence is a logical consequence of a set of premises if there is no way for the premises to be true without the conclusion also being true.

The question of **completeness** is whether our methods of proof are sufficient so that we can always find a proof of such a conclusion from the premises.

Completeness

The first completeness proof was given by Gödel in his thesis in 1929, for a system much like our FOL.

It is also possible to prove the FOL is sound: if there is a proof of S from a set of sentences P, then S is a first-order consequence of P.

To say FOL is complete and sound is to say we can prove all the theorems and only the theorems.


Completeness

The first completeness proof was given by Gödel in his thesis in 1929, for a system much like our FOL.


It is also possible to prove the FOL is sound: if there is a proof of S from a set of sentences P, then S is a first-order consequence of P.

To say FOL is complete and sound is to say we can prove all the theorems and only the theorems.

FOL, however, is not powerful enough to develop even elementary arithmetic. When we move to more powerful systems, things don't go so well...



Now, at last,
The Incompleteness Theorem



Now, at last,

The Incompleteness Theorem

The Incompleteness Theorem actually involves a slightly different notion of completeness.

We say that a set of sentences T is formally complete if for any sentence S in the language, either S or $\neg S$ is provable from T .

The set of sentences in question are the axioms of Peano arithmetic, and Gödel showed that Peano arithmetic is not formally complete.

To begin, we need to realize that as well as writing expressions within a formal system, we can make statements about that system and those expressions.

For example, the expression

$$2 + 3 = 5$$

belongs to mathematics, but the statement

' $2 + 3 = 5$ ' is an arithmetical formula

does not express an arithmetical fact and does not belong to the formal language of arithmetic. It belongs to **meta-mathematics**.

The formulas

$$x = x$$

$$0 = 0$$

belong to mathematics, but the statement

' x is a variable

belongs to meta-mathematics, since it characterizes a certain arithmetical sign as belonging to a specific class of signs, namely, variables.

The formulas

$$x = x$$

$$0 = 0$$

belong to mathematics, but the statement

' x is a variable

belongs to meta-mathematics, since it characterizes a certain arithmetical sign as belonging to a specific class of signs, namely, variables.

The following statement also belongs to meta-mathematics:

The formula ' $0 = 0$ ' is derivable from the formula ' $x = x$ ' by substituting the numeral '0' for the variable ' x '.

Gödel devised a scheme that let him express meta-mathematical statements as purely arithmetical relations.

This provided a way to make statements about the system inside the system itself.

Gödel devised a scheme that let him express meta-mathematical statements as purely arithmetical relations.

This provided a way to make statements about the system inside the system itself.

The starting point is a way to assign a unique number to every sign, formula (sequence of signs), and proof (sequence of formulas). We call these **Gödel numbers**.

Constant sign	Gödel number	Usual meaning
\sim	1	not
\vee	2	or
\supset	3	if...then...
\exists	4	there exists
$=$	5	equals
0	6	zero
s	7	successor of
(8	punctuation
)	9	punctuation
,	10	punctuation

There are three kinds of variables. First are numerical variables like 'x', 'y', and 'z', for which numerals and numerical expressions can be substituted.

Numerical variable	Gödel number	Possible substitution
x	11	0
y	13	s0
z	17	y

Numerical variables are associated with primes greater than 10.

Next are sentential variables like 'p', 'q', and 'r', for which formulas (sentences) can be substituted.

Sentential variable	Gödel number	Possible substitution
p	11^2	$0 = 0$
q	13^2	$(\exists x)(x = sy)$
r	17^2	$p \supset q$

Sentential variables are associated with squares of primes greater than 10.

Next are predicate variables like 'P', 'Q', and 'R', for which predicates like 'Prime' or 'Greater than' can be substituted.

Predicate variable	Gödel number	Possible substitution
P	11^3	Prime
Q	13^3	Composite
R	17^3	Greater than

Predicate variables are associated with cubes of primes greater than 10.

$(\exists x) (x = s y)$

$(\exists x) (x = s y)$
 8 4 11 9 8 11 5 7 13 9

$$\begin{aligned}
 & (\exists x) (x = sy) \\
 & 2^8 \cdot 3^4 \cdot 5^{11} \cdot 7^9 \cdot 11^8 \cdot 13^{11} \cdot 17^5 \cdot 19^7 \cdot 23^{13} \cdot 29^9
 \end{aligned}$$

$(\exists x) (x = sy)$
 $2^8 \cdot 3^4 \cdot 5^{11} \cdot 7^9 \cdot 11^8 \cdot 13^{11} \cdot 17^5 \cdot 19^7 \cdot 23^{13} \cdot 29^9$

This scheme allows us to represent every formula with a unique number. Given a number, we can determine whether it is a Gödel number, and if so, we can recover the formula, since every number has a unique prime factorization.

Note that if our calculus requires formulas like ' $p \wedge q$ ', we would just use ' $\sim(p \vee \sim q)$ ' instead, and instead of '3', we would use 'sss0' in determining the Gödel number of the formula.

Suppose m is the Gödel number of $(\exists x) (x = sy)$
 Suppose n is the Gödel number of $(\exists x) (x = s0)$

Suppose m is the Gödel number of $(\exists x) (x = sy)$
 Suppose n is the Gödel number of $(\exists x) (x = s0)$

Then the Gödel number of the sequence of formulas m, n is

$$2^m \cdot 3^n$$

Here is an example of going backwards:

$$\begin{aligned}
 & 243,000,000 \\
 & 64 \cdot 243 \cdot 15625 \\
 & 2^6 \cdot 3^5 \cdot 5^6 \\
 & 0 = 0
 \end{aligned}$$

Here is an example of going backwards:

$$\begin{aligned}
 & 243,000,000 \\
 & 64 \cdot 243 \cdot 15625 \\
 & 2^6 \cdot 3^5 \cdot 5^6 \\
 & 0 = 0
 \end{aligned}$$

Now a meta-mathematical statement about ' $0 = 0$ ' can be represented by an arithmetical statement (within the system) about 243,000,000.

Correspondence between syntactic properties of sentences and arithmetic properties of Gödel numbers

S: $(\exists x)(x = s y)$

$2^8 \cdot 3^4 \cdot 5^{11} \cdot 7^9 \cdot 11^8 \cdot 13^{11} \cdot 17^5 \cdot 19^7 \cdot 23^{13} \cdot 29^9$

Syntactic remark: S begins with '('

Arithmetic statement: The Gödel number of S is divisible by 2^8 but not by 2^9

Correspondence between syntactic properties of sentences and arithmetic properties of Gödel numbers

Argument A: S_1, S_2, \dots

Syntactic remark: Argument A has three steps

Arithmetic statement: The Gödel number of A is divisible by 2, 3, and 5, but not by any prime > 5 , and in its factorization, the exponents are Gödel numbers.

Correspondence between syntactic properties of sentences and arithmetic properties of Gödel numbers

Statement A: $(p \vee p) \supset p$

Statement B: $(p \vee p)$

Syntactic remark: B is an initial part of A
 \updownarrow

Arithmetic statement: The Gödel number of A is divisible by the Gödel number of B

Correspondence between syntactic properties of sentences and arithmetic properties of Gödel numbers

Gödel showed that all of the important syntactic notions of first-order logic can be represented in the language of Peano Arithmetic, such as:

- n is the Gödel number of a wff
- n is the Gödel number of a sentence
- n is the Gödel number of an axiom of Peano Arithmetic
- n is the Gödel number of a proof
- n is the Gödel number of a proof of the sentence whose Gödel number is m

Consider the statement:

'n is the Gödel number of a proof of the sentence whose Gödel number is m'

This statement is represented by a definite formula in the arithmetic calculus that expresses a purely arithmetic relation between n and m.

So the truth or falsity of the statement hangs on whether n and m bear the proper arithmetic relationship.

Next time:

- The Incompleteness Theorem
- What does it say?
- What does it not say?

