

1 (Online). Five couples go to the movies together and sit in a row of ten seats. In how many ways can the 10 people be arranged if:

(a) They may sit in any order.

Solution

This is a basic permutation of all the seats:

$$P(10, 10) = 10!$$

(b) All the men sit together and all the women sit together.

Solution

In this case, we break the problem up into two parts. First, we consider the fact that we can place the gender blocks in one of two ways: men left, women right, or vice versa. Second, we consider the arrangements within those gender blocks, each of which is a smaller version of the same problem as part (a), a simple permutation. Thus, we get:

$$2 \cdot P(5, 5) \cdot P(5, 5) = 2 \cdot 5!^2$$

(c) Each couple sits together (i.e., for each couple, the two people are in adjacent seats).

Solution

This problem is similar to part (b) in that we are seating in blocks, except the blocks here are couples. First, we consider the different ways of arranging the couples themselves, which is a simple permutation of five blocks. Second, within each couple, there are two ways to seat individuals. Since there are five couples, we multiply times two five times:

$$P(5, 5) * 2 * 2 * 2 * 2 * 2 = 5! \cdot 2^5$$

(d) One couple is arguing and they refuse to sit together. The other couples can sit in any way—together or not.

Solution

Here we want to find the number of seating arrangements with the fighting couple sitting together and then subtract that from the total number of seating arrangements. We know the total number from part (a). The number of seating arrangements with the fighting couple sitting together is a three part problem. First, consider the number of places that couple can be seated (there are nine). Next, consider the number of ways to seat the remaining eight individuals (a simple permutation of eight). Finally, multiply by two for seating within the couple itself:

$$P(10, 10) - 9 \cdot P(8, 8) \cdot 2 = 10! - 9 \cdot 8! \cdot 2$$

2 (Online). This question concerns a standard deck of 52 playing cards. A deck of 52 cards consists of 4 suits and within each suit there are 13 cards of different ranks: Ace, 2, 3, ..., 10, Jack, Queen, King (the last three are referred to as face cards). A hand is a selection of some number of cards from the deck. Within a hand, the order of the cards is not important. Note: make sure that the expression for your final answer is clearly identified.

(a) How many different seven-card hands are there that contain only face cards?

Solution

There are 12 face cards, 3 in each of four suits. So: $C(12,7)$

(b) How many different seven-card hands are there that contain no 5's but have four cards of the same rank?

Solution

There are 12 ways to choose the rank, then 44 cards other than that rank and the 5's. So: $12 \cdot C(44,3)$

(c) How many different seven-card hands are there that contain two or more cards of the same rank?

Solution

There are $C(52,7)$ total hands. To subtract the ones that don't have pairs, we observe that such hands have cards of 7 different ranks, and there are $C(13,7)$ ways to select those. Then there are 4 choices for each card. So:

$$C(52,7) - C(13,7) \cdot 4^7$$

3 (Online). Consider the sequence of the first $2n$ positive integers. In how many ways can you order it such that no two consecutive terms have a sum divisible by 2?

Solution

Two consecutive terms will sum to an even number if both are odd or both are even, so consecutive terms must always have opposite parities. Since there will be n numbers of each parity, the only way to arrange them is to have all odd-numbered terms be odd and all even-numbered terms even, or vice versa, so 2 possibilities. After that decision, you can have any arrangement of the n odd and n even numbers, so the total number of ways is $2(n!)^2$.

4 [Pigeon-hole Principle] (Online). Prove that among any five points with integer coordinates in the plane, there are two such that the center of the line segment that connects them also has integer coordinates.

Solution

There are four possible cases for any point p in a plane: (Even, Even), (Even, Odd), (Odd, Even), (Odd, Odd). By pigeonhole principle, at least two of them must have the same parity. Since Even + Even = Even, and Odd + Odd = Even, if there are two points p_1, p_2 with the same parity, we will have $p_1 + p_2 = (\text{Even}, \text{Even})$, which means p_1 and p_2 have a midpoint with integer coordinates.

5. Challenge Problem

A stressed-out computer science professor consumes at least one espresso every day of a particular year, drinking 500 overall. Prove that on some consecutive sequence of whole days exactly 100 espressos were consumed.

Solution

Let a_i be the total number of drinks consumed up to and including the i th day, for $i = 1, \dots, 365$. Combine these with the numbers $a_1 + 100, \dots, a_{365} + 100$, providing $365 \times 2 = 730$ numbers, all positive and less than or equal to $500 + 100 = 600$. Hence two of these 730 numbers are identical. Since all a_i 's and, hence, $(a_i + 100)$'s are distinct, then $a_j = a_i + 100$, for some $i < j$. Thus, on days $i + 1$ to j , the person consumes exactly 100 drinks.

6. The game of Yahtzee! involves rolling 5 dice, each with 6 sides numbered 1 to 6, on each turn. For each of the following calculations, the ordering of the dice **does** matter, i.e. 1-2-3-4-5 is not considered the same roll as 5-4-3-2-1. . **(Exact solutions are shown just for interest—it is not necessary to multiply any of these out to a final answer.)**

a) How many total Yahtzee rolls are there?

Solution

This is an ordered selection of five values in the range (1-6), with repetition allowed. Thus there are $6^5 = 7776$ unique rolls

b) How many ways are there to roll a “large” straight (the dice can be re-arranged to make a sequence of five consecutive values)?

Solution

Unordered, the dice will have either the values 1-2-3-4-5 or the values 2-3-4-5-6, so there are 2 choices to make. Next, either choice can be permuted in $5!$ ways. So the total number is $(2 \cdot 5!) = 240$.

c) How many rolls contain at least two dice which have the same value?

Solution

It is easier to count complement of this set, that is, the number of rolls for which all five dice have different values. This is simply an ordered selection of 5 objects from a group of 6 without repetition, or $P(6, 5)$. All other rolls must contain at least two dice of the same value, so there are $6^5 - P(6, 5) = 7056$ such rolls.

d) How many ways are there to roll a four-of-a-kind (including five-of-a-kinds)?

Solution

The number of ways to get four-of-a-kind with the fifth value different is the number of values times the number of ways to place them among the five dice times the number of values for the fifth die. Add to this the 6 ways of getting five-of-a-kind:

$$6 \cdot C(5,4) \cdot 5 + 6 = 156$$

e) How many ways are there to roll a full house (three dice of the same value and the other two dice of the same value, which also includes five-of-a-kinds)?

Solution

First consider the number of full houses that are not five-of-a-kinds. There are 6 ways to choose the value for the threesome, then 5 ways to choose it for the pair, then $C(5,3)$ ways to place the threesome (which determines the placement of the pair). Add to this the 6 ways of getting five-of-a-kind:

$$6 \cdot 5 \cdot C(5,3) + 6 = 306$$

7 (Online). Consider a regular 4×4 grid of sixteen points, as in this picture:



How many triangles can be formed whose corners lie on the grid? A triangle has to have nonzero area.

Solution

The number of ways of picking three points from the grid is $16C3 = 560$. These three points will form a degenerate triangle (one with zero area) if and only if they all lie in a straight line. This straight line may be one of the rows, one of the columns, or one of the diagonals. The number of ways of picking three points, all from one row of the grid, is $4 \times 4C3 = 16$ (pick the row, then pick 3 points within the row). Symmetrically, the number of ways of picking three points, all from the same column, is 16. Now let's try the diagonals. There are three "bottom-left to top-right" diagonals with at least three points — the middle one has 4 points and the others have 3 each. The number of ways of picking three points, all from one of these diagonals, is $4C3 + 2 \times (3C3) = 6$. The situation for "bottom-right to top-left" diagonals is identical and also gives 6 possibilities. So the total number of ways of picking a degenerate triangle is $2 \times 16 + 2 \times 6 = 44$, and the number of ways of picking a non-degenerate triangle is $560 - 44 = 516$.

8 (Online). Prove that from a set of 10 distinct two-digit natural numbers, it is possible to select 2 disjoint nonempty subsets whose members have the same sum.

Solution See Review Session Broadcast (online)