Problem Set #10

Due: Wed., December 3 (Turn in by noon, Dec. 5 at Bob's office for one late day)

Combinatoric problems can be very misleading. They seem very simple, but may turn out to be surprisingly involved (of course, some of them *are* simple too). Many of the following problems will require careful thought and analysis to make sure you are counting everything, and at the same time, not counting too much.

The best way to solve these problems is to: a) reduce the numbers (i.e., try simpler cases, with fewer alternatives) so you can actually count what needs to be counted, and then watch for patterns and ways to generalize; and, b) map them to problems you have seen before. There are many examples of counting problems in the Combinatorics handout. These can serve as models for many of the problems below.

On all combinatoric problems, show your work so we can award partial credit. Proofs are not required here; just show us your reasoning in deriving solutions. Also, it is not necessary to multiply formulas out to get exact values. We prefer answers that have terms like C(10, 3) or 26^5 to answers that have the actual numeric values of these terms.

1) A shoe store has 30 styles of shoes. If each style is available in 12 different lengths, 4 different widths, and 6 different colors, how many pairs of shoes must be kept in stock in order to have one pair for every possibility?

2) a) How many binary numbers of length 8 have exactly four 0's or exactly three 1's?

b) How many binary numbers of length 10 contain either 5 consecutive 1's or 5 consecutive 0's?

3) A "scone shop" (only in Palo Alto....) has plain scones, raisin scones, blueberry scones, raspberry scones, apple scones, and cherry scones. The scones within a type are indistinguishable and the order of selection does not matter. You may assume that there are plenty of scones of each type. How many ways are there to choose:

a) 3 dozen scones?
b) 2 dozen scones with at least two of each type?
c) 2 dozen scones with at least one plain scone, at least two raisin scones, at least three blueberry scones, at least one raspberry scone, at least two apple scone, and no more than three cherry scones?

4) This problem concerns 5-digit decimal numbers, where numbers with leading 0's are allowed.

(a) How many 5-digit numbers are composed of distinct digits?
(b) How many odd 5-digit numbers are composed of distinct digits?
(c) How many odd 5-digit numbers composed of distinct digits do not contain the digit 9?

5) How many properly written decimal numbers (i.e., leading 0's are NOT allowed) use each digit exactly twice?

6) If a coin is tossed 10 times,

(a) how many ways are there of having exactly 4 heads and 6 tails?
(b) how many ways are there of having at least three heads?

7) You have just been given the charge of 8 eight-year-old boys for an afternoon. You could take them to a baseball game or a movie, but you decide to expand their horizons and take them to the ballet. Elliot, Brian, Matt, Miles, and Adam are fairly well-behaved boys (for eight-year-olds). But, Thomas, Alex and Connor are little time bombs ready to go off when you least expect it. It is critical that you do not allow any two of these three to sit next to each other. So, here you are at the ballet with 8 seats (assuming you are sitting in the row behind them). Find the number of ways of seating the 8 boys so that Thomas, Alex and Connor do not sit together.

8) What is wrong with the following argument, which claims to show that there are $13^4 \cdot 48$ (unordered) five-card poker hands containing cards of all four suits, and what is the correct answer?

Pick one card from each suit. This can be done in 13^4 ways. Since the fifth card can be chosen in 48 ways from the remaining cards, the answer is $13^4 \cdot 48$.

9) It's that time of the year again: the college football postseason (regretfully, Stanford will not be a part of it). Consider that there are 118 NCAA Division 1-A football teams.

(a) Suppose the NCAA decided to have a 16-team tournament to determine a champion. This would require "seeding" the tournament with the top 16 teams, each ranked 1-16. How many ways can this be done?

(b) Back to reality, the NCAA instead has 5 Bowl Championship Series (BCS) bowl games, in which 10 teams get to participate. How many ways are there to select the 10 BCS teams?

(c) Now, the 10 teams must be assigned to 5 games. How many ways can this be done if the games themselves are indistinguishable, and two teams playing each other don't have an ordering (ie, there is no "home" and "away" team)? By saying that the games are indistinguishable, we mean that there is no ordering or naming of the games. Thus if two teams are picked to play, you aren't concerned with which bowl they are in, because we don't distinguish one bowl from another.

(d) Finally, consider that the 5 games are NOT really indistinguishable, one game is considered the BCS “championship” and is therefore separate. How many ways can the 10 teams now be assigned to the five games, if one of the games is the championship and the other four are still indistinguishable?

10) This problem involves 5-card poker hands dealt from a standard pack of 52 cards, with 13 different ranks (ace, 2, 3, 4,..., king). In counting the number of hands of a certain type, the order of the cards in a hand is not significant. For example, the number of hands with all four kings and the ace of spades is 1.

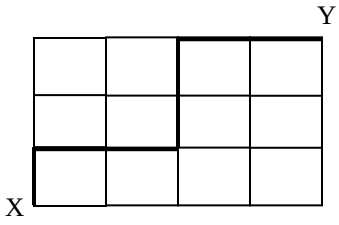
(a) How many ways can you get a full house in a 5 card poker hand? A "full house" is a hand with three cards of one rank and two cards of different rank, such as three kings and two 4's.

(b) How many hands do not have two cards of the same rank in a 5 card poker hand?

11) (a) How many binary sequences (i.e., sequences of 0's and 1's) are there of length 7 that contain four 0's and three 1's.

(b) Use the Bijection Principle to solve the following problem:

A student wishes to walk from corner X to corner Y through the streets shown in the following map. One such route is indicated by the heavy lines.



How many shortest routes are there from X to Y?