Handout #55
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CS103A
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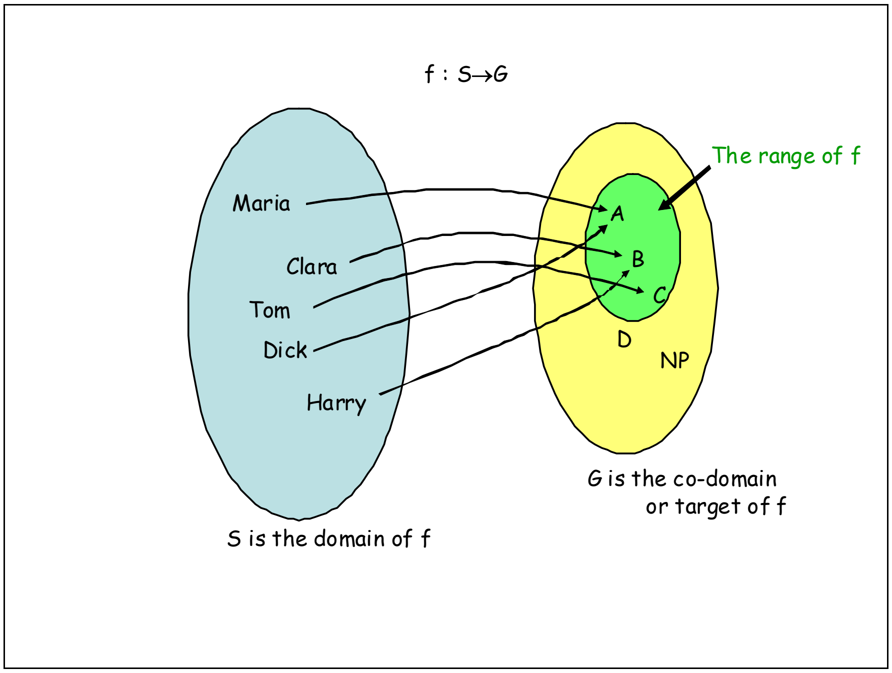
Functions

As used in ordinary language, the word *function* indicates dependence of one quantity on another. More specifically, suppose two sets of objects are given: set A and set B; and suppose that with each element of A there is associated a particular element of B. These three things: the two sets and the correspondence between elements comprise a function. We would say that the functions "maps" elements of A into elements of B.

A **function** f is a mapping from a set D to a set T with the property that for each element d in D, f maps d to a unique element of T, denoted f(d). Here D is called the **domain** of f, and T is called the **target** or **co-domain**. We write f: D → T. We also say that f(d) is the **image** of d under f, and we call the set of all images the **range** R of f.

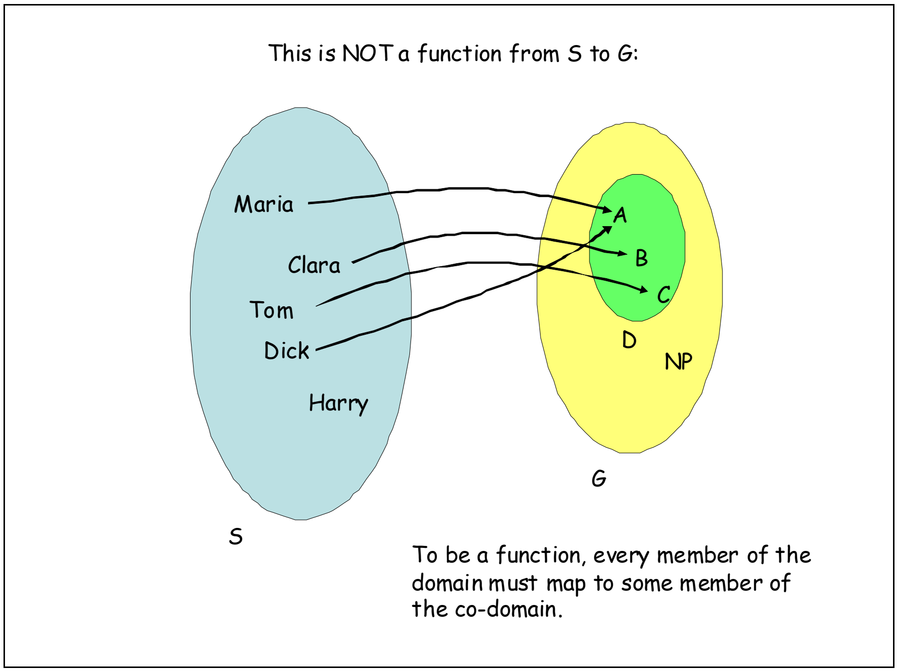
Here is an example that maps students to grades:



Here you can see how the function f "maps" students into grades. Notice the following:

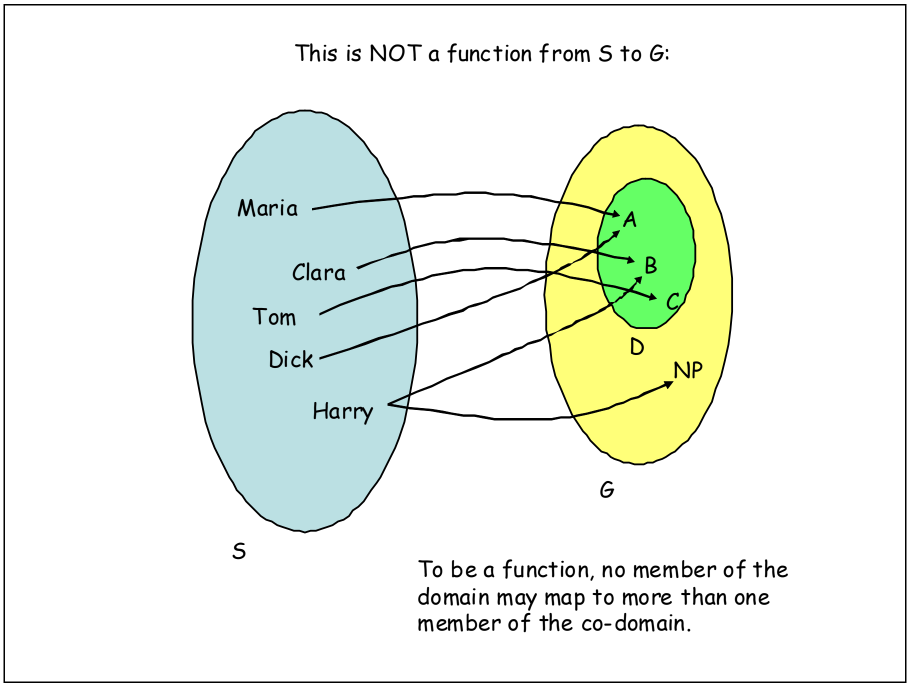
1. Every element of the domain (a set of students) maps into an element in the co-domain (the set of possible grades).
2. It is possible for more than one element in the domain to map into the same element of the co-domain.
3. Not every element in the co-domain has an element in the domain that maps to it. That is, not every element in the co-domain is the image of an element in the domain.
4. The set of co-domain elements that <u>are</u> images of domain elements is called the range.

A mapping might fail to be a function if it is not defined at every element of the domain, or if it maps an element of the domain to two or more elements in the range:



This is NOT a function from S to G:

To be a function, every member of the domain must map to some member of the co-domain.



This is NOT a function from S to G:

To be a function, no member of the domain may map to more than one member of the co-domain.

As we have seen, one way to define a function is to specify its domain, co-domain, and the correspondences between the two. Another way is to specify a rule for how the function operates, rather than listing out what maps to what.. For example, using the common notations

$\mathbb{N}$: the set of natural numbers {1, 2, 3, ...}
$\mathbb{Z}$: the set of all integers {..., -2, -1, 0, 1, 2, ...}

We could define a function $f : \mathbb{N} \to \mathbb{N}$ with the rule $f(a) = 2a$. The specification of the domain and co-domain are considered to be part of definition, so the function $g : \mathbb{Z} \to \mathbb{Z}$ where $g(a) = 2a$ is not the same function as f, even though the rules are the same.

Note that in these definitions, the range of the functions is not the same as the co-domain. For example, the range of f is positive even integers, which is not the same as $\mathbb{N}$. You might wonder why we often specify a set as the co-domain that includes elements that no member of the domain maps to. This is just a matter of convenience. We write $f : \mathbb{N} \to \mathbb{N}$ to convey the information that everything in the range of f is a natural number, without commenting (until we give the rule) on whether every natural number is actually in the range.

Example 1

Let $\mathbb{B}$ be the set of all bit strings, i.e., all finite strings of 0's and 1's (excluding the empty string)
f, g, h, j are the following functions from $\mathbb{B}$ to $\mathbb{N}$:

f(s) = the integer represented by the string s
g(s) = number of bits in s
h(s) = number of ones in s
j(s) = rightmost bit of s

If s = 110010 then f(s) = 50, g(s) = 6, h(s) = 3, j(s) = 0. The range of f, g, h, is $\mathbb{N}$, but the range of j is {0,1}. Here are two mappings from $\mathbb{B}$ to $\mathbb{N}$ that are not functions:

k(s) = the fifth bit in s counting from the left

l(s) = 1 if s ends with 1
2 if s ends with 0
4 if s ends with 00

Why do these mappings fail?

Which of the following are functions from $\mathbb{N}$ to $\mathbb{N}$ where r is an integer?

m(r) = the number of digits in the decimal representation of r

n(r) = 0 if r is even
1 if r is odd

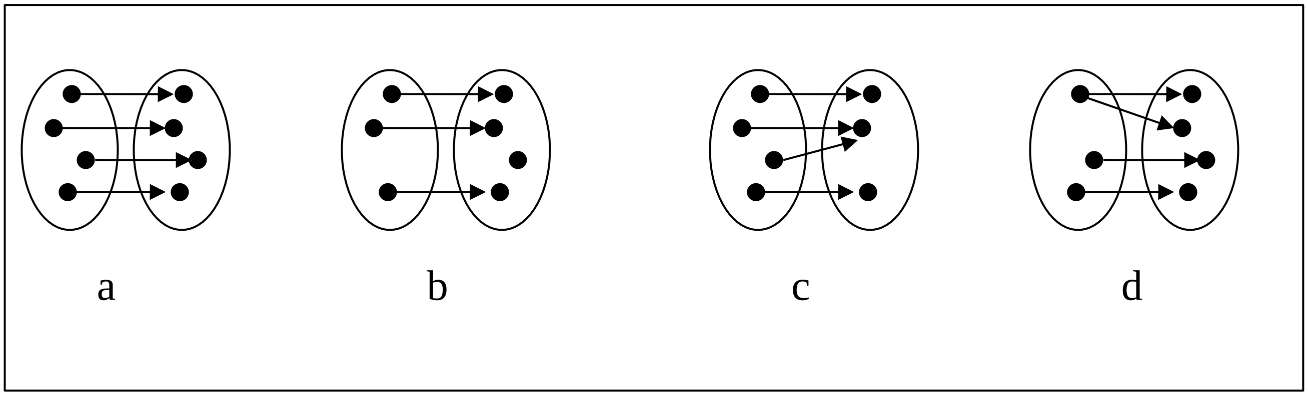
o(r) = the number whose decimal representation is r consecutive 1's

p(r) = 0 if 2 divides r
1 if 3 divides r
1 if neither 2 nor 3 divide r

Types of Functions

A function is said to be **onto** if its range is equal to its target (as in figures a and c below, but not b. Also, functions f, g, h above are onto; but j is not if the mapping is from $\mathbb{B}$ to $\mathbb{N}$). Another way of saying this is if for every element y of the co-domain, there is at least one element x in the domain such that f(x) = y.

Two functions q and r are **equal** if they have the same domain D and q(d) = r(d) for every d in D. A function is **one-to-one** if it maps distinct elements of the domain to distinct elements of the range. A formal definition of one-to-one is: if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ (or, equivalently, the contrapositive: if $f(x_1) = f(x_2)$ then $x_1 = x_2$). In the diagram below, functions a and b are **one-to-one**. A function is a **one-to-one correspondence** if it is both one-to-one and onto. Function a in the diagram below is an example.



Summarizing the diagram:

a is one-to-one and onto, and is thus a one-to-one correspondence
b is one-to-one but not onto
c is not one-to-one but is onto
d is not a function

Other terms you may have seen: a one-to-one function is **injective**; an onto function is **surjective**; a function that is both (a one-to-one correspondence) is **bijective**. Function f in Example 1 is one-to-one; g is not (g(101) = g(111)); h is not. Functions f, g, and h are all onto.

For a function $f: X \to Y$, we can express these categories like this:

Surjective: $\forall y \in Y\ (\exists x \in X\ (y = f(x))$

Injective: $\forall x_1, x_2 \in X\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2)\ ))$

Bijective: Surjective and Injective

Another way to look at these definitions involves the notion of a pre-image. For a function $f:X \to Y$, given an element $y \in Y$, a <u>pre-image</u> of y (under f) is an element $x \in X$ such that $y = f(x)$. So

> f is an injection iff every element of Y has at most one pre-image (a and b)
>
> f is a surjection iff every element of y has at least one pre-image (a and c)
>
> f is a bijection iff every element of y has precisely one pre-image (a)

Example 2

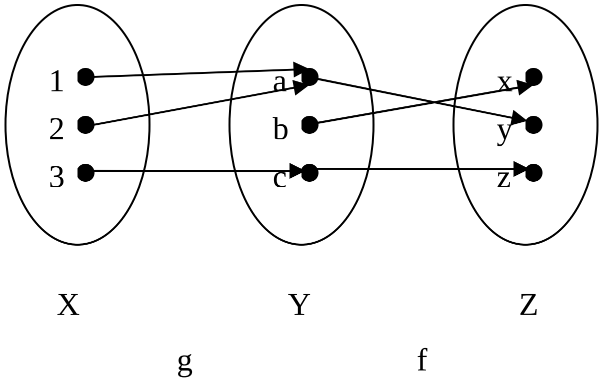
> Define $f : \mathbb{Z} \to \mathbb{Z}$ with the rule that $f(x) = x^2$ for all $x \in \mathbb{Z}$. Is f(x) one-to-one?
>
> We must show that for all integers, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
> But this is not true when $x_1 = -1$ and $x_2 = 1$, so f(x) is not one-to-one.

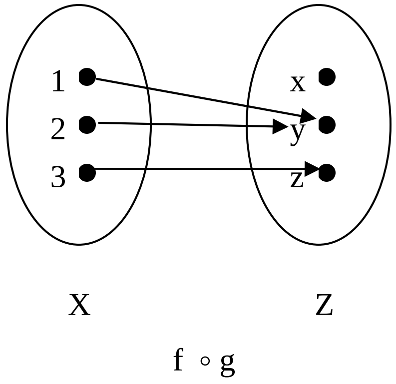
Composition of Functions

Consider two functions: the successor function as applied to integers (i.e., $g(x) = x + 1$), and a sqr function (i.e., $f(x) = x^2$). If we take the output of the successor function and use it as input to the sqr function, the two functions are operating together. They take as input an integer n, add one to it and then square n+1. We call this a **composition** of two functions.

Let g be a function $g : X \to Y$ and f be a function $f : Y' \to Z$ where Y is a subset of Y'. A new function (denoted by $f \circ g$) is defined by the following rule: For all $x \in X$, $(f \circ g)(x) = f(g(x))$. This is called the **composition of f and g**.

Here is a diagram to illustrate:





Notice in the successor-sqr example mentioned earlier that composition is not commutative.

Example 3

Let g be the function from the set {a,b,c} to itself such that $g(a) = b$; $g(b) = c$ and $g(c) = a$. Let f be a function from the set {a,b,c} to the set {1,2,3} such that $f(a) = 3$; $f(b) = 2$; $f(c) = 1$. What is the composition of f and g? What is the composition of g and f?

$(f \circ g)$ is defined by $(f \circ g)(a) = f(g(a)) = f(b) = 2$; $(f \circ g)(b) = f(g(b)) = f(c) = 1$; $(f \circ g)(c) = f(g(c)) = f(a) = 3$.

$(g \circ f)$ is not defined because the co-domain of f is not a subset of the domain of g.

Example 4

$f(x) = 2x + 3$; $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

$(f \circ g)(x) = (f(g(x)) = f(3x+2) = 2(3x+2) + 3 = 6x + 7$

$(g \circ f)(x) = (g(f(x)) = g(2x+3) = 3(2x+3) + 2 = 6x + 11$

The Growth of Functions

Now that we have all the definitions out of the way, we should point out that our main concern is often with certain functions, and how these functions behave when we increase the x in f(x). The functions we are most interested in are as follows:

$f(x) = 1$ (i.e., f is constant)
$f(x) = \log x$
$f(x) = x$
$f(x) = x \log x$
$f(x) = x^2$
$f(x) = 2^x$
$f(x) = x!$

These functions are listed in increasing order of growth rate. Recall that logarithms are defined as follows: If b is a positive real number not equal to 1 and y is a positive real number, then the logarithm to the base b of y ($\log_b y$) is the unique real number x such that $b^x = y$. In other words, $\log_b y$ = the power to which b must be raised in order to obtain y.

The performance of algorithms can often be described in terms of one of these functions. For example, we might be able to show that if n is the size of the data set for a particular algorithm, then as n becomes larger, the time taken by the algorithm is goes up in a manner that is proportional to n log n. Growth rates of n^2 and larger are not practical for large data sets. You will study this topic in detail in CS103B and in other CS theory classes.

Bibliography

For additional information on functions, consult Discrete Math textbooks or Calculus textbooks. The following are excellent references:

S. Epp, *Discrete Mathematics with Applications*, Belmont, CA: Wadsworth, 1990.

R. McEliece, R. Ash, C. Ash, *Introduction to Discrete Mathematics*, New York: Random House, 1989.

K. Rosen, *Discrete Mathematics and its Applications 4th Ed.*, New York: McGraw-Hill, 1999.

A reference that specifically addresses the growth of functions:

G. Hardy, E. Wright, *Introduction to the Theory of Numbers*, London: Oxford University Press, 1938.

Historical Notes

The term "function" and the form (x, f(x)) is attributed to Leibniz who used it to refer to several of his mathematical formulas, but the concept of a function was first presented by René Descartes (1596-1650) in his *Geometry* of 1637. In this book, he presents the concept of defining an equation in terms of one variable (i.e., as a function of...), and how to graph equations in the Cartesian plane. The formal definition of a function (as a mapping from domain to target) was first formulated for sets of numbers by Lejeune Dirichlet (1805-1859) in 1837.